2. Mehrdimensionale Analysis

Grundidee: Funktionen mehrerer Variablen

- Höhe eines Gebirges $h = h(x,y)$
- Temperatur im Raum $T = T(x,y,z)$
- zeitabhängige Potenziale $V = V(t,x,y,z)$
- thermodynamische Zustandsgleichungen: $f(p,V,T) = 0$

etwa: $pV = nRT$ (ideales Gas)

Bem: hier (Kap 2) nur skalare Funktionen. Für vektorielle Funktionen s. Kap. 4 Vektoranalysis

2.1 Partielle Ableitung

Def: Sei $f: \mathbb{R}^n \to \mathbb{R}$, $x \mapsto f(x_1, \dots, x_n)$. Dann heißt die Ableitung nach einer Variablen (bei Festhalten aller anderen) partielle Ableitung:

$$\frac{\partial f}{\partial x_i} := \lim_{\Delta x_i \to 0} \frac{f(x_1, \dots, x_i + \Delta x_i, \dots, x_n) - f(x_1, \dots, x_i, \dots x_n)}{\Delta x_i}$$

Bem.: • "∂" : "de", "del", Ableitungskringel ...

• $n=2$, $f(x,y)$: Notation

$$\frac{\partial f}{\partial x}(x,y) \equiv \partial_x f(x,y) \equiv f_x(x,y) := \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$\frac{\partial f}{\partial y}(x,y) \equiv \partial_y f(x,y) \equiv f_y(x,y) := \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Bsp.: (i) $f(x,y) = x^2 + xy^3$

$$\Rightarrow \frac{\partial f}{\partial x} = 2x + y^3 \qquad (y \text{ als Konstante behandelt!})$$

$$\frac{\partial f}{\partial y} = 3xy^2 \qquad (x \text{ ——— " ———})$$

(ii) höhere Ableitungen:

$$\frac{\partial}{\partial x}\frac{\partial}{\partial x} f \equiv \frac{\partial^2}{\partial x^2} f = f_{xx} = 2, \quad \frac{\partial}{\partial x}\frac{\partial}{\partial y} f \equiv \frac{\partial^2}{\partial x \partial y} f \equiv f_{xy} = 3y^2$$

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x} f \equiv \frac{\partial^2}{\partial y \partial x} f \equiv f_{yx} = 3y^2, \quad \frac{\partial}{\partial y}\frac{\partial}{\partial y} f \equiv \frac{\partial^2}{\partial y^2} f \equiv f_{yy} = 6xy$$

Satz von Schwarz: Für stetig differenzierbare Funktionen ist die Reihenfolge der partiellen Ableitungen egal:

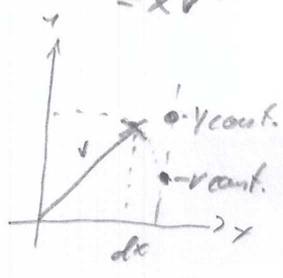
$$\frac{\partial^2}{\partial x \partial y} f(x,y) = \frac{\partial^2}{\partial y \partial x} f(x,y).$$

Bem.: • $\partial_x \partial_y = \partial_y \partial_x$: symmetrischer Operator

• ggf. festgehaltene Koordinaten explizit angeben: $f(x,y) = x(x^2+y^2) = xr^2$

$$\underbrace{\left(\frac{\partial f}{\partial x}\right)_y}_{y\text{ const.}} = 3x^2 + y^2 \neq \underbrace{\left(\frac{\partial f}{\partial x}\right)_r}_{r\text{ const.}} = r^2 = x^2 + y^2$$

(etwa Nebenbeding.) (→2.7)



2.2 Das totale Differenzial

Idee: Änderung einer Funktion $f(x,y)$ $(f: \mathbb{R}^2 \to \mathbb{R})$ bei gleichzeitigen kleinen Änderungen von x und y um Δx bzw. Δy:

$$\Delta f = f(x+\Delta x, y+\Delta y) - f(x,y)$$

$$= f(x+\Delta x, y+\Delta y) \overbrace{- f(x, y+\Delta y) + f(x, y+\Delta y)}^{=0} - f(x,y)$$

$$= \underbrace{\frac{f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y)}{\Delta x}}\Delta x + \underbrace{\frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}}\Delta y$$

$$\approx \frac{\partial f}{\partial x}(x, \underbrace{y+\Delta y}) \Delta x + \frac{\partial f}{\partial y}(x,y) \Delta y$$

$\to y$ für Δy infinitesimal

Für infinitesimale $\Delta x, \Delta y$ $(\Delta x \to dx, \Delta y \to dy)$ gilt:

$$\boxed{df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy}$$

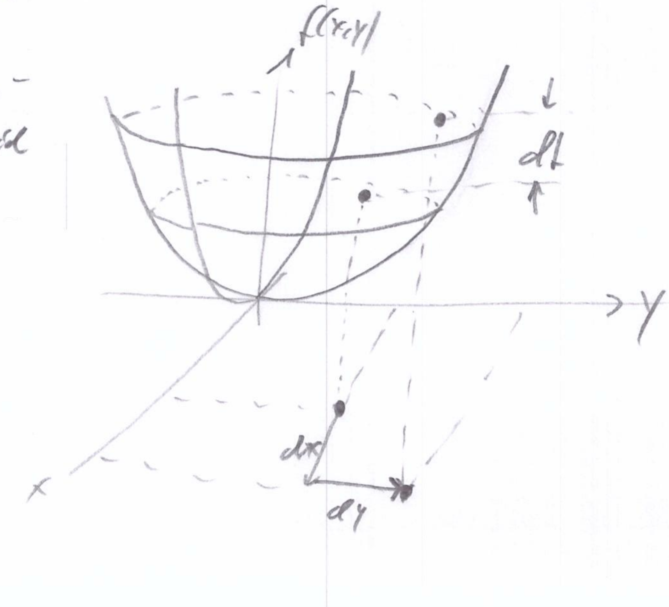
Def: Das **totale Differenzial** von $f: \mathbb{R}^n \to \mathbb{R}$ ist gegeben durch:

$$df := \frac{\partial f}{\partial x_1} dx_1 + \ldots + \frac{\partial f}{\partial x_n} dx_n = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

Bsp.: Totales Differenzial von $f(x,y) = x^2 + y^2$:

$$df(x,y) = 2x\,dx + 2y\,dy$$

Rotations-paraboloid



2.3 Totales Differenzial & Ableitungsregeln

- mehrdimensionale Kettenregel für $f:\mathbb{R}^n \to \mathbb{R}$, $f=f(x_1,\dots,x_n)$ mit $x_i = \overset{\text{i. Koordinate}}{x_i}(\overset{\text{Zeit}}{t})$:

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, dx_i \quad \text{mit } dx_i = \frac{\partial x_i}{\partial t}\, dt \quad i=1,\dots,n$$

$$\Rightarrow df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial t}\, dt$$

$$\Rightarrow \boxed{\frac{df}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial t}}$$

Bem: $n=1$ ✓ siehe 1.4. Ableitungsregeln $h(x)' \overset{h(g(x))}{=} \frac{dh}{dy}\Big|_{y=g(x)} \frac{dg}{dx} = \frac{dh}{dg}\frac{dg}{dx}$

