

5. Komplexe Zahlen

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5.1 Definitionen & formales Rechnen

5.2 Graphische Darstellung in komplexer Ebene

5.3 Komplexe Exponentialfunktionen

5.1 Definitionen & formales Rechnen

Recap: (i) **natürliche Zahlen** $\mathbb{N} = \{1, 2, 3, \dots\}$

$+, \cdot : a, b \in \mathbb{N} \Rightarrow a+b, a \cdot b \in \mathbb{N}$, aber kein Inverses

(ii) **ganze Zahlen** $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

$\hookrightarrow$ Inverses Element bzgl. Addition, aber nicht Multiplikation

(iii) **rationale Zahlen** $\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}$

$\hookrightarrow$ Inverses Element bzgl. Multiplikation

$\hookrightarrow$ Grenzwert konvergenter Reihen ggf. nicht in $\mathbb{Q}$

(iv) **reelle Zahlen** $\mathbb{R}$

$\hookrightarrow$ Abgeschlossen von $\mathbb{Q}$ durch irrationale Zahlen wie $\sqrt{2}, \pi, e, \dots$

$\hookrightarrow$ nicht alle Polynome in Linearform darstellbar:

$$x^2 + 1 = 0 \text{ nicht in } \mathbb{R} \text{ lösbar}$$

Idee: **imaginäre Einheit** $i^2 := -1$ (häufig eingeführt als $i = \sqrt{-1}$)
aber Vorsicht!

$\Rightarrow$ Menge der **komplexen Zahlen** $\mathbb{C} := \{z = a + ib \mid a, b \in \mathbb{R}\}$

Realteil $a = \operatorname{Re}(z)$, **Imaginärteil** $b = \operatorname{Im}(z)$

$$\Rightarrow \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

4.11 Integration by parts

recap: $\int_a^b f'g \, dx = fg|_a^b - \int_a^b fg' \, dx$, $f, g: \mathbb{R} \rightarrow \mathbb{R}$

For scalar fields $f: \mathbb{R}^n \rightarrow \mathbb{R}$ and vector fields $\underline{v}: \mathbb{R}^n \rightarrow \mathbb{R}^n$:

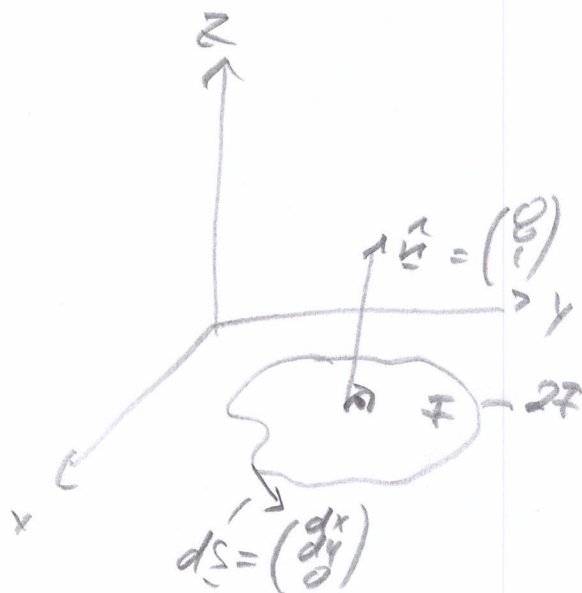
$$\int_V (\nabla f) \cdot \underline{v} \, dV = \int_{\partial V} f \underline{v} \cdot d\underline{A} - \int_V f (\nabla \cdot \underline{v}) \, dV$$

4.12 Green's theorem

$\mathcal{F}$: plane region (surface in $\mathbb{R}^2$) $\Rightarrow$ normal vector $\underline{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

