Vorlesung 2 - Algebra 2

Friday, 19 September 2025 12:35

Wie kann man 100 durch die Summe aufeinanderfolgender natürlicher Zahlen darstellen? Warum geht dies nicht mit 1024?

1. Fall $\underbrace{\text{Anzahl an Summanden}}_{\boxed{= u}}$ ungerade

31 32 33

$\uparrow$

m . mittlere Zahl

$$100 \overset{!}{=} u \cdot m$$

$$100 = 10 \cdot 10 = 2 \cdot 5 \cdot 2 \cdot 5$$

$$u = 5 \ \underset{\text{oder}}{\lor} \ u = 25$$

a) $u = 5$

$m = 2 \cdot 5 \cdot 2 = \frac{100}{5} = 20 \rightarrow$ Reihe $\overbrace{18 \ \underbrace{19 \ 20 \ 21}_{40} \ 22}^{40}$ ✓

b) $u = 25$

$m = \frac{100}{25} = 4$ $\qquad \underleftarrow{-2 \ -1 \ 0}_{12} \ 1 \ \underrightarrow{2 \ 3 \ 4}_{12}$ keine Lösung!

2. Fall $\underbrace{\text{Anzahl an Summanden}}_{= g}$ gerade

$$\Rightarrow m = \frac{u}{2}$$

$$100 \overset{!}{=} \frac{u}{2} \cdot g = \boxed{u \cdot \frac{g}{2}}$$

$100 = 2 \cdot 5 \cdot 2 \cdot 5$

a) $u = 5$ $\quad \frac{g}{2} = 20 \Rightarrow g = 40$

$m = \frac{5}{2} = 2{,}5$ $\qquad$ -- $\underleftarrow{-1 \ 0 \ 1}_{20} \ \overset{m}{\downarrow}\!2 \ \underrightarrow{3 \ 4 \ 5}_{20}$ --- keine Lösung!

b) $u = 25$ $\quad \frac{g}{2} = 4 \Rightarrow g = 8$

$m = \frac{u}{2} = 12{,}5$ $\qquad 9 \ 10 \ 11 \ 12 \ \overset{m}{\downarrow} \ 13 \ 14 \ 15 \ 16$ ✓

→ es gibt 2 Lösungen

Wie ist es mit $1024 = 2^{10} = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ hat keinen ungeraden Teiler!

Logarithmen $\log_a b = x$ „Logarithmus zur Basis a"

Umkehrung von Potenz: $a^x = b$

$a = 10 \quad \log_{10} = \log = \lg$ dekadische Logarithmus

$a = e \quad e = 2{,}718\ldots \quad \log_e = \ln$ natürliche Logarithmus

Logarithmusgesetze $\quad \log_c(a \cdot b) = \log_c(a) + \log_c(b)$

$$\log_c(a/b) = \log_c(a) - \log_c(b)$$

$$\log_c(b^a) = a \log_c(b) \quad \text{für } a \in \mathbb{R}$$

<u>Bsp</u>

$$\ln\left(\sqrt[4]{\frac{a^2 c}{b d^2}}\right) = \ln\left(\frac{a^{1/2} c^{1/4}}{b^{1/4} d^{1/2}}\right) = \ln\left(a^{1/2} c^{1/4}\right) - \ln\left(b^{1/4} d^{1/2}\right)$$

$$= \ln(a^{1/2}) + \ln(c^{1/4}) - \ln(b^{1/4}) - \ln(d^{1/2})$$

$$= \frac{1}{2}\ln(a) + \frac{1}{4}\ln(c) - \frac{1}{4}\ln(b) - \frac{1}{2}\ln(d)$$

1.5. Gleichungen mit einer Variablen

1.5.1 Lineare Gleichungen

$$a \cdot x = b \qquad (a, b \in \mathbb{R})$$

„Paul ist 15 a, seine Mutter 40 a, in welcher Zeit ist Paul halb so alt wie seine Mutter?"