Bsp: Schwerpunkt von n Massen m_i an den Orten $x_i(t)$ mit Gesamtmasse

$$M = \sum_{i=1}^{n} m_i : \quad R(t) = \frac{1}{M}\sum_{i=1}^{n} m_i\, x_i(t)$$

Massen m_i mit x_i gewichtet

Frage: Geschwindigkeit des Schwerpunkts:

(i) Linearität: $$\frac{dR(t)}{dt} = \frac{d}{dt} M^{-1}\sum_{i=1}^{n} m_i\, x_i(t)$$

$$= M^{-1}\sum_{i=1}^{n} m_i\, \dot{x}_i(t) \quad \text{mit } m_i \text{ gewichteter Mittelwert}$$

(ii) Kettenregel: $$\frac{d}{dt}R(x(t)) = \sum_{i=1}^{n} \frac{\partial R}{\partial x_i}\underbrace{\frac{\partial x_i}{\partial t}}_{= \dot{x}_i = \frac{dx_i}{dt}} \quad x_i \text{ hängt nur von } t \text{ ab}$$

$$\frac{\partial R}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{1}{M}\sum_{j=1}^{n} m_j x_j \quad \text{(Achtung: Summationsindex hier } j\text{)}$$

$$= \frac{1}{M}\sum_{j=1}^{n} m_j \underbrace{\frac{\partial x_i}{\partial x_j}}_{= \delta_{ij} := \begin{cases} 1 & \text{für } j=i \\ 0 & \text{sonst} \end{cases}} \quad \text{Kronecker-Delta}$$

nur $j=i$ bleibt

$$= \frac{1}{M} m_i$$

$$\Rightarrow \frac{dR}{dt} = \sum_{i=1}^{n} \frac{m_i}{M} x_i$$

2. Multidimensional analysis

2.1. Partial derivatives

Let $f: \mathbb{R}^n \to \mathbb{R}$, $\underline{x} \mapsto f(x_1, \dots, x_n)$ be a function on $\mathbb{R}^n$ (or an open subset $\Omega \subset$ ...

Then, the partial derivative of f with respect to the i-th variable is

defined as
$$\frac{\partial f}{\partial x_i} := \lim_{\Delta x_i \to 0} \frac{f(x_1, \dots, x_i + \Delta x_i, \dots, x_n) - f(x_1, \dots, x_n)}{\Delta x_i}$$

notation
$$\frac{\partial f}{\partial x_i} \equiv \partial_{x_i} f \equiv f_{x_i} \equiv D_{x_i} f$$

Schwarz's theorem: For $f: \Omega \to \mathbb{R}$ defined on $\Omega \subset \mathbb{R}^n$, if $\underline{p} \in \Omega$
(Clairaut's)

and f has continuous second partial derivatives on a neighbourhood of $\underline{p}$,

then
$$\frac{\partial^2}{\partial x_i \partial x_j} f(\underline{p}) = \frac{\partial^2}{\partial x_j \partial x_i} f(\underline{p}) \quad \forall\, i, j \in \{1, \dots, n\}$$

2.2 Total differential

The sum of all partial differentials w.r.t. all independent variables $x_1, \dots, x_n$ is called total differential:

$$df = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

2.3 Total differentiation and rules of differentiation

- chain rule: $f: \mathbb{R}^n \to \mathbb{R}$, $f = f(x_1, \dots, x_n)$ with $x_i = x_i(t)$.

Then:
$$\frac{df}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial t}$$

• Totale vs. partielle Zeitableitung:

Mechanik: oft implizite & explizite Zeitableitungen:

$$f = f(x(t),\ y(t),\ t)$$

Bsp.: $f = x(t)^2 + y(t)^2 + \alpha t$

↳ partielle Ableitung: $\frac{\partial f}{\partial t} \equiv \left(\frac{\partial f}{\partial t}\right)_{x,y\ \text{const}} = \alpha$

↳ totale zeitliche Ableitung: tatsächliche/im Experiment beobachtete Änderung von f:

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial t}$$

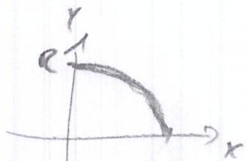
$$\Rightarrow \frac{df}{dt} = 2x\dot{x} + 2y\dot{y} + \alpha$$

• implizite Differentiation:

Idee: Zusammenhang von x und y nur als $f(x,y)=0$, aber nicht $y=y(x)$

Frage: $\frac{dy}{dx} = ?$

↳ Bsp.: (i) Kreis: $f(x,y) = R^2 - x^2 - y^2 = 0$, etwa $x,y \geq 0$



↳ $df = -2x\,dx - 2y\,dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$

↳ check $(R,0)$, $(0,R)$

(ii) Ableitung von $y = x^x$, $x>0$:

$\ln y = \ln x^x \Rightarrow f(x,y) = \ln y - x \ln x = 0$

$\Rightarrow df = \frac{1}{y}dy - \left(\ln x + x\frac{1}{x}\right)dx = 0$

$\Rightarrow \frac{dy}{dx} = y(\ln x + 1) = x^x(\ln x + 1)$

$\lim_{x\to 0} x \ln x = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0} -x$

Bem.: $\lim_{x\to 0} x^x = 1$ z.B. mit Regel von l'Hospital

$$\lim_{x\to 0} x^x = e^{\ln \lim_{x\to 0} x^x} = e^{\lim_{x\to 0} \ln x^x} = e^{\lim_{x\to 0} x \ln x} = e^0 = 1$$

2.4 Der Gradient

Für eine Funktion $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ lautet das totale Differenzial

bei $\underset{\uparrow \text{ Vektor}}{\underline{a}} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$: $df(\underline{a}) = \underset{\uparrow \; \equiv \frac{\partial f}{\partial x_1}}{\partial_{x_1}} f(\underline{a})\, dx_1 + \dots + \underbrace{\partial_{x_n} f(\underline{a})}_{\partial_{x_n} f(\underline{x})|_{\underline{x}=\underline{a}}} dx_n$

Alternative Schreibweise mit Vektoren: $d\underline{x} \equiv \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix}$

Wir definieren den **Gradienten** von f als

$$\operatorname{grad} f(\underline{a}) := \begin{pmatrix} \partial_{x_1} f(\underline{a}) \\ \vdots \\ \partial_{x_n} f(\underline{a}) \end{pmatrix} = \begin{pmatrix} \partial_{x_1} \\ \vdots \\ \partial_{x_n} \end{pmatrix} f(\underline{a}) .$$

$$\Rightarrow df(\underline{a}) = \operatorname{grad} f(\underline{a}) \overset{\text{Skalarprodukt}}{\cdot} d\underline{x} = \begin{pmatrix} \partial_{x_1} f(\underline{a}) \\ \vdots \\ \partial_{x_n} f(\underline{a}) \end{pmatrix} \cdot \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix} = \sum_{i=1}^{n} \partial_{x_i} f(\underline{a})\, dx_i$$

Merke: grad f liefert durch Skalarprodukt mit $d\underline{x}$ die Änderung von f unter einer infinitesimalen Verschiebung $d\underline{x}$

Bem.: übliche Schreibweise ist $\underline{\nabla} f(\underline{a}) \equiv \operatorname{grad} f(\underline{a})$ mit dem