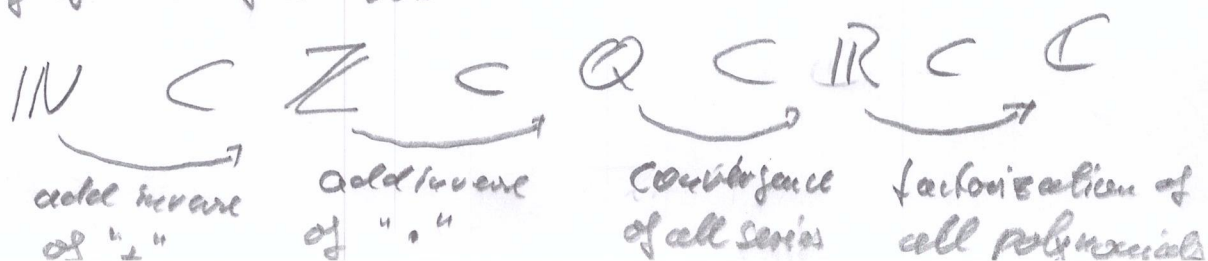
$\underline{v}: \mathcal{C}^1$ vector field with $\underline{v} = \begin{pmatrix} f(x,y) \\ g(x,y) \\ v_z \end{pmatrix}$

$$\Rightarrow \int_{\partial \mathcal{F}} f(x,y) dx + g(x,y) dy = \int_{\mathcal{F}} \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy$$



5 Complex numbers

Hierarchy of sets of numbers:



Rechenregeln:

$$(i) \text{ Addition: } \underbrace{(a+ib)}_{z_1} + \underbrace{(c+id)}_{z_2} = \underbrace{(a+c)}_{\operatorname{Re}(z_1+z_2)} + \underbrace{i(b+d)}_{\operatorname{Im}(z_1+z_2)}$$

$$(ii) \text{ Multiplikation: } (a+ib)(c+id) = (ac-bd) + i(ad+bc)$$

$$(iii) \text{ Komplex konjugierte Zahl } z^* = \bar{z} = a-ib$$

$$\hookrightarrow \operatorname{Re}(z) = \frac{1}{2}(z+z^*), \quad \operatorname{Im}(z) = \frac{1}{2i}(z-z^*)$$

$$(iv) \text{ Division: } \frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{z_2^*}{z_2^*} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd) + i(bc-ad)}{c^2+d^2}$$

$$(v) \text{ Betrag: } |z| = \sqrt{a^2+b^2} = \sqrt{(a+ib)(a-ib)} = \sqrt{z z^*}$$

$$\text{Bsp: Löse } x^2 + 2x + 2 = 0$$

quadratische Ergänzen:

$$x^2 + 2x + 1 - 1 + 2 = 0$$

$$\Leftrightarrow (x+1)^2 = -1$$

$$\Leftrightarrow x+1 = \pm i$$

$$\Rightarrow x_{1,2} = -1 \pm i$$

pq-Formel:

$$x_{1,2} = -\frac{2}{2} \pm \sqrt{\left(\frac{2}{2}\right)^2 - 2}$$

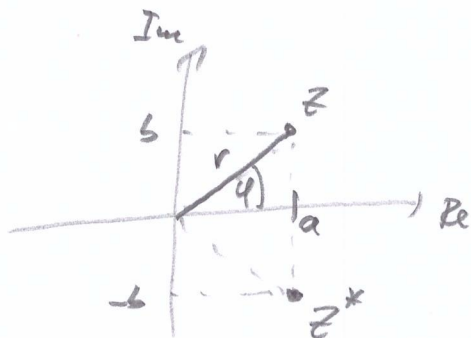
$$= -1 \pm \sqrt{-1}$$

$$= -1 \pm i$$

5.2 Graphische Darstellung in komplexer Ebene

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Idea: $x = \operatorname{Re}(z)$, $y = \operatorname{Im}(z) \Rightarrow$ Polardarstellung