$A_P = 15 + x \qquad A_M = 40 + x$

gesucht: $\quad A_P = \frac{1}{2} A_M \Rightarrow 15 + x = \frac{40 + x}{2} \Rightarrow x = 10$

$A_P = 25$

1.5.2 Quadratische Gleichungen

$$ax^2 + bx + c = 0$$

Lösung: $\underbrace{(x - x_1)}\underbrace{(x - x_2)} = 0$

Linearfaktor

1. $\underbrace{x^2 - (x_1 + x_2)x + x_1 x_2 = 0}$

$$\overset{!}{=} \; x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad (a \neq 0)$$

p-q-Formel / <u>Mitternachtsformel</u>

↙

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x_{1,2} = -\frac{b}{2a} \pm \frac{1}{2}\sqrt{\frac{b^2}{a^2} - 4\frac{c}{a}}$$

Bsp: $2x^2+6x-14=6 \Rightarrow x^2+3x-10=0 \Rightarrow x_{1,2} = \begin{cases} 2 \\ -5 \end{cases}$

kubische Gleichungen: $ax^3+bx^2+cx+d=0$ → es existiert etwas wie Mitternachtsformel

höhere Potenzen ab x^4 : kein geschlossener Ausdruck

1.5.3 Exponentialgleichungen

$$e^x = a \Rightarrow \ln(e^x) = \ln(a) \Rightarrow x = \ln(a)$$

Bsp. $3^{x^2-4} = 6^{-x}$

$\Rightarrow (x^2-4)\ln(3) = -x\ln(6)$ → quadratische Gleichung

$$\Rightarrow x^2-4 = -x\underbrace{\frac{\ln(6)}{\ln(3)}}_{\log_3(6)} \qquad \left| \frac{\log_c(x)}{\log_c(a)} = \log_a(x) \right|$$

$x^2 + \log_3(6)x - 4 = 0 \Rightarrow x_1 = -2{,}98 \quad x_2 = 1{,}34$

$x^2-4 = -x\log_3(6) \quad | + x\log_3(6)$

1.5.4 Faktorisieren

Bsp: $x^4-x^2=0 \underset{\substack{\uparrow \\ \text{äquivalent}}}{\Leftrightarrow} x^2(x^2-1)=0$

Nullproduktsatz

Ist ein Produkt = 0

⇒ mindestens ein Faktor = 0

(die Umkehrung gilt auch)

$\Rightarrow x^2=0 \quad \vee \quad x^2-1=0$

$\Rightarrow x=0 \qquad \Rightarrow x^2=1 \Rightarrow x=\pm 1$

$L = \{-1, 0, 1\}$

1 5.5 Wurzelgleichungen

Lösung durch Quadrieren → keine Äquivalenzumformung ↯

Bsp: $\sqrt{4x^2+\sqrt{x^2}+1} = x+1$

Achtung: ab jetzt andere Gültigkeit

KÄ

$4x^2+\sqrt{x^2}+1 = (x+1)^2 \Rightarrow 4x^2+\sqrt{x^2}+\cancel{1} = x^2+2x+\cancel{1}$

KÄ

$$4x^2 + \sqrt{x^2} + 1 = (x+1)^2 \Rightarrow 4x^2 + \sqrt{x^2} + \cancel{1} = x^2 + 2x + \cancel{1}$$

Ä

$$\Rightarrow 3x^2 - 2x = -\sqrt{x^2}$$

$$\Rightarrow \sqrt{x^2} = 2x - 3x^2$$

kÄ

$$x^2 = (2x - 3x^2)^2 = 4x^2 - 12x^3 + 9x^4$$

$$\Rightarrow 0 = 3x^2 - 12x^3 + 9x^4 \quad | : 3 \quad \Rightarrow \quad 0 = x^2 - 4x^3 + 3x^4$$

Faktorisieren: $0 = x^2 \cdot (1 - 4x + 3x^2)$ Nullproduktsatz: $x^2 = 0 \vee 1 - 4x + 3x^2 = 0$

$\Rightarrow x = 0$ ①

② $x^2 - \frac{4}{3}x + \frac{1}{3} = 0 \overset{p,q}{\Rightarrow} x \in \{1/3, 1\}$

Prüfen durch Einsetzen: $x = 0$ $\sqrt{4 \cdot 0^2 + \sqrt{0^2} + 1} = 0 + 1 \Rightarrow \sqrt{1} = 1$ ✓

$x = 1/3$ $\sqrt{\underbrace{4/9 + 3/9 + 9/9}} = \underbrace{\frac{3}{3} + \frac{1}{3}}_{\frac{4}{3}}$ ✓

$\sqrt{\frac{16}{9}} = \frac{4}{3}$

$x = 1$ $\underbrace{\sqrt{4 + \sqrt{1} + 1}} = 1 + 1$

ursprünglich gesuchte Lösungsmenge $\sqrt{6} = 2$ falsch!