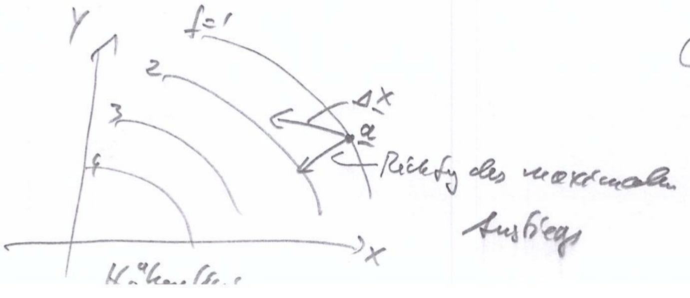
Nabla-Operator $\underline{\nabla} = \begin{pmatrix} \partial_{x_1} \\ \vdots \\ \partial_{x_n} \end{pmatrix} \Rightarrow df(\underline{a}) = \underline{\nabla} f(\underline{a}) \cdot d\underline{x}$

- $\underline{\nabla} f(\underline{a})$ zeigt in Richtg des maximalen Anstiegs von f bei $\underline{a}$:

$$\Delta f = \underline{\nabla} f \cdot \Delta \underline{x} = |\underline{\nabla} f|\, |\Delta \underline{x}| \cos \overset{\text{Winkel zwischen } \Delta\underline{x} \text{ und } \underline{\nabla} f(\underline{a})}{\theta}$$ (Kap. 3.3 Skalarprodukt)

↳ bei $\theta = 0$ ($\Delta \underline{x} \parallel \underline{\nabla} f(\underline{a})$) ist $\cos\theta = 1$

$\Rightarrow \Delta f$ am größten



2.4 Gradient

The gradient of a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ at $\underline{a} \in \Omega$ is defined as

$$\operatorname{grad} f(\underline{a}) = \begin{pmatrix} \partial_{x_1} f(\underline{a}) \\ \vdots \\ \partial_{x_n} f(\underline{a}) \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} f(\underline{a})$$

- Using vector notation, the total differential can be rewritten as

$$df(\underline{a}) = \operatorname{grad} f(\underline{a}) \underset{\substack{\uparrow \\ \text{scalar} \\ \text{product}}}{\cdot} d\underline{x} \quad \text{with } d\underline{x} = \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix}$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(\underline{a}) \cdot dx_i$$

- notation: $\operatorname{grad} f(\underline{a}) \equiv \underline{\nabla} f(\underline{a})$ with the Nabla operator $\underline{\nabla} = \begin{pmatrix} \partial_{x_1} \\ \vdots \\ \partial_{x_n} \end{pmatrix}$

  $\hookrightarrow df(\underline{a}) = \underline{\nabla} f(\underline{a}) \cdot d\underline{x}$

- geometric intuition: (i) $\underline{\nabla} f(\underline{a})$ points in the direction of steepest slope of f at $\underline{a}$.

  (ii) $\underset{\substack{\uparrow \\ \text{change of } f}}{\Delta f} = \underline{\nabla} f \cdot \Delta \underline{x} = |\underline{\nabla} f| \cdot |\Delta \underline{x}| \cos \Theta$ ← angle between $\underline{\nabla} f$ and $\Delta \underline{x}$

- viele Anwendungen:
  - ↳ Physik: $\underline{E} = -\underline{\nabla} V$ (E-Feld; V: elektrostatisches Potenzial)
  - ↳ Physik: konservative Kräfte: $\underline{F} = -\underline{\nabla} V$
  - ↳ Optimierung (gradient descent)

2.5 Mehrdimensionale Taylor-Entwicklung

Erinnerung: Taylor-Reihe für $f: \mathbb{R} \to \mathbb{R}$: $f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0) \underbrace{(x-x_0)}^k$
(vgl. 1.6)

Δx: Abstand vom Entwicklungspunkt x_0

$$\left(= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\Delta x \frac{d}{dx}\right)^k f(x_0) \right)$$

Verallgemeinerung auf $f: \mathbb{R}^n \to \mathbb{R}$:

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \Delta\underline{x} = \begin{pmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{pmatrix}, \quad \underline{x}_0 = \begin{pmatrix} x_{01} \\ \vdots \\ x_{0n} \end{pmatrix}$$

$$f(\underline{x}) = \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{\left(\sum_{i=1}^{n} \Delta x_i \frac{\partial}{\partial x_i} \right)^k}_{(\Delta\underline{x} \cdot \underline{\nabla})^k} f(\underline{x}_0)$$

($\cdot$: "Skalarprodukt")

Bsp.: 0-te Ordnung: $k=0$: $P_0(\underline{x}) = f(\underline{x}_0)$

1. Ordnung: $k=1$: $P_1(\underline{x}) = \sum_{i=1}^{n} \left(\partial_{x_i} f(\underline{x}_0)\right) \Delta x_i$

2. Ordnung: $k=2$: Nebenrechnung: $\left(\sum_{i=1}^{2} a_i\right)\left(\sum_{i=1}^{2} a_i\right) = (a_1+a_2)(a_1+a_2)$

$$= a_1 a_1 + a_1 a_2 + a_2 a_1 + a_2 a_2$$

$$= \sum_{i=1}^{2} \sum_{j=1}^{2} a_i a_j = \sum_{i,j=1}^{2} a_i a_j$$

(Doppelsumme)

Damit: $$P_2(\underline{x}) = \frac{1}{2!}\left(\sum_{i=1}^{n} \Delta x_i \partial_{x_i}\right)^2 f(x_0)$$

$$= \frac{1}{2!}\sum_{i,j=1}^{n}\left(\partial_{x_i}\partial_{x_j} f(x_0)\right)\Delta x_i \Delta x_j$$

$$\Rightarrow f(\underline{x}) = P_0(\underline{x}) + P_1(\underline{x}) + P_2(\underline{x}) + \mathcal{O}(|\Delta x|^3)$$

$$= f(x_0) + \sum_{i=1}^{n}\left(\partial_{x_i} f(x_0)\right)\Delta x_i + \frac{1}{2}\sum_{i,j=1}^{n}\left(\partial_{x_i}\partial_{x_j} f(x_0)\right)\Delta x_i \Delta x_j + \mathcal{O}(|\Delta x|^3)$$

- Bem: Kurzschreibweise:

$$f(\underline{x}) = f(x_0) + (\nabla f(x_0))\cdot\Delta\underline{x} + \frac{1}{2}\Delta\underline{x}^{\overset{\text{transponiert}}{T}}\,\underline{\underline{H}}(x_0)\,\Delta\underline{x} + \mathcal{O}(|\Delta x|^3)$$

mit der Hesse-Matrix

$$\underline{\underline{H}}(x_0) = \begin{pmatrix} \partial_{x_1}\partial_{x_1} f(x_0) & \dots & \partial_{x_1 x_n} f(x_0) \\ \vdots & & \vdots \\ \partial_{x_n}\partial_{x_1} f(x_0) & \dots & \partial_{x_n}\partial_{x_n} f(x_0) \end{pmatrix}$$

