Musterlösung
Integrale

$\boxed{A1}$

a) $\int \sin(x)\, dx = -\cos(x) + c$

b) $\int 2x^3\, dx = 8x^4 + c$

c) $\int \sqrt{x}\, dx = \frac{2}{3} x^{\frac{3}{2}}\, dx = \frac{2}{3}\sqrt{x^3}$

d) $\int \frac{1}{\sqrt{2x}}\, dx = \int (2x)^{-\frac{1}{2}}\, dx = (2x)^{\frac{1}{2}} + c = \sqrt{2x} + c$

e) $\int \tan(x)\, dx = \int \frac{\sin(x)}{\cos(x)}\, dx = -\int \frac{\overbrace{\sin(x)}^{f'(x)}}{\underbrace{-\cos(x)}_{f(x)}}\, dx = -\ln(|-\cos(x)|) + c$

$= -\ln(|\cos(x)|) + c$ (cos(x) ist symetrisch)

f) $\int \sin^2(x)\, dx = -\sin(x)\cos(x) + \int \cos^2(x)\, dx = -\sin(x)\cos(x) + \int 1 - \sin^2(x)\, dx$

$= -\sin(x)\cos(x) + x - \int \sin^2(x)\, dx + c = \int \sin^2(x)\, dx$

$\Leftrightarrow 2\int \sin^2(x)\, dx = x - \sin(x)\cos(x) + c$

$\Leftrightarrow \int \sin^2(x)\, dx = \frac{x - \sin(x)\cos(x)}{2} + c$

g) $\int \ln(x)\, dx = \int 1 \cdot \ln(x)\, dx = x \cdot \ln(x) - \int \frac{x}{x}\, dx = x \cdot (\ln(x) - 1) + c$

h) $\int \frac{2x+3}{x^2-x-2}\, dx \overset{!}{=} \int \frac{A}{x-2} + \frac{B}{x+1}\, dx \Rightarrow A = \frac{7}{3}\ ;\ B = -\frac{1}{3}$

$\Rightarrow \int \frac{2x+3}{x^2-x-2}\, dx = \frac{7}{3}\int \frac{1}{x-2}\, dx - \frac{1}{3}\int \frac{1}{x+1}\, dx = \frac{7}{3}\ln|x-2| - \frac{1}{3}\ln|x+1| + c$

i) $\int \frac{x^2+1}{x^3+3x}\, dx$ Sub.: $u = x^3 + 3x \quad \frac{du}{dx} = 3x^2 + 3 \Rightarrow dx = \frac{du}{3x^2+3}$

$\Rightarrow \int \frac{x^2+1}{x^3+3x}\, dx = \int \frac{x^2+1}{3x^2+3} \cdot \frac{1}{u}\, du = \frac{1}{3}\int \frac{1}{u}\, du = \frac{1}{3}\ln(|u|) + c$

$= \frac{1}{3}\ln(|x^3+3x|)$

$\boxed{A2}$

a) $\int_3^1 x^2\, dx = -\int_1^3 x^2\, dx = -\left[\frac{1}{3}x^3\right]_1^3 = -9 + \frac{1}{3} = -\frac{26}{3}$

b) $\int_0^{2\pi} \cos(x)\, dx = 0$

c) $\int_0^{\pi} x\cos(x)\, dx = x\sin(x)\Big|_0^{\pi} - \int_0^{\pi} \sin(x)\, dx = \cos(x)\Big|_0^{\pi} = -2$

$\boxed{A3}$

a) $\int_1^{\infty} \frac{1}{x}\, dx = \lim_{n\to\infty} \ln(x)\Big|_1^n = \infty$

b) $\int_1^{\infty} \frac{1}{x^2}\, dx = \lim_{n\to\infty} \int_1^n x^{-2}\, dx = \lim_{n\to\infty} -\frac{1}{x}\Big|_1^n = 1$

c) $\int_0^{\infty} x e^{-x^2}\, dx$ Sub.: $u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$

$\Rightarrow \int_0^{\infty} x e^{-x^2}\, dx = \lim_{n\to\infty} \frac{1}{2}\int_0^n e^{-u}\, du = \lim_{n\to\infty} \frac{1}{2} \cdot (e^{-n} + 1) = \frac{1}{2}$

$\boxed{A4}$

a) $f(x) = \sqrt{1-x^2}$ $\quad f'(x) = \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot (-2x) = \frac{-x}{\sqrt{1-x^2}}$

$L = \int_{-1}^{1} \sqrt{1+(f'(x))^2}\, dx$

NR: $1 + f'(x)^2 = 1 + \frac{x^2}{1-x^2} = \frac{1-x^2+x^2}{1-x^2} = \frac{1}{1-x^2}$

$\Rightarrow L = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx = \arcsin(x)\Big|_{-1}^{1} = \pi$ (Kreisumfang mit Radius 1)

b) Kreisgleichung: $x^2+y^2=R^2 \Leftrightarrow y = \sqrt{R^2-x^2} = f(x)$

$V = \pi \int_{-R}^{R} R^2 - x^2\, dx = \pi \cdot \left(2R^3 - \frac{2}{3}R^3\right) = \frac{4}{3}\pi R^3$

c) $\varrho(x) = (x+1)$

$M = \int_{-1}^{1} x+1\, dx = \left[\frac{1}{2}x^2 + x\right]_{-1}^{1} = 2\ \mathrm{kg}$

$X_s = \frac{1}{2}\int_{-1}^{1} x\cdot(x+1)\, dx = \frac{1}{2}\int_{-1}^{1} x^2+x\, dx = \frac{1}{2}\left[\frac{1}{3}x^3 + \frac{1}{2}x^2\right]_{-1}^{1} = \frac{1}{2}\cdot\left(\frac{2}{3}\right) = \underline{\underline{\frac{1}{3}}}\ \mathrm{m}$