Vorlesung 5 - Trignometrie

Friday, 26 September 2025 09:07

Wer viel Spaß hat lebt viel,
wer viel lebt schafft viel. ?!

Sex, Drugs & ~~Rock'n Roll~~ ?

Bio, Physik, Biophysik,
Chemiezeichen, ...

$$f(x) = \frac{p_n(x)}{p_m(x)} \qquad n \geq m \rightarrow \text{unecht gebrochen}$$

$$= g_e(x) + h(x), \quad h(x) = \frac{p_i(x)}{p_j(x)}, \quad j > i \rightarrow \text{echt gebrochene}$$

erster Ansatz: Polynomdivision

Jedes Polynom lässt sich in Linearfaktoren zerlegen (nur in $\mathbb{C}$?!)

$$p_j(x) = \underbrace{\prod_{k=1}^{j}}_{\text{„Produkt"}} (x - x_k) = (x-x_1)(x-x_2)(x-x_3) \dots (x-x_j)$$

$x^2+1 = ?$ nicht in $\mathbb{R}$ in LF zerlegbar

$$h(x) = \frac{p_i(x)}{(x-x_1)(x-x_2)(x-x_3)\dots(x-x_j)} = \frac{a_0 + a_1 x + a_2 x^2 + \dots + a_i x^i}{(x-x_1)(x-x_2) \dots (x-x_j)}$$

Idee: diese Form ist die Summe von Brüchen rückwärts   gemeinsamer Nenner

$$h(x) = \frac{A}{x-x_1} + \frac{B}{x-x_2} + \frac{C}{x-x_3} + \dots + \frac{M}{x-x_j}$$

Aufgabe: finde $A, B, C, .., M$

$$\frac{p_n(x)}{p_m(x)} = g_e(x) + \frac{A}{x-x_1} + \frac{B}{x-x_1} + \dots$$ maximale Vereinfachung

$$\frac{p_n(x)}{p_m(x)} = q(x) + \frac{A}{x-x_1} + \frac{B}{x-x_2} + \ldots$$

maximale Vereinfachung einer gebrochenen rationalen Funktion

$\to$ nur noch Terme der Potenz 1

2.15 Irrationale Funktionen

Bsp: $y = \sqrt[n]{x}$, $n \in \mathbb{N}$, $n \geq 2$

Wurzelfunktion $D = \{x \mid x \in \mathbb{R}, x \geq 0\}$ für gerade n

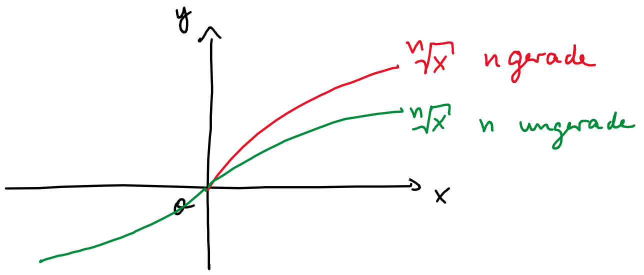
$D = \mathbb{R}$ für ungerade n

für ungerade n ist $\sqrt[n]{x}$ Umkehrfunktion von x^n auf ganz $D = \mathbb{R}$

$\uparrow$ bijektiv $\qquad \uparrow$ bijektiv

für gerade n ist $\sqrt[n]{x}$ auch Umkehrfunktion von x^n allerdings nur auf $[0,\infty) = \mathbb{R}_0^+$

(ansonsten nicht bijektiv)



insbesondere gilt $\sqrt[n]{x^n} = |x|$ für gerade n.

Exponentialfunktion

$y = a^x$, $a \in \mathbb{R}^+$ Spezialfall: $a = e$ Eulerzahl $e = 2{,}718\ldots$

$D = \mathbb{R}$, $W = \mathbb{R}^+$

Reihendarstellung: $e^x = \exp(x) = \sum_{k=0}^{n} \frac{x^k}{k!}$

Logarithmusfunktion

$y = \log_a(x)$, $a \in \mathbb{R}^+$, $a \neq 1$

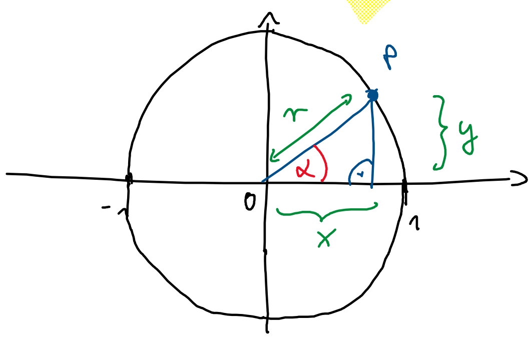
$$= \frac{\log(x)}{\log(a)}$$

für $a = e$ $\quad y = \log_e(x) = \ln(x)$ $\quad \underbrace{D = \mathbb{R}^+}$, $W = \mathbb{R}$

Umkehrfunktion $e^x > 0$

3. Trigonometrische Funktionen

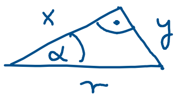
3.1 Motivation



Jedem Punkt auf dem Einheitskreis ordnen wir ein rechtwinkliges Dreieck zu.

(Durch Projektion auf x-Achse)

Den Winkel zwischen Hypotenuse und x-Achse nennen wir α.