Bsp: • Fernbeobachtungen für $n=2$: $f: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto f(x,y)$

$$f(\underline{x}) = f(x_0) + \partial_x f(x_0)\Delta x + \partial_y f(x_0)\Delta y$$

$$+ \frac{1}{2}\partial_x\partial_x f(x_0)\Delta x^2 + \frac{1}{2}\partial_{yy} f(x_0)\Delta y^2$$

$$+ \frac{1}{2}\overset{\text{S.v.Schwarz}}{2}\,\partial_x\partial_y f(x_0)\Delta x\Delta y + \mathcal{O}(|\Delta x|^3)$$

$$= f(x_0) + \begin{pmatrix}\partial_x f(x_0) \\ \partial_y f(x_0)\end{pmatrix}\cdot\begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix} + \frac{1}{2}(\Delta x\ \Delta y)\begin{pmatrix}\partial_x\partial_x f(x_0) & \partial_x\partial_y f(x_0) \\ \partial_y\partial_x f(x_0) & \partial_y\partial_y f(x_0)\end{pmatrix}\begin{pmatrix}\Delta x \\ \Delta y\end{pmatrix}$$

$$+ \mathcal{O}(|\Delta x|^3)$$

2.6 Extremwerte / -Stellen

• Erinnerung: $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x)$: $f'(a) = 0$ notwendige Bedingung für ein Extremum bei a.

↳ $f''(a) > 0$: positive Krümmung ⇒ Minimum

$f''(a) < 0$: negative Krümmung ⇒ Maximum

$f''(a) = 0$: Maximum, Minimum oder Sattelpunkt (z.B.: $f(x) = x^4$ bei $a = 0$)

• $f'(a) = 0$ ⇒ a ist ein stationärer Punkt (Fixpunkt) $\dot{x} = f(x)$

• Verallgemeinerung auf $f: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto f(x,y)$

↳ notwendige Bedingung: $df = \frac{\partial f}{\partial x}(\underline{a})\, dx + \frac{\partial f}{\partial y}(\underline{a})\, dy = 0$ für beliebige dx, dy.

$$\Rightarrow \frac{\partial f}{\partial x}(\underline{a}) = 0 \quad \wedge \quad \frac{\partial f}{\partial y}(\underline{a}) = 0$$

Art des stationären Punkts hängt von der Krümmung ab: (Eigenwerte Kap. 6)

(i) Maximum: $\partial_x^2 f(\underline{a}) < 0$, $\partial_y^2 f(\underline{a}) < 0$, $D := (\partial_x^2 f)(\partial_y^2 f) - (\partial_x \partial_y f)^2 > 0$

(ii) Minimum: $\partial_x^2 f(\underline{a}) > 0$, $\partial_y^2 f(\underline{a}) > 0$, $D > 0$

(iii) Sattelpunkt: $D < 0$

(iv) $D = 0$: höhere Ordnungen zu untersuchen

Bsp.: $f(x,y) = xy + x^2 + y^2 - 6y$

↳ notwendig:
$$\left.\begin{aligned} \partial_x f &= y + 2x = 0 \\ \partial_y f &= x + 2y - 6 = 0 \end{aligned}\right\} \quad \begin{aligned} x &= -2 \\ y &= 4 \end{aligned}$$

2. Ableitungen:
$$\left.\begin{aligned} \partial_x^2 f &= 2 \quad > 0 \\ \partial_y^2 f &= 2 \quad > 0 \\ \partial_x \partial_y f &= 1 \Rightarrow D > 0 \end{aligned}\right\} \text{ Minimum}$$

• Verallgemeinerung auf $f: \mathbb{R}^n \to \mathbb{R}$:

↳ $df(\underline{a}) = 0 \quad \forall\, dx_1, \dots, dx_n \Rightarrow \partial_{x_1} f = 0 = \partial_{x_2} f = \dots = \partial_{x_n} f$

oder kompakt: $\underline{\nabla} f(\underline{a}) = 0$ ← Nullvektor

↳ Art des stationären Punkts über Eigenwerte der Hesse-Matrix

↳ alle Eigenwerte $> 0 \Rightarrow$ Minimum

↳ alle Eigenwerte $< 0 \Rightarrow$ Maximum

2.7 Extrema unter Nebenbedingungen / Methode der Lagrange-Multiplikatoren

hier: $n = 2$

Idee: Extrema von $f(x,y)$ unter einer Nebenbedingung $g(x,y) = 0$

Bsp.: Maximum von $f(x,y) = -(x^2+y^2)$ unter Nebenbedingung $y = x + c$

$\Rightarrow g(x,y) = y - x - c = 0$

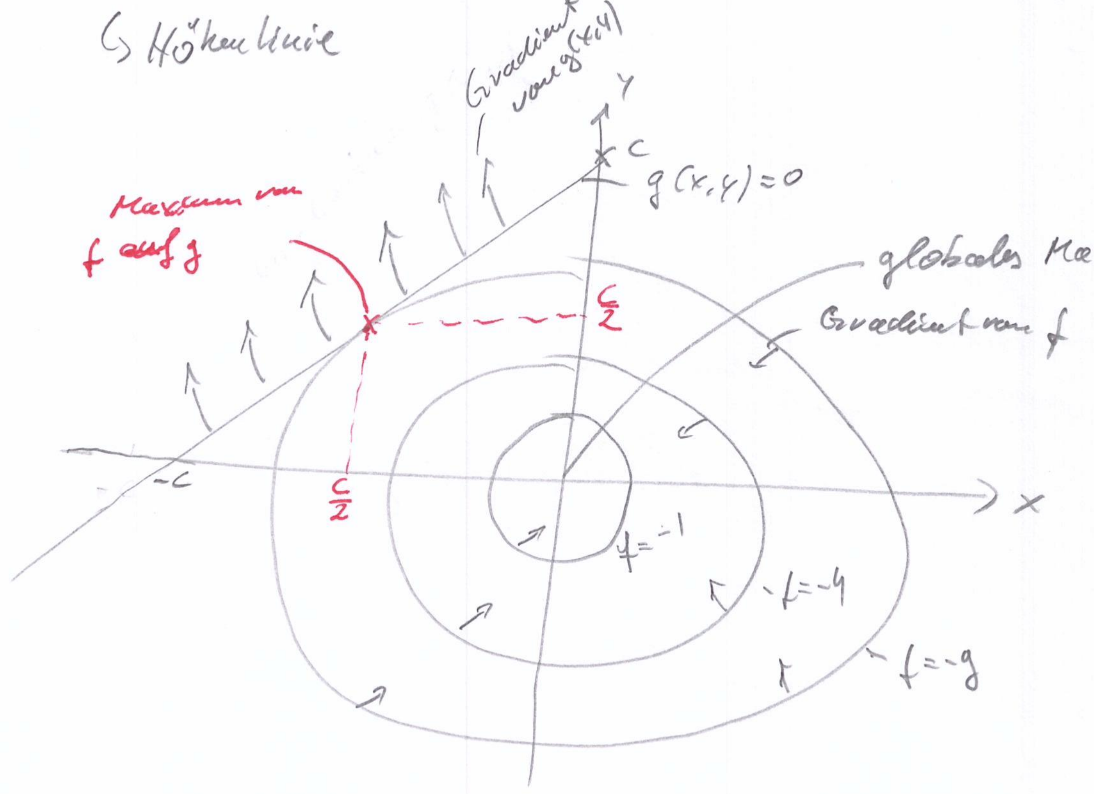
$\partial_x g = -1$

$\partial_y g = 1$

nach unten geöffneter Rotationsparaboloid

↳ s. 22 (VL6)

↳ Höhenlinie



$\partial_x^2 f = -2 < 0$

$\partial_y^2 f = -2 < 0$

$D = 4 > 0$

↳ Gradienten stehen senkrecht auf Höhenlinien.