$L = \{0, 1/3\}$

1.5.6 Betragsgleichungen

→ wir wollen Betragsstriche loswerden

→ Fallunterscheidung bereichsweise

allgemein: $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

Bsp. $|x-1| + |x-2| = 5$

$$|x-1| = \begin{cases} x-1, & x-1 \geq 0 \\ 1-x, & x-1 < 0 \end{cases}$$

$$|x-2| = \begin{cases} x-2, & x-2 \geq 0 \\ 2-x, & x-2 < 0 \end{cases}$$

① $|x-1| = x-1$, $|x-2| = x-2 \Rightarrow x \geq 1 \wedge x \geq 2 \Rightarrow x \geq 2$

$x \in [2, +\infty[$

$x - 1 + x - 2 = 5 \Rightarrow 2x = 8 \Rightarrow x = 4$

② $|x-1| = x-1$, $|x-2| = 2-x \Rightarrow x \geq 1 \wedge x < 2 \Rightarrow x \in [1, 2[$

② $|x-1| = x-1$, $|x-2| = 2-x$ $\Rightarrow$ $x \geq 1$ $\wedge$ $x < 2$ $\Rightarrow$ $x \in [1, 2[$

$x-1 + 2-x = 5$ $\Rightarrow$ $1 = 5$ falsch → keine Lösung

③ $\underline{|x-1| = 1-x}$, $|x-2| = x-2$, $\Rightarrow$ $x < 1$ $\wedge$ $x \geq 2$ → keine Lösung

④ $|x-1| = 1-x$, $|x-2| = 2-x$ $\Rightarrow$ $x<1$ $\wedge$ $x < 2$ $\Rightarrow$ $x \in]-\infty, 1[$

$1-x + 2-x = 5$ $\Rightarrow$ $-2 = 2x$ $\Rightarrow$ $x = -1$

gesamte Lösungsmenge : es gilt nun ① v ② v ③ v ④

$$(x=4 \wedge x \in [2,+\infty[) \vee \overbrace{(\underset{f}{1=5} \wedge \underbrace{x \in [1,2[}_{})}^{f}$$

$$\vee \underbrace{(x \in \{\})}_{\text{kein Beitrag}} \vee (x = -1 \wedge x \in]-\infty, 1[ )$$

$$x \in \{-1; 4\} \vee x \in \{\} \Rightarrow x \in \{-1, 4\} = L$$

Lösungen

a) 1 a $$\left[\frac{x^{3^3}}{x^{3^2} (x^3)^2} \right]^{\frac{1}{24}} = \left[\frac{x^{27}}{x^9 \cdot x^{3\cdot 2}} \right]^{\frac{1}{24}} = \left(\frac{x^{27}}{x^{15}} \right)^{\frac{1}{24}}$$

$$= x^{\frac{12}{24}} = \sqrt{x}$$

b) $$\frac{1}{x} - \frac{5}{x+3} - \frac{3}{x^2+3x} + \frac{2x}{x-2} - \frac{20}{x^2+x-6}$$

$$\frac{x+3-5x-3}{x^2+3x} + \frac{2x}{x-2} - \frac{20}{x^2+x-6}$$

$$= \frac{-4x}{x^2+3x} + \frac{2x}{x-2} - \frac{20}{x^2+x-6}$$

$$= \frac{(-4)(x-2) + 2x(x+3)}{(x+3)(x-2)} - \frac{20}{x^2+x-6}$$

$$= \frac{-4x+8+2x^2+6x-20}{x^2+x-6}$$

$$= \frac{2x^2+2x-12}{x^2+x-6}$$

$$= \frac{2(x^2+x-6)}{}$$

$$-\quad x^2+x-6$$

$$= \frac{2(x^2+x-6)}{x^2+x-6}$$

$$= 2$$

c) $$\frac{\frac{a}{a-b} - \frac{b}{a+b}}{\frac{a}{a+b} + \frac{b}{a-b}} = \frac{\frac{a^2+ab-ab+b^2}{a^2-b^2}}{\frac{-ab+a^2+ab+b^2}{a^2-b^2}} = \frac{\frac{a^2+b^2}{a^2-b^2}}{\frac{a^2+b^2}{a^2-b^2}} = 1$$