$$r = |z| = \sqrt{a^2 + b^2}$$

$$\operatorname{Re} z = r \cos \varphi$$

$$\operatorname{Im} z = r \sin \varphi$$

$$\Rightarrow z = r(\cos \varphi + i \sin \varphi)$$

↑
Betrag

↑
Phase $\varphi = \arg(z)$

Wichtig: i) $\varphi \rightarrow \varphi + n 2\pi$, $n \in \mathbb{Z}$ liefert die gleiche komplexe Zahl
 $\Rightarrow$ nicht eindeutig

iii) $r \neq 0$: z reell $\Leftrightarrow \varphi = n\pi$, $n \in \mathbb{Z}$

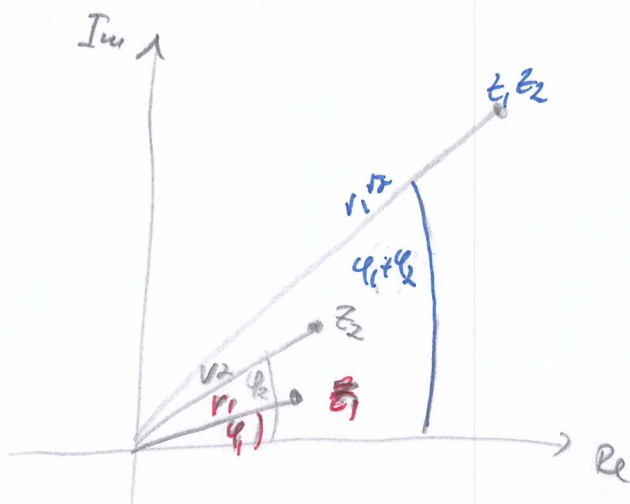
Multiplikation: $z_1 = r_1(\cos \varphi_1 + i \sin \varphi_1)$, $z_2 = r_2(\cos \varphi_2 + i \sin \varphi_2)$

$$\Rightarrow z_1 z_2 = r_1 r_2 \left[\underbrace{(\cos \varphi_1 \cos \varphi_2 - \sin \varphi_1 \sin \varphi_2)}_{\cos(\varphi_1 + \varphi_2)} + i \underbrace{(\cos \varphi_1 \sin \varphi_2 + \sin \varphi_1 \cos \varphi_2)}_{\sin(\varphi_1 + \varphi_2)} \right]$$

$$= r_1 r_2 [\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2)]$$

Beträge
multiplizieren sich

Phasen
addieren sich



Damit lässt sich der vermeintliche Widerspruch für $i = \sqrt{-1}$ auflösen:

$$-1 = i^2 = \sqrt{-1} \sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$$

$$i = 0 + i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$$

$$\Rightarrow i^2 = i i = 1 \cdot 1 \left(\underbrace{\cos \pi}_{=-1} + i \underbrace{\sin \pi}_{=0} \right) = -1$$

5.3 Komplexe Exponentialfunktionen

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Idee: Definiere die **komplexe Exponentialfunktion** über ihre Potenzreihe:

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

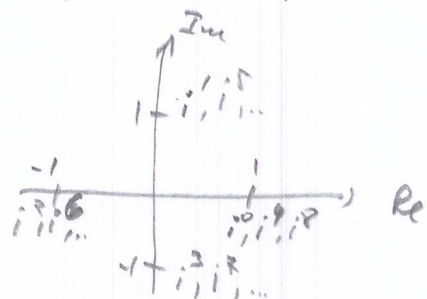
$$\hookrightarrow e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y)$$

Beweis: $e^{iy} = \sum_{n=0}^{\infty} \frac{1}{n!} i^n y^n$

n	0	1	2	3	4	5	6	7	8
i^n	1	i	-1	-i	1	i	-1	-i	1

$$= \underbrace{\left(1 - \frac{1}{2!} y^2 + \frac{1}{4!} y^4 - \frac{1}{6!} y^6 + \frac{1}{8!} y^8 + \dots \right)}_{\cos y}$$

$$+ i \underbrace{\left(y - \frac{1}{3!} y^3 + \frac{1}{5!} y^5 - \frac{1}{7!} y^7 + \dots \right)}_{\sin y}$$