↳ Extrema mit Nebenbedingungen, wo Gradient von f und g parallel sind

$\Rightarrow \underline{\nabla} f(x,y) = -\lambda\, \underline{\nabla} g(x,y) \Leftrightarrow \underline{\nabla}(f + \lambda g) = 0$

$\nabla\underbrace{(f+\lambda g)}_{=:h(x,y,\lambda)}=0 \quad \wedge \quad g(x,y)=0$: 3 Gleichungen für x, y, λ

$h(x,y,\lambda) = -(x^2+y^2) + \lambda(y-x-c)$

$\Rightarrow$

$$\nabla h = 0 \quad \Leftrightarrow \quad \left.\begin{aligned} \frac{\partial h}{\partial x} &= -2x-\lambda &= 0 \\ \frac{\partial h}{\partial y} &= -2y+\lambda &= 0 \\ y-x-c &&= 0 \end{aligned}\right\}$$

$\Rightarrow$ Lösung: $\lambda = c, \quad x = -\frac{c}{2}, \quad y = \frac{c}{2}$

- Verallgemeinerung auf s Nebenbedingungen: $g_i(\underline{x})=0 \quad, i=1,\dots,s$

$\hookrightarrow \quad h(\underline{x}, \lambda_1, \dots, \lambda_s) := f(\underline{x}) + \sum_{i=1}^{s} \lambda_i \, g_i(\underline{x})$

s Lagrange-Parameter / -Multiplikator

$$\left.\begin{aligned} &\Rightarrow \; n \text{ Gleichungen: } \nabla h(\underline{x}, \lambda_1, \dots, \lambda_s) = 0 \\ & s \text{ Gleichungen: } g_i(\underline{x}) = 0 \quad i=1,\dots,s \end{aligned}\right\} n+s \text{ Unbekannte } \checkmark$$

2.5 High(er)dimensional Taylor expansion

reminder: $f: \mathbb{R} \to \mathbb{R}$: $f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0) \underbrace{(x-x_0)^k}_{\Delta x^k}$

generalization to $f: \mathbb{R}^n \to \mathbb{R}$ using vector notation:

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \Delta x = \begin{pmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{pmatrix}, \quad x_0 \in \mathbb{R} \longrightarrow \underline{x}_0 \in \mathbb{R}^n, \quad \frac{d}{dx} \to \underline{\nabla} = \begin{pmatrix} \partial_{x_1} \\ \vdots \\ \partial_{x_n} \end{pmatrix}$$

$$\Rightarrow \quad f(\underline{x}) = \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{\left(\sum_{i=1}^{n} \Delta x_i \frac{\partial}{\partial x_i} \right)^k}_{(\Delta \underline{x} \cdot \underline{\nabla})^k} f(\underline{x}_0)$$

scalar product

up to order 2: $k = 0, 1, 2$

transposed: $\Delta \underline{x}^T = (x_1, \dots, x_n)$

$$f(\underline{x}) = f(\underline{x}_0) + \underline{\nabla} f(\underline{x}_0) \cdot \Delta \underline{x} + \frac{1}{2} \Delta \underline{x}^T \underline{\underline{H}}(\underline{x}_0) \Delta \underline{x} + O(|\Delta \underline{x}|^3)$$

with the Hessian / Hesse matrix (of 2^{nd} derivatives)

$$\underline{\underline{H}}(\underline{x}_0) = \begin{pmatrix} \partial_{x_1} \partial_{x_1} f(\underline{x}_0) & \cdots & \partial_{x_1} \partial_{x_n} f(\underline{x}_0) \\ \vdots & & \vdots \\ \partial_{x_n} \partial_1 f(\underline{x}_0) & \cdots & \partial_{x_n} \partial_{x_n} f(\underline{x}_0) \end{pmatrix}$$

2.6 Extreme values

reminder: $f: \mathbb{R} \to \mathbb{R}$ with extreme value/extremum at a: $f'(a) = 0$

↳ $f''(a) > 0$ ◡ : Minimum (positive curvature)

↳ $f''(a) < 0$ ⌒ : Maximum (negative curvature)

↳ $f''(a) = 0$: anything possible (including saddles)

generalization to $f: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto f(x,y)$

$$df(\underline{a}) = \frac{\partial}{\partial x} f(\underline{a})\, dx + \frac{\partial}{\partial y} f(\underline{a})\, dy = 0 \Rightarrow \frac{\partial}{\partial x} f(\underline{a}) = 0 \wedge \frac{\partial f}{\partial y}(\underline{a}) = 0$$

↳ Minimum: $\partial_x^2 f(\underline{a}) > 0$, $\partial_y^2 f(\underline{a}) > 0$, $D := (\partial_x^2 f)(\partial_y^2 f) - (\partial_x \partial_y f)^2 > 0$

$\partial_x \partial_x f = \frac{\partial^2}{\partial x^2} f$

↳ Maximum: $\partial_x^2 f(\underline{a}) < 0$, $\partial_y^2 f(\underline{a}) < 0$, $D > 0$

↳ Saddle: $D < 0$

↳ unclear (higher orders needed): $D = 0$

$n > 2$:
$df(\underline{a}) = 0$
Eigenvalues of $\underline{\underline{H}}(\underline{a})$:
↳ all negative: Maximum
↳ all positive: Minimum

2.7 Extremum with constraints

idea: extremum of $f(x,y)$ for a constraint $g(x,y) = 0$

↳ $\underline{\nabla} f(x,y) = -\lambda \underline{\nabla} g(x,y)$ (parallel gradients)

λ: Lagrange multiplier

2.8 Integralrechnung (skalare Funktionen) in $\mathbb{R}^n$

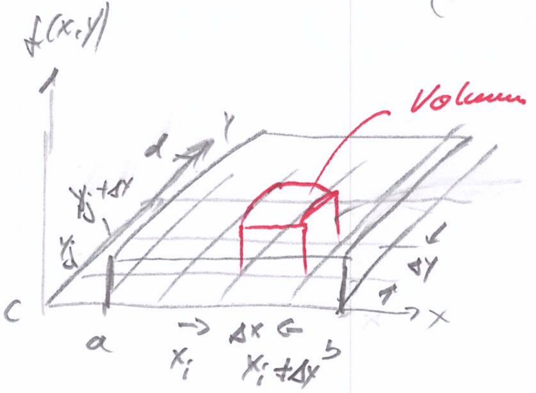
(1) Definition: ($\mathbb{R}^2$)

Sei $f: \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^2$ stetig in Ω (etwa $[a,b] \times [c,d]$). Wir definieren das **zweidimensionale Integral** als

$$\underbrace{\int_\Omega f(x,y)\,dx\,dy}_{\text{Volumen unter } f(x,y)} := \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \sum_{j=0}^{\frac{d-c}{\Delta y}-1} \sum_{i=0}^{\frac{b-a}{\Delta x}-1} f(x_i, y_j)\,\Delta x\,\Delta y$$