d) $$\ln\left[e^{3(\ln e^2+\ln e^6)}\right]^{\frac{1}{2}} = \frac{1}{2}\ln e^{\cdots}$$

$$= \frac{1}{2}\left(3(\ln e^2+\ln e^6)\right)$$

$$= \frac{1}{2}\,3\,(2+6)$$

$$= 12$$

e) $$\sqrt{x+16} - \sqrt{x-12} = 2 \quad \left| + \sqrt{x-12}\right.$$

$$\sqrt{x+16} = 2+\sqrt{x-12} \quad |\,(\,)^2$$

$$x+16 = 4+4\sqrt{x-12}+x-12 \quad |\,-x \;-4 \;+12$$

$$24 = 4\sqrt{x-12} \quad |\cdot 4$$

$$6 = \sqrt{x-12} \quad |\,(\,)^2$$

$$36 = x-12 \quad |\,+12$$

$$x = 48$$

f) $$8x^2-14x = 9 \quad |\;8$$

$$x^2-\frac{7}{4}x = \frac{9}{8} \quad |\left(\frac{7}{8}\right)^2$$

$$\left(x-\frac{7}{8}\right)^2 = \frac{72+49}{64}$$

$$\left(x-\frac{7}{8}\right)^2 = \frac{121}{64} \quad |\sqrt{\;}$$

$$x-\frac{7}{8} = \pm\frac{11}{8} \quad |\,+\frac{7}{8}$$

$x - \frac{7}{8} = \pm \frac{11}{8} \quad | + \frac{7}{8}$

$x_1 = -\frac{1}{2}$

$x_2 = \frac{9}{4}$

g) $x^4 - \frac{7}{4}x^2 - \frac{9}{8} = 0$

$\left(x^2 - \frac{7}{8}\right)^2 = \frac{121}{64} \quad | \sqrt{}$

$x^2 - \frac{7}{8} = \pm \frac{11}{8} \quad | + \frac{7}{8}$

$x_{1,2}^2 = -\frac{1}{2} \quad | \sqrt{} \qquad \Rightarrow x_{12} = \pm i \frac{1}{\sqrt{2}}$

$x_{3,4}^2 = \frac{9}{4} \quad | \sqrt{}$

$x_{34} = \pm \frac{3}{2}$

h) $|x+1| + |x+2| \leq 2$

fall 1: $x+1 \geq 0 \Rightarrow x+2 \geq 0$

$x+1 + x+2 \leq 2$

$2x + 3 \leq 2 \quad | -3$

$2x \leq -1 \quad | : 2$

$x \leq -\frac{1}{2} \Rightarrow -1 \leq x \leq -\frac{1}{2}$

fall 2: $x+1 = 0, \; x = -1$

$x+2 \leq 2$

$1 \leq 2$

fall 3 $\quad x+1 \leq 0 \quad x+2 \geq 0$

$$|x| := \begin{cases} x, & x \geq 0 \\ -x, & x \leq 0 \end{cases}$$

$|x+1| + |x+2| \leq 2$

$[-2, -1]$

$-(x+1) + x+2 \leq 2$

$-x - 1 + x + 2 \le 2$

$1 \le 2$

Fall 4 $\quad x+1 \le 0 \quad x+2 \le 0 \quad x \le -2$

$-x - 1 - x - 2 \le 2$

$-2x - 3 \le 2 \quad | +3$

$-2x \le 5 \quad | (-1)$

$2x \ge -5 \quad | \cdot 2$

$x \ge -\frac{5}{2} \qquad [-\frac{5}{2} ; -2]$

$\mathbb{L} = [-\frac{5}{2} ; -\frac{1}{2}]$

i) $\frac{x+2}{x^2-x-2} < -1$

fall 1 $\quad x^2 - x - 2 > 0$

$x+2 < -x^2 - x - 2 \quad | +x+2$

$0 < -x^2 \quad | (-1)$

$0 > x^2$ ↯

fall 2 $\quad x^2 - x - 2 < 0$

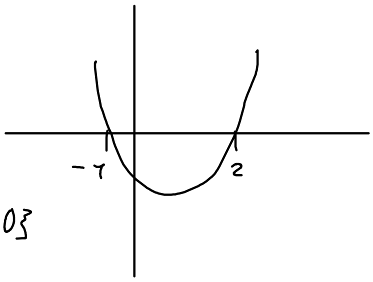
$0 > -x^2 \quad | (-1)$

$x^2 > 0$

$x^2 - x - 2 < 0$

$(x+1)(x-2)$

$(-1,2)\setminus\{0\}$ $\quad]-1,2[\setminus\{0\}$



A2 $\quad L = a+b$

$\frac{L}{a} = \frac{a}{b}$

$\frac{a+b}{a} = \frac{a}{b} \Rightarrow \frac{a}{a} + \frac{b}{a} = \frac{a}{b}$

