1. [*7 Punkte*] **Discrete and Continuous Probability Distributions**

The probability that an event with success probability p occurs n times in N trials is given by the binomial distribution

$$p_n = \binom{N}{n} p^n (1-p)^{N-n} = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n} .$$

A continuous distribution is the uniform distribution

$$p_X(x) = \begin{cases} \frac{1}{2a} & \text{for } |x| < a \\ 0 & \text{otherwise} \end{cases} .$$

2 (a) Show that both distributions are normalized.

4 (b) Calculate the mean $\langle X \rangle$ and the variance $\mathrm{Var}(X) = \langle X^2 \rangle - \langle X \rangle^2$.

1 (c) How does the variance scale with N for the binomial distribution? What does the scaling of the variance imply for fluctuations?

2. [*3 Punkte*] **Cauchy Distribution**

Consider the Cauchy distribution

$$p_X(x) = \frac{1}{\pi} \frac{\gamma}{(x-x_0)^2 + \gamma^2}$$

with the location parameter x_0 and the half-width at half-maximum γ.

1 (a) Show that $p_X(x)$ is normalized.

2 (b) Discuss whether the mean $\langle X \rangle$ and variance $\mathrm{Var}(X)$ exist.

3. [*5 Punkte*] **Characteristic Function and Moments**

The characteristic function of a probability density function $p_X(x)$ is defined as

$$G(k) = \int_{-\infty}^{\infty} e^{ikx} p_X(x) \, \mathrm{d}x .$$

Using this characteristic function one can compute the moments of the distribution:

$$\langle X^m \rangle = (-i)^m \frac{\mathrm{d}^m}{\mathrm{d}k^m} G(k) \Big|_{k=0} ,$$

where $\frac{\mathrm{d}^m}{\mathrm{d}k^m} G(k)$ is the m-th derivative of $G(k)$ with respect to k. Consider the following distributions: the uniform distribution

$$p_X(x) = \begin{cases} \frac{1}{2a} & \text{for } |x| < a \\ 0 & \text{otherwise} \end{cases}$$

and the Gaussian distribution

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} \right) ,$$

with parameters μ and σ.

2 (a) Compute $G(k)$ for both distributions.

2 (b) Use $G(k)$ to compute $\langle X \rangle$ and $\langle X^2 \rangle$. Express the parameters μ and σ of the Gaussian distribution in terms of the moments.

0.5 (c) Which property makes the Gaussian distribution special in terms of its moments or characteristic function?

0.5 (d) What is the advantage of using the characteristic function to compute moments?

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4. [*5 Punkte*] **Joint probability density**

Given a probability density function $p_{X,Y}(x,y)$ of two random variables X and Y:

$$p_{X,Y}(x,y) = \begin{cases} 6\,(1-x-y) & \text{if } x \geq 0,\ y \geq 0, \text{ and } x+y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

1 (a) Check if $p_{X,Y}(x,y)$ is normalized.

2 (b) Compute the marginal distribution $p_X(x)$ and the conditional probability function $p(y|x)$.

1 (c) Are X and Y independent? Check whether $p_{X,Y}(x,y) = p_X(x)p_Y(y)$ or not.

1 (d) Does $\langle XY\rangle = \langle X\rangle\langle Y\rangle$ hold? Is this condition sufficient for independence?

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