↳ Anzahl der Stützstellen: $N_x = \frac{b-a}{\Delta x}$ bzw. $N_y = \frac{d-c}{\Delta y}$

$\left(\Delta x = \frac{b-a}{N_x}\right)$ $\left(\Delta y = \frac{d-c}{N_y}\right)$



$x_i = a + i\Delta x$

$y_j = c + i\Delta y$

(2) Satz von Fubini:

$$\text{Notation: } \int_\Omega f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy \equiv \int_c^d dy \int_a^b dx\, f(x,y)$$

$$\int_\Omega f(x,y)\,dx\,dy = \lim_{N_y \to \infty} \sum_{j=0}^{N_y-1} \Bigg(\underbrace{\lim_{N_x \to \infty} \sum_{i=0}^{N_x-1} f(x_i,y_j)\,\Delta x}_{\int_a^b (x, y_j)\,dx} \Bigg) \Delta y$$

$$= \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$$

$$= \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$$

(3) Beispiele:

(i) $\Omega = [0,2] \times [0,1]$ (Rechteck), $f(x,y) = xy + y^2$

$$\Rightarrow \int_\Omega f(x,y)\,dx\,dy = \int_0^2 dx \int_0^1 dy\,(xy+y^2)$$

$$= \int_0^2 dx \left(\frac{1}{2}xy^2 + \frac{1}{3}y^3\right)\Big|_{y=0}^{y=1}$$

$$= \int_0^2 dx \left(\frac{1}{2}x + \frac{1}{3}\right)$$

$$= \frac{1}{4}x^2 + \frac{1}{3}x\Big|_{x=0}^{2}$$

$$= \frac{1}{4}4 + \frac{1}{3}2 - 0$$

$$= \frac{5}{3}$$

$$= \int_0^1 dy \int_0^2 dx\,(xy+y^2)$$

$$= \int_0^1 dy \left(\frac{1}{2}x^2y + xy^2\right)\Big|_{x=0}^{2}$$

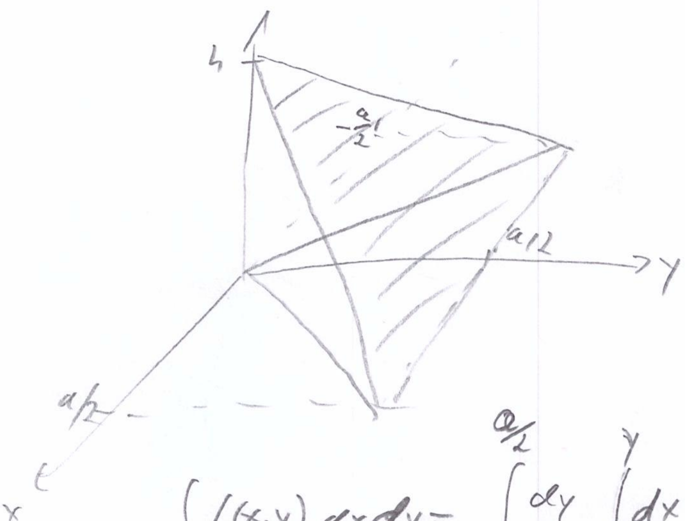
$$= \int_0^1 dy\,(2y+2y^2)$$

$$= y^2 + \frac{2}{3}y^3\Big|_{y=0}^{1}$$

$$= 1 + \frac{2}{3} = \frac{5}{3}$$

(erst x- dann y-Integration liefert dasselbe Ergebnis)

(ii) Dreieck:



$$f(x,y) = h - \frac{h}{a/2}\,y$$

$$\Omega = \{(x,y) \mid 0 \le y \le \frac{a}{2} \wedge -y \le x \le y\}$$

x-Richtung hängt von y ab!

$$\int_\Omega f(x,y)\,dx\,dy = \int_0^{a/2} dy \int_{-y}^{y} dx \left(-\frac{2h}{a}y + h\right) = h\int_0^{\frac{a}{2}} dy \left(-\frac{2}{a}y + 1\right)x\Big|_{x=-y}^{x=y}$$

$$= h\int_0^{\frac{a}{2}} dy \left(-\frac{4}{a}y^2 + 2y\right) = h\left(-\frac{4}{3}y^3 + y^2\right)\Big|_{y=0}^{y=\frac{a}{2}}$$

$$= \frac{1}{12}a^2h$$

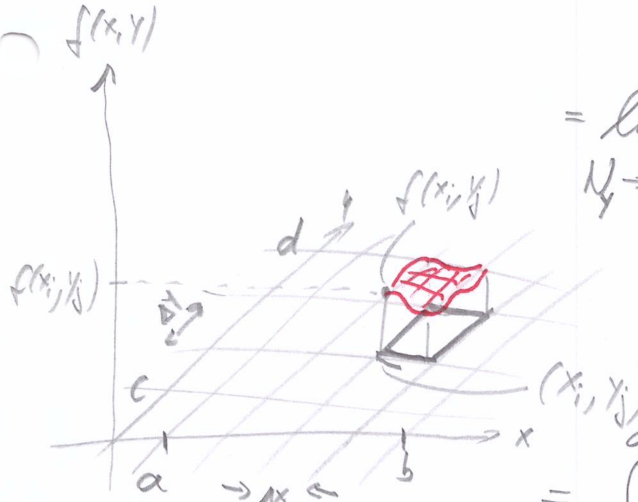
(Vgl. Viertel Pyramide $\frac{1}{3}a^2h$)

2.8 Integration (of scalar functions) in $\mathbb{R}^n$

Def.: The 2-dimensional integral of $f: \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ (continuous) over $\Omega = [a,b] \times [c,d]$ is defined as:

$$\int_\Omega f(x,y)\,dx\,dy := \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \sum_{j=0}^{\frac{d-c}{\Delta y}-1} \sum_{i=0}^{\frac{b-a}{\Delta x}-1} f(x_i, y_j)\,\Delta x\,\Delta y .$$

$$\left(\begin{array}{llll} \text{with} & x_i = a + i\,\Delta x & \text{and} & N_x = \frac{b-a}{\Delta x} \\ & y_j = c + j\,\Delta y & & N_y = \frac{d-c}{\Delta y} \end{array} \right)$$



$$= \lim_{N_y \to \infty} \lim_{N_x \to \infty} \sum_{j=0}^{N_y-1} \sum_{i=0}^{N_x-1} f(x_i, y_j)\,\Delta x\,\Delta y$$

volume of cuboid: $f(x_i, y_j)\,\Delta x\,\Delta y$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Fubini's theorem:

$$\int_\Omega f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

(4) Verallgemeinerung auf $\mathbb{R}^n$:

$f: \Omega \subset \mathbb{R}^n \to \mathbb{R}$, $f = f(x_1, \dots, x_n)$, $\Omega = I_1 \times I_2 \times \dots \times I_n$

stetig in Ω

$$\Rightarrow \int_\Omega f(x_1, \dots, x_n)\, dx_1 \dots dx_n \equiv \int_\Omega f(x_1, \dots, x_n)\, d^n x$$

$$= \int_{I_1} dx_1 \int_{I_2} dx_2 \dots \int_{I_n} dx_n\, f(x_1, \dots, x_n)$$

Bsp./Notation:

(i) Flächenintegrale: $\displaystyle\int_\Omega f(x,y)\, dx\, dy \equiv \int_\Omega f(x,y) \underbrace{dA}_{\text{Flächenelement}}$

(ii) Volumenintegrale: $\displaystyle\int_\Omega f(x,y,z)\, dx\, dy\, dz \equiv \int_\Omega f(\underline{x})\, d^3x = \int_\Omega f(x_1, x_2, x_3)\, d^3x$

$$\equiv \int_\Omega f(x,y,z) \underbrace{dV}_{\text{Volumenelement}}$$

(iii) Dichte: diskret $\rho = \dfrac{\Delta m}{\Delta V}$

$\hookrightarrow$ Gesamtmasse: $\displaystyle M = \int_{\text{Volumen}} \rho\, dV$

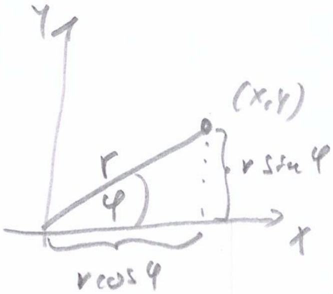
(iv) Volumen einer Teilmenge des $\mathbb{R}^3$: $\Omega \subset \mathbb{R}^3$

$$V = \int_\Omega 1\, d^3x = \int_\Omega dV$$

2.9 Variablen-/Koordinatentransformationen

Idee: Koordinaten, die an die Symmetrie des Systems/Problems angepasst sind. $\Rightarrow$ Rechnungen vereinfachen

2.9.1 Polarkoordinaten im $\mathbb{R}^2$



$x = x(r,\varphi) = r\cos\varphi$ mit $r \in [0,\infty)$, $\varphi \in [0,2\pi)$

$y = y(r,\varphi) = r\sin\varphi$

$\Rightarrow$ totale Differenziale: $dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \varphi}d\varphi$

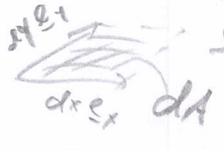
$= \cos\varphi\, dr - r\sin\varphi\, d\varphi$

$dy = \sin\varphi\, dr + r\cos\varphi\, d\varphi$

Zusammengefasst als Jacobi-Matrix $\underline{\underline{J}} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$

Flächenelement: $dA = dx\,dy = \det \underline{\underline{J}}\, dr\, d\varphi = \cos\varphi\, r\cos\varphi - (-r\sin\varphi)\sin\varphi\; dr\,d\varphi$

Achtung: nicht Produkt aus dx & dy, sondern $(dx\,e_x \times dy\,e_y)$



$= r(\cos^2\varphi + \sin^2\varphi)\,dr\,d\varphi$

$= r\,dr\,d\varphi$

$$\Rightarrow \int_\Omega f(x,y)\,dA = \int_{(x,y)\in\Omega} f(x,y)\,dx\,dy = \int_{(r,\varphi)\in\Omega} f(x(r,\varphi), y(r,\varphi))\, r\,dr\,d\varphi$$

Bsp.: (i) Fläche eines Kreises $\{(r,\varphi) \mid r<R,\ 0\le\varphi<2\pi\}$

$$A = \int_{\text{Kreis}} dA = \int_{(r,\varphi)\in\text{Kreis}} r\,dr\,d\varphi = \int_0^R dr\, r \int_0^{2\pi} d\varphi = \int_0^R dr\, r\, 2\pi = \frac{1}{2}\pi R^2 2\pi$$

$$= \pi R^2$$

(ii) Gauß-Integral: $I = \int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$ (S. 4B 3)

Trick: Berechne $I^2 = \left(\int_{-\infty}^{\infty} dx\, e^{-ax^2}\right)\left(\int_{-\infty}^{\infty} dy\, e^{-ay^2}\right)$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{-a(x^2+y^2)}$$ (Integral über (x,y)-Ebene)

$x = r\cos\varphi$
$y = r\sin\varphi$

Kettenregel!

$$= \int_0^{2\pi} d\varphi \int_0^{\infty} dr\, r\, e^{-ar^2}$$

$$= \int_0^{2\pi} d\varphi \left(-\frac{1}{2a} e^{-ar^2}\right)\Big|_{r=0}^{\infty}$$

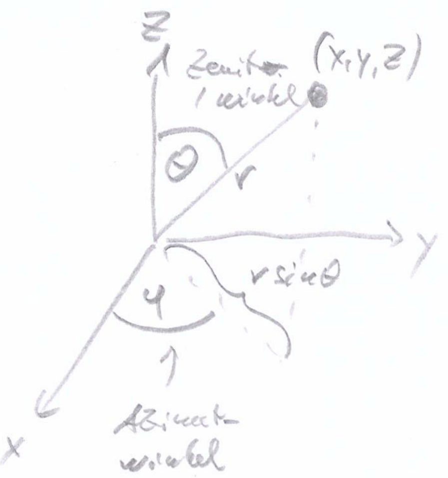
$$= \frac{1}{2a} \int_0^{2\pi} d\varphi$$

$$= \frac{1}{2a}\, 2\pi = \frac{\pi}{a}$$

$\Rightarrow I = \sqrt{\frac{\pi}{a}}$

normierte Gauß-Funktion: $\int_{-\infty}^{\infty} \sqrt{\frac{a}{\pi}}\, e^{-ax^2} dx = 1$

2.9.2 Kugelkoordinaten



$x = x(r,\Theta,\varphi) = r\sin\Theta\cos\varphi$

$y = y(r,\Theta,\varphi) = r\sin\Theta\sin\varphi$

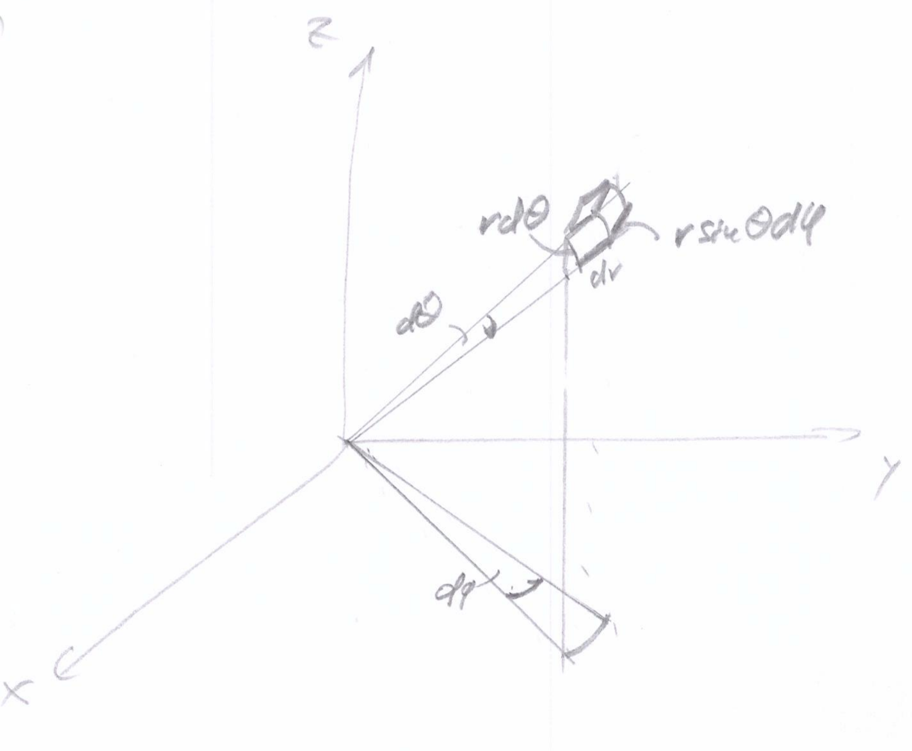
$z = z(r,\Theta,\varphi) = r\cos\Theta$

mit $r \in [0,\infty)$, $\Theta \in [0,\pi]$, $\varphi \in [0,2\pi)$

Jacobi-Matrix: $$\underline{\underline{J}} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \Theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \Theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial r} \end{pmatrix} = \begin{pmatrix} \sin\Theta\cos\varphi & r\cos\Theta\cos\varphi & -r\sin\Theta\sin\varphi \\ \sin\Theta\cos\varphi & r\cos\Theta\sin\varphi & r\sin\Theta\cos\varphi \\ \cos\Theta & -r\sin\Theta & 0 \end{pmatrix}$$