$1 + \frac{b}{a} = \frac{a}{b}$ $\qquad \phi = \frac{a}{b}$

$1 + \frac{1}{\phi} = \phi$

$\phi - 1 - \frac{1}{\phi} = 0 \quad | \cdot \phi$

$\phi^2 - \phi - 1 = 0 \quad | + \left(\frac{1}{2}\right)^2$

$\left(\phi - \frac{1}{2}\right)^2 = \frac{5}{4} \quad | \sqrt{}$

$\phi - \frac{1}{2} = \pm\frac{\sqrt{5}}{2} \quad | + \frac{1}{2}$

$\phi = \frac{1}{2} + \frac{\sqrt{5}}{2} \approx 1{,}618$

3 $\quad (a+b)^2$

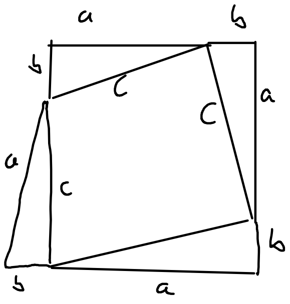
a)

A 1 Δ Dreieck : $\frac{1}{2}ab$

4mal $\Rightarrow 2ab$

A kl. Quadr. $= c^2$

groß $\quad 2ab + c^2$



$(a+b)^2 = 2ab + c^2$

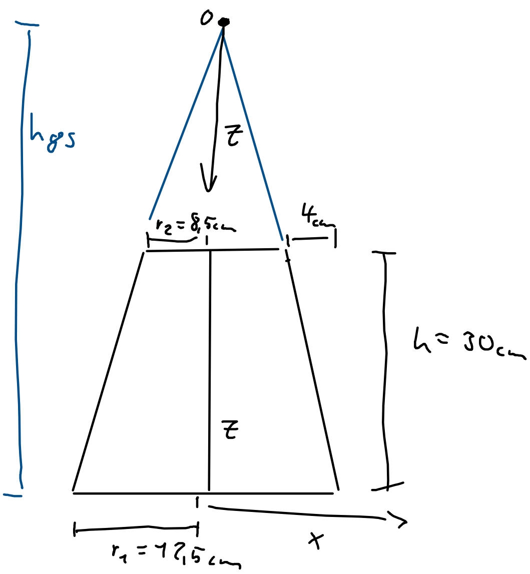
$a^2 + 2ab + b^2 = 2ab + c^2 \quad | -2ab$

$a^2 + b^2 = c^2$

$a^2+b^2=c^2$

b) $V=\frac{1}{3}G\cdot h$

$\overset{cm}{12{,}5}-8{,}5\,cm=4\,cm$



$\frac{4}{30}h_{ges}=12{,}5cm$

$h_{ges}=93{,}75cm$

$V_{groß}$ $\frac{1}{3}\pi\,12{,}5cm^{2\,2}\cdot h_{ges}=15339cm^3$

V_{klein} $\frac{1}{3}\pi(8{,}5cm)^2(h_{ges}-h)=4823cm^3$

$V_{tat}=V_{groß}-V_{klein}=10516cm^3$

$=10{,}516\,l$

$M_{groß}=\pi r_1\sqrt{h_{ges}^2+r_1^2}$

$=3714cm^2$

$M_{klein}=\pi r_2\sqrt{(h_{ges}-h)^2+r_2^2}$

$=1747cm^2$

$M_{tat}=1997cm^2$

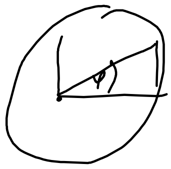
$=0{,}1997m^2=19{,}97dm^2$

$A\text{ Boden}=\pi r_2^2=\pi(8{,}5cm)^2=2{,}27dm^2$

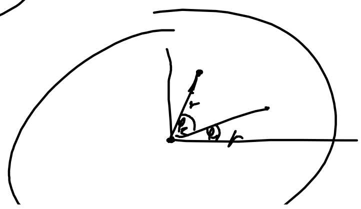
$A_{ges}\Rightarrow 22{,}24dm^2$

c) $\begin{pmatrix}x\\y\\z\end{pmatrix}$ $\begin{pmatrix}r\\\varphi\\z\end{pmatrix}$ $r=\sqrt{x^2+y^2}$ $\tan\varphi=\frac{y}{x}$