Das Gradmaß

α in Grad (°) $\quad 0° \leq \alpha < 360°$ oder $-180° \leq \alpha < 180°$

Das Bogenmaß

Verhältnis von Umfang U zu Durchmesser $\quad U = \pi \cdot d$

$$\pi = \frac{U}{d} = 3{,}1415\ldots \quad \text{Kreiszahl}$$

Man gibt α in Radianten an (rad) $\quad 0 \leq \alpha < 2\pi$

$-\pi \leq \alpha < \pi$

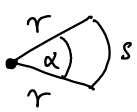
$$2\pi \mathrel{\hat{=}} 360°$$

<u>Umrechnung</u>

$$1 \text{ rad} = \frac{360°}{2\pi} = \frac{180°}{\pi} \approx 57{,}2958°$$

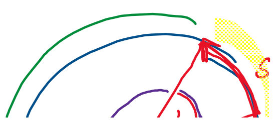
$$1° = \frac{2\pi}{360} \text{ rad} \approx 0{,}01745 \text{ rad}$$

<u>Bogenlänge Kreissegment</u>



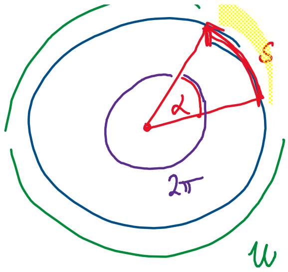
$s = r \cdot \alpha$, wenn man α in rad einsetzt

$[\alpha] = \text{rad}$



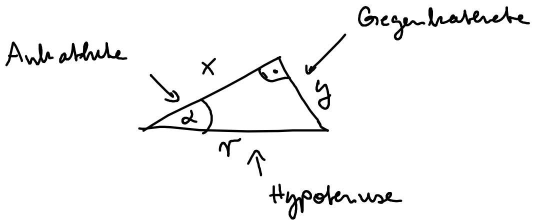
Anteilig berechnet:

Anteilung berechnen.



$$\frac{S}{U} = \frac{s}{2\pi r} = \frac{\alpha}{2\pi} \Rightarrow s = r \cdot \alpha$$

Definitionen: trigonometrische Funktionen



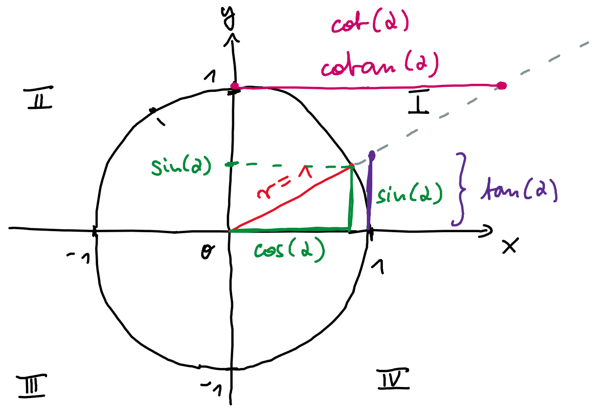
Sinus: $\sin(\alpha) = \frac{\text{Gegenkathete}}{\text{Hypotenuse}} = \frac{y}{r}$

Kosinus: $\cos(\alpha) = \frac{\text{Ankathete}}{\text{Hypotenuse}} = \frac{x}{r}$

Tangens: $\tan(\alpha) = \frac{\text{Gegenkathete}}{\text{Ankathete}} = \frac{y}{x}$

Kotangens: $\cot(\alpha) = \frac{\text{Ankathete}}{\text{Gegenkathete}} = \frac{x}{y}$

geometrisch

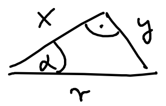


Vorzeichen trigonometrischer Funktionen

Quadrant	sin	cos	tan	cot
I	+	+	+	+
II	+	−	−	−
III	−	−	+	+
IV	−	+	−	−

Beziehungen für gleiche Winkel und Umrechnungsformeln

Satz des Pythagoras im Kreis: $x^2 + y^2 = r^2$



$r = 1 \Rightarrow x^2 + y^2 = 1$

$\Rightarrow \sin^2(\alpha) + \cos^2(\alpha) = 1$

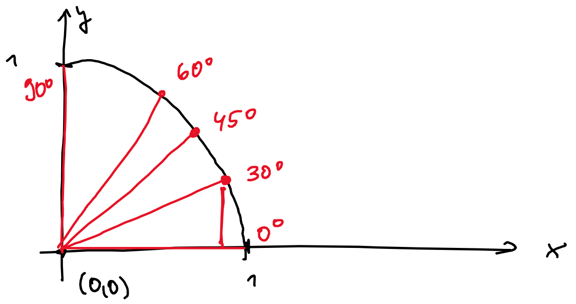
$\sin(\alpha) = \pm\sqrt{1 - \cos^2(\alpha)}$, $\cos(\alpha) = \pm\sqrt{1 - \sin^2\alpha}$

$$1 + \tan^2\alpha = 1 + \frac{\sin^2\alpha}{\cos^2\alpha} = \frac{\cos^2\alpha}{\cos^2\alpha} + \frac{\sin^2\alpha}{\cos^2\alpha} = \frac{\overset{1}{\cos^2\alpha + \sin^2\alpha}}{\cos^2\alpha}$$

$$= \frac{1}{\cos^2\alpha}$$

analog: $1 + \cot^2\alpha = \dfrac{1}{\sin^2\alpha}$

Spezielle Winkel / Stützstellen