Sheet 2

1. [*5 Punkte*] **The damped harmonic oscillator**

The equation of motion of the damped harmonic oscillator is

$$m\ddot{x} + 2\gamma m\dot{x} + kx = 0\,.$$

Assume that $\gamma < \sqrt{k/m}$ so that the oscillator is underdamped.

1 (a) Show that $x(t) = e^{-\gamma t}\big(a\cos(\omega t) + b\sin(\omega t)\big)$ is a soluation of the equation of motion, provided that ω is chosen appropriately. Determine ω in terms of m, k, and γ.

2 (b) Introduce the vector

$$\mathbf{x}(t) = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}$$

and rewrite the equation of motion as a first-order vector differential equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}\,.$$

Determine the matrix $\mathbf{A}$.

2 (c) Solve the vector differential equation by determining the eigenvalues and eigenvectors of $\mathbf{A}$. Show that your result is equivalent to the solution obtained in part (a).

2. [*2 Punkte*] **Transformation of Random Variables**

Consider the following transformation of a random variable. Let X be an exponentially distributed random variable, with probability density function (PDF):

$$p_X(x) = e^{-x}, \quad x \geq 0\,.$$

We define a new random variable $Y = e^X$.

Determine the range of Y, calculate the PDF of Y, and verify that the resulting PDF is properly normalized.

3. [*9 Punkte*] **Detailed derivation of Ito's rule**

This exercise shows how the rule $(\mathrm{d}W)^2 = \mathrm{d}t$ *emerges from a discrete-time description. The main steps of this derivation are outlined in the textbook. Fill in the missing steps in detail.*

Consider the quantity

$$Y(T) = \sum_{n=0}^{N-1} (\Delta W_n)^2\,,$$

where the increments $\Delta W_n = W((n+1)\Delta t) - W(n\Delta t)$ with $\Delta t = T/N$ are independent Gaussian random variables with

$$\langle \Delta W_n \rangle = 0 \quad \text{and} \quad \mathrm{Var}(\Delta W_n) = \Delta t\,.$$

2 (a) Determine the probability density $p_{Z_n}(z)$ of the random variable $Z_n = (\Delta W_n)^2$.

2 (b) Compute the characteristic function of Z_n, namely

$$\chi_{Z_n}(s) = \left\langle e^{isZ_n} \right\rangle = \int_0^\infty e^{isz} p_{Z_n}(z)\,\mathrm{d}z\,.$$

1 (c) Use independence to determine the characteristic function of

$$Y(T) = \sum_{n=0}^{N-1} Z_n\,.$$

Hint: Recall that characteristic functions of independent variables multiply.

1 (d) Show that

$$\lim_{N\to\infty} \chi_Y(s) = e^{isT}\,.$$

2 (e) What is the corresponding probability density $p_Y(y)$?

1 (f) Explain why this result justifies the rule $(\mathrm{d}W)^2 = \mathrm{d}t$.

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4. [*4 Punkte*] **Discrete stochastic dynamics**

The following equation is a discrete-time approximation of a linear stochastic differential equation. For $c < 0$, its continuum limit corresponds to the Ornstein–Uhlenbeck process. We analyze its behavior for the first two time steps.

Consider the discrete stochastic equation

$$x_{n+1} = x_n + c x_n \Delta t + b \Delta W_n \quad \text{with} \quad x_0 = 0\,,$$

where the increments ΔW_n are independent Gaussian random variables with

$$\langle \Delta W_n \rangle = 0 \quad \text{and} \quad \mathrm{Var}(\Delta W_n) = \Delta t\,.$$

The probability density of ΔW is

$$p(\Delta W) = \frac{1}{\sqrt{2\pi\Delta t}} \exp\left(-\frac{\Delta W^2}{2\Delta t}\right)\,.$$

Determine the probability density of $x_2 = x(2\Delta t)$.

Hint: Determine first x_1 and from this x_2. Calculate $\langle x_2 \rangle$ and $\mathrm{Var}(x_2)$. Use your result to determine the full probability density $p(x_2)$. Recall that a linear combination of independent Gaussian random variables is again Gaussian.

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