$\underset{\text{später}}{\Rightarrow} \det \underline{\underline{J}} = r^2\sin\Theta \Rightarrow \underset{\text{Volumenelement}}{dV} = dx\,dy\,dz = r^2\sin\Theta\,dr\,d\Theta\,d\varphi$

$$\int_\Omega f(x,y,z)\,dV = \int_{(x,y,z)\in\Omega} f(x,y,z)\,dx\,dy\,dz = \int_{(r,\Theta,\varphi)\in\Omega} f\big(x(r,\Theta,\varphi), y(r,\Theta,\varphi), z(r,\Theta,\varphi)\big)\, r^2\sin\Theta\,dr\,d\Theta\,d\varphi$$



Bsp.: (i) Volumen einer Kugel: $\{(r,\theta,\varphi) \mid r \le R, 0 \le \theta \le \pi, 0 \le \varphi < 2\pi\}$

$$V = \int_{\text{Kugel}} dV = \int_{(r,\theta,\varphi)\in \text{Kugel}} r^2 \sin\varphi \, dr \, d\theta \, d\varphi$$

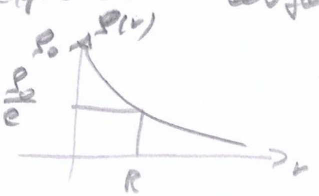
$$= \int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\varphi \; r^2 \sin\theta$$

$$= \underbrace{\int_0^R dr\, r^2}_{\frac{1}{3}R^3} \; \underbrace{\int_0^\pi d\theta \sin\theta}_{-\cos\theta\big|_0^\pi = +2} \; \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}$$

$$= \frac{4\pi}{3} R^3$$

(ii) Gesamtmasse einer exponentiell abfallenden Dichte:

$$\rho(r) = \rho_0 e^{-\frac{r}{R}}$$



$$M = \int_{\text{ganzen Raum}} \rho(r) \, dV$$

$$= \underbrace{\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi}_{\int d\Omega = 4\pi} \; \underbrace{\int_0^\infty dr \; r^2 \rho_0 e^{-\frac{r}{R}}}_{= \rho_0 R^3 \underbrace{\int_0^\infty du \; u^2 e^{-u}}_{=2! \text{ (vgl. 1.10)}}}$$

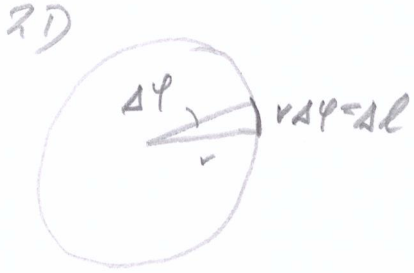
Raumwinkel

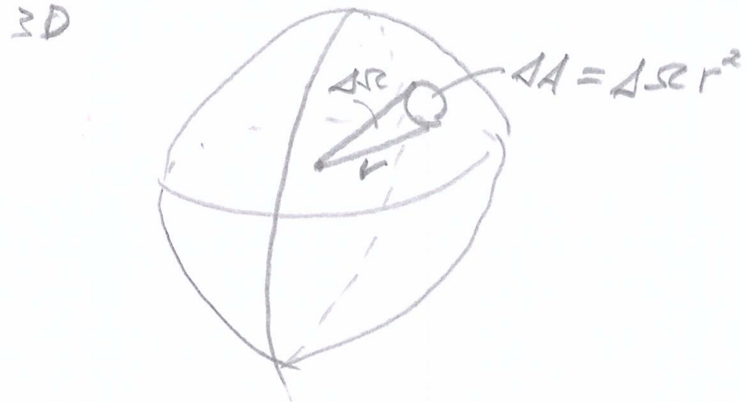
$u = \frac{r}{R}$

$R\,du = dr$

$$= 8\pi \rho_0 R^3$$

Raumwinkel:





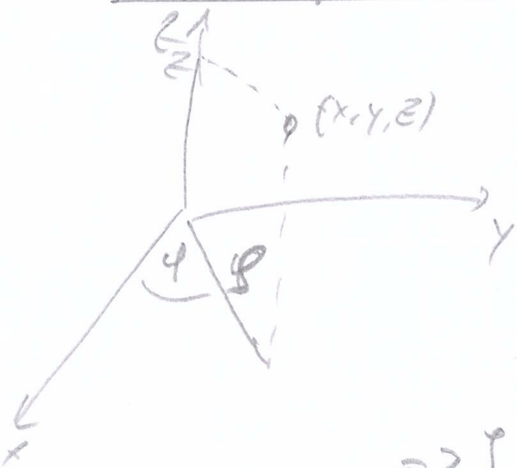
gesamter Raumwinkel:

$$\Omega = \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi = 4\pi = \frac{\text{Kugeloberfläche}}{r^2} = \frac{4\pi r^2}{r^2}$$

$$\Rightarrow dV = dA\,dr = r^2 dr\, d\Omega \underset{\substack{\uparrow \\ d\Omega = \sin\theta\, d\theta\, d\varphi}}{=} r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$\int \underset{\substack{\uparrow \\ \text{unabhängig} \\ \text{von Winkeln } \theta, \varphi}}{f(r)}\, dV = \int d\Omega \int dr\, f(r) = 4\pi \int dr\, f(r)$$

2.9.3 Zylinderkoordinaten



$$x = x(\rho,\varphi) = \rho\cos\varphi$$

$$y = y(\rho,\varphi) = \rho\sin\varphi$$

$$z = z$$

$$\Rightarrow \underline{\underline{J}} = \begin{pmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \det \underline{\underline{J}} = \rho(\cos^2\varphi + \sin$$

$$= \rho$$

$\rightarrow$ Volumenelement: $dV = dx\,dy\,dz = \rho\, d\rho\, d\varphi\, dz$

Bsp: Zylindervolumen:

Höhe

$$V = \int_{\text{Zylinder}} dV = \int_{(\rho,\varphi,z)\in \text{Zylinder}} \rho\, d\rho\, d\varphi\, dz = \int_0^R d\rho\, \rho \int_0^{2\pi} d\varphi \int_0^H dz$$

$$= \frac{1}{2}R^2 \quad 2\pi \quad \pi$$

$$= \pi R^2 H$$