$\varphi=\tan^{-1}\frac{y}{x}$



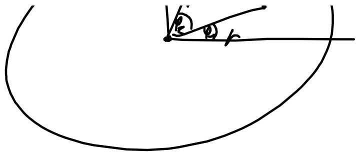
a) $\begin{pmatrix}1\\0\\6\end{pmatrix}$ b) $\begin{pmatrix}\sqrt{2}\\\frac{\pi}{4}\\0\end{pmatrix}$ c) $\begin{pmatrix}\sqrt{2}\\\frac{\pi}{4}\\1\end{pmatrix}$ d) $\begin{pmatrix}\sqrt{3}\\54{,}74^0\\6\end{pmatrix}$



$(\sqrt{13}$ $(0$

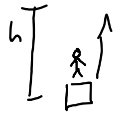
6) 0) 1) 0

e) $\begin{pmatrix} \sqrt{13} \\ 56{,}31^\circ \\ 4 \end{pmatrix}$ f $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$



4 $F = m \cdot a$

$= \frac{kg\ m}{s^2}$



$F\,h = W$

$E_{kin} = \frac{1}{2} m v^2$

$= \frac{1}{2} kg \frac{m^2}{s^2}$

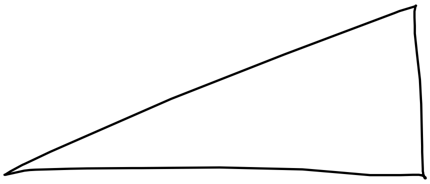
$$E = \frac{Q}{4\pi \frac{As}{Vm}} \frac{m}{m^3} \Rightarrow \frac{Q}{4\pi As} \frac{1}{m^2} Vm$$

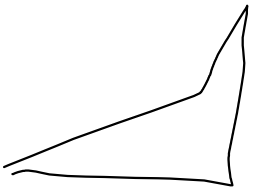
$$\Rightarrow \frac{Q}{4\pi As} \frac{1}{m} V$$

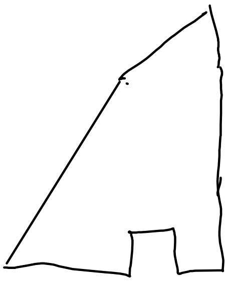
$$\Rightarrow \frac{A}{A} \frac{V}{m} \frac{1}{4\pi}$$

$$\Rightarrow \frac{V}{m}$$

5







6 $$f_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$$

$$f_0 = \frac{(1+\sqrt{5})^0 - (1-\sqrt{5})^0}{2^0\sqrt{5}}$$

$$= \frac{1-1}{2^n\sqrt{5}} = 0$$

$$f_1 = \frac{1+\sqrt{5} - (1-\sqrt{5})}{2\sqrt{5}}$$

$$= \frac{2\sqrt{5}}{2\sqrt{5}} = 1$$

$$2^n \sqrt{5}\, f_n = (1+\sqrt{5})^n - (1-\sqrt{5})^n \qquad n = n+2-2$$

$$= (1+\sqrt{5})^{n-2}(1+\sqrt{5})^2 - (1-\sqrt{5})^{n-2}(1-\sqrt{5})^2$$

$$= (1+\sqrt{5})^{n-2}(\overset{+1}{2}+2\sqrt{5}+\overset{-1}{4}) - (1-\sqrt{5})^{n-2}(\overset{+1}{2}-2\sqrt{5}+\overset{-1}{4})$$

$$= 2(1+\sqrt{5})^{n-1} + 4(1+\sqrt{5})^{n-2} - 2(1-\sqrt{5})^{n-1} - 4(1-\sqrt{5})^{n-2}$$

$$= 2(1+\sqrt{5})^{n-1} - 2(1-\sqrt{5})^{n-1} + 4(1+\sqrt{5})^{n-2} - 4(1-\sqrt{5})^{n-2}$$

$$= 2^n \sqrt{5}\, f_{n-1} + 2^n \sqrt{5}\, f_{n-2}$$

$$f_n = f_{n-1} + f_{n-2}$$