(Vgl. Teil 1 & Auf 3)

Euler'sche Formel: $e^{iy} = \cos y + i \sin y, \quad y \in \mathbb{R}$

• $e^{i\pi} = -1, \quad |e^{iy}| = 1 \quad (|e^z| = e^x), \quad \cos y = \frac{e^{iy} + e^{-iy}}{2}, \quad \sin y = \frac{e^{iy} - e^{-iy}}{2i}$
 vgl. $\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$

• Polardarstellung: $z = r e^{iy}$

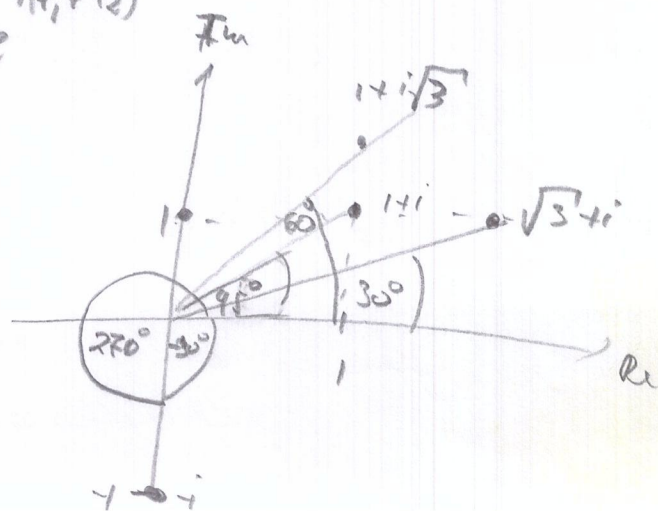
$$\hookrightarrow z_1, z_2 = r_1 e^{iy_1} r_2 e^{iy_2} = r_1 r_2 e^{i(y_1 + y_2)}$$

Bsp: $1+i = \sqrt{2} e^{i\frac{\pi}{4}}$

$$-i = e^{-\frac{\pi}{2}i} = e^{\frac{3\pi}{2}i}$$

$$\sqrt{3}+i = 2 e^{i\frac{\pi}{6}}$$

$$1+i\sqrt{3} = 2 e^{i\frac{\pi}{3}}$$



• Komplexer Logarithmus:

$$\begin{aligned}
 z = r e^{i\varphi} &= r e^{i(\varphi + 2\pi n)} \Rightarrow \ln z = \ln r e^{i(\varphi + 2\pi n)} \\
 &= \ln r + \ln e^{i(\varphi + 2\pi n)} \\
 &= \ln r + \underbrace{i(\varphi + 2\pi n)}_{\text{mehrwertig!}}
 \end{aligned}$$

• Kettenzeichenschreibweise:

$$Z = a \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\substack{\underline{E} \in \mathbb{R} \\ \cong \underline{I}}} + b \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\substack{\underline{I} \in \mathbb{R} \\ \cong \underline{I}_m}} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \stackrel{!}{=} a + ib$$

Test: (i) Einheitsmatrix: $\underline{E} \underline{E} = \underline{E}$

$$(ii) \underline{I} \underline{I} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\underline{E} \stackrel{!}{=} i^2 = -1$$

(iii) Multiplikationssatz: (oder auch mit $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}$) $\left[\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} + \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix} \right]$

$$Z_1 Z_2 = \left[\begin{pmatrix} a_1 & 0 \\ 0 & a_1 \end{pmatrix} + \begin{pmatrix} 0 & b_1 \\ b_1 & 0 \end{pmatrix} \right] \left[\begin{pmatrix} a_2 & 0 \\ 0 & a_2 \end{pmatrix} + \begin{pmatrix} 0 & b_2 \\ b_2 & 0 \end{pmatrix} \right]$$

$$= \begin{pmatrix} a_1 a_2 & 0 \\ 0 & a_1 a_2 \end{pmatrix} + \begin{pmatrix} 0 & -a_1 b_2 \\ a_1 b_2 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -b_1 a_2 \\ b_2 a_1 & 0 \end{pmatrix} + \begin{pmatrix} -b_1 b_2 & 0 \\ 0 & -b_1 b_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 a_2 - b_1 b_2 & 0 \\ 0 & a_1 a_2 - b_1 b_2 \end{pmatrix} + \begin{pmatrix} 0 & -a_1 b_2 - b_1 a_2 \\ +a_1 b_2 + b_1 a_2 & 0 \end{pmatrix}$$

$$= (a_1 a_2 - b_1 b_2) \underline{E} + (a_1 b_2 + a_2 b_1) \underline{I} \stackrel{!}{=} (ac + bd) + i(bc + ad)$$

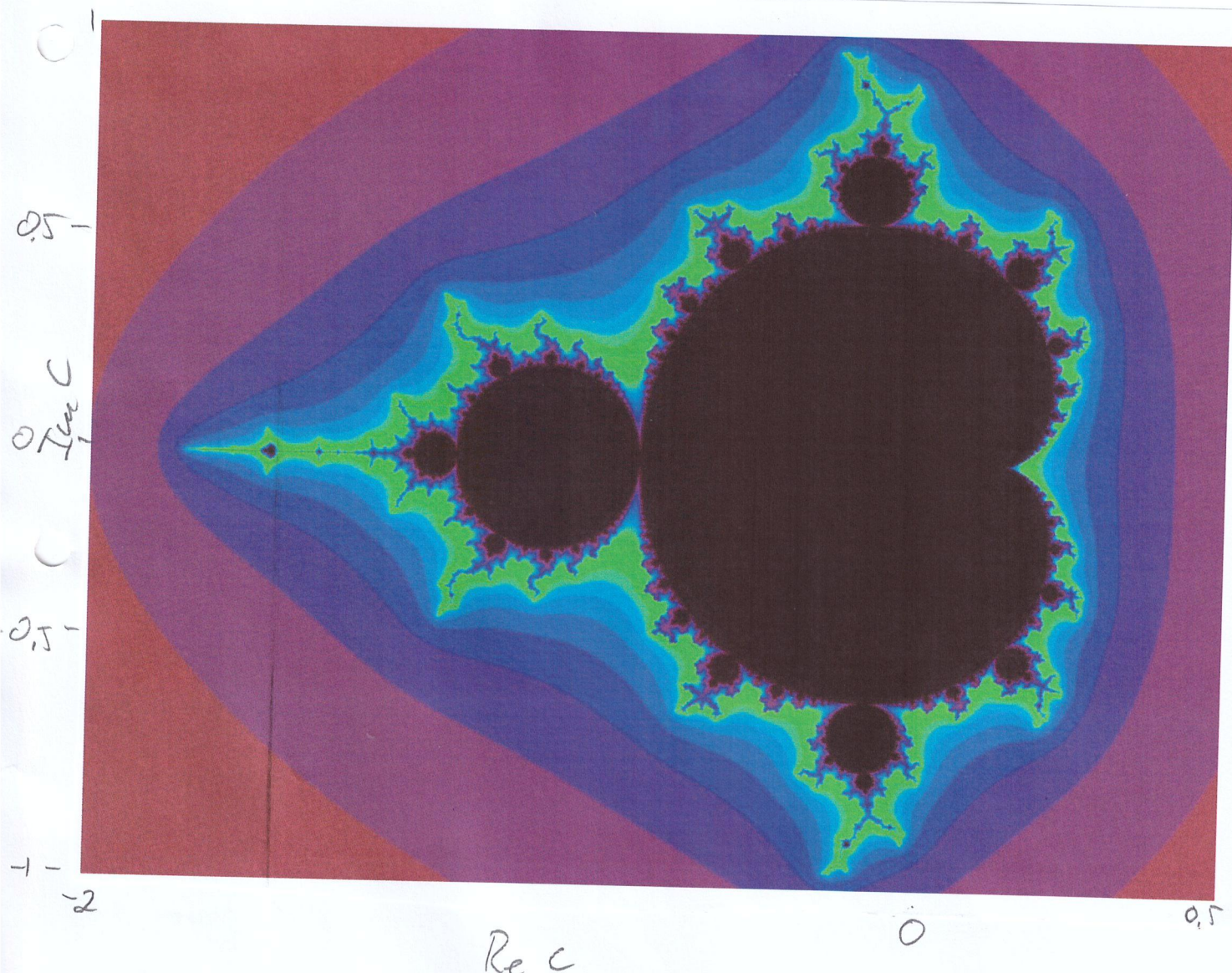
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$$z = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} \Rightarrow z^* = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

$$z = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad \text{vgl.:} \quad \begin{pmatrix} r \cos \varphi & -r \sin \varphi \\ r \sin \varphi & r \cos \varphi \end{pmatrix} \quad \begin{array}{l} \text{Drehstreckung!} \\ \text{Skalierung } r, \text{ Drehwinkel } \varphi \end{array}$$

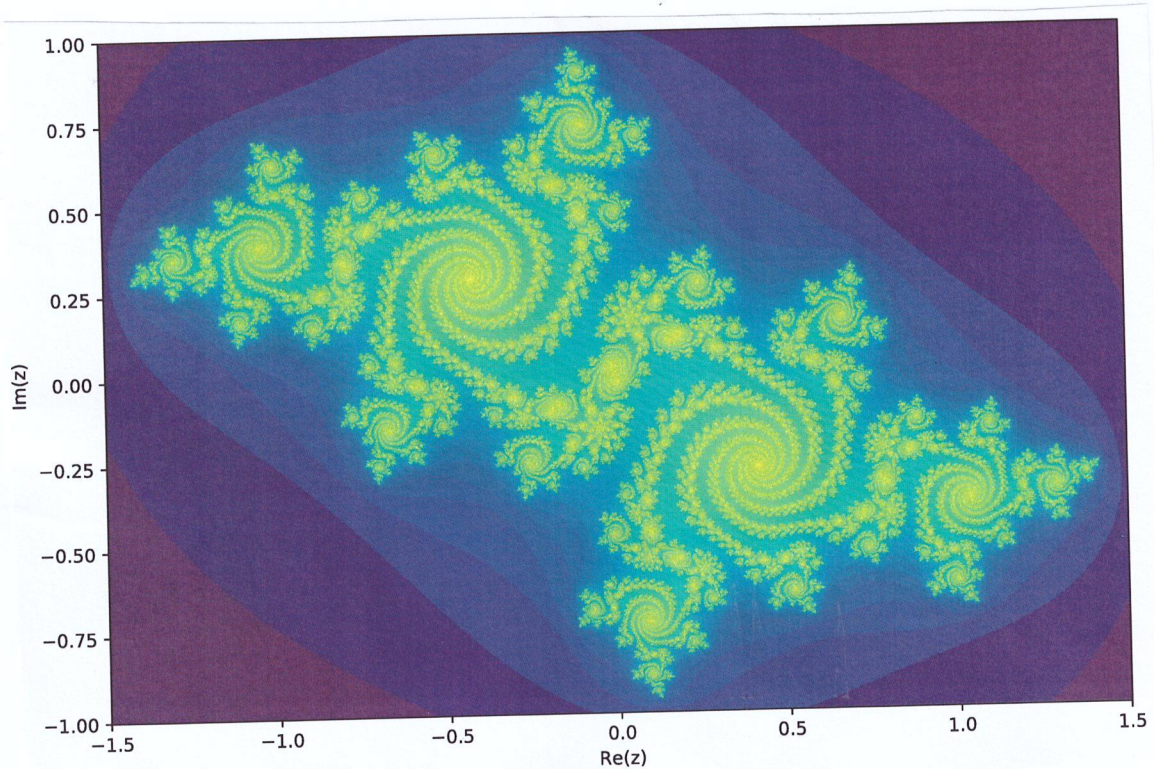
$$\hat{=} r e^{i\varphi} \quad \text{in komplexer Ebene}$$

Mandelbrot-Menge: Iteration von $f_c(z) = z^2 + c$: $z_0 = 0, z_1 = c, z_2 = c^2 + c$
 $z_3 = f_c(z_2) \dots$



Farbcode: Schwarz für $|z_n| < 1000$

braun für $|z_n| \geq 1000$, in welchem Schritt n
 (hier: rot $\hat{=} n=5$)



hier : C fest (etwa: $C = -0.51251449 + i 0.5212955$)

und z variable : $f_c(z) = z^2 + C$

Farbcode : gelb $\hat{=}$ beschränkt $z_{n+1} = z_n^2 + C$ bleibt $|z| \leq 2$

blau: Iterationsschritt, bei dem $|z| > 2$

Erweiterung auf andere Polynome möglich

↳ Fraktale Strukturen (Selbstähnlichkeit bei Zoom)

↳ Anknüpfung an dynamische Systeme und Chaos

§.1 Complex numbers (definitions and basic calculations)

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$$z = a + ib = r e^{i\varphi} \in \mathbb{C} \quad \text{with } a, b, r, \varphi \in \mathbb{R}, r \geq 0, i^2 = -1$$

$a = \operatorname{Re}(z)$: real part, $b = \operatorname{Im}(z)$: imaginary part

$$r = |z| : \text{absolute value}, \quad \varphi = \arg(z) = \arctan\left(\frac{b}{a}\right) : \text{argument}$$

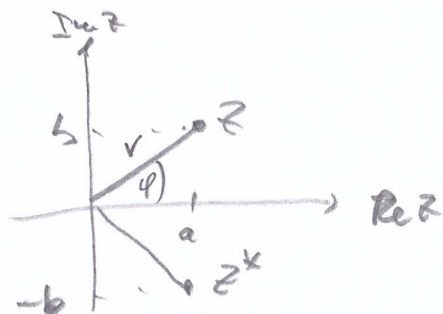
$$= \sqrt{a^2 + b^2}$$

↑
Watch out of signs of a, b

$$= \sqrt{z z^*}, \quad z^* = a - ib \text{ complex conjugate} \quad \hookrightarrow \arctan 2\text{-function!} : \arctan 2(x, y)$$

§.2 Complex plane

multi-valued phase: $z = r e^{i\varphi} = r e^{i(\varphi + n2\pi)}, n \in \mathbb{Z}$



multiplication:

$$z_1, z_2 = (a_1, a_2 - b_1, b_2) + i(a_1, b_2 + b_1, a_2) \\ = r_1 r_2 e^{i(\varphi_1 + \varphi_2)}$$

§.3 complex exponential

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = e^x e^{iy} = e^x (\cos y + i \sin y)$$

Euler's formula: $e^{i\varphi} = \cos \varphi + i \sin \varphi$

complex logarithm: $\ln z = \ln r + i(\varphi + n2\pi), n \in \mathbb{Z}$ multi-valued!

Knodelbrodt set: $f_c(z) = z^2 + c$ with varying c and $z_0 = 0$

Julia set: $f_c(z) = z^2 + c$ with fixed c and varying z_0

Complex numbers as matrices:

$$z = a \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbb{I}} + b \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\mathbb{J}} \triangleq a + ib$$

$\mathbb{I}^2 = \mathbb{I}$ $\mathbb{J}^2 = -\mathbb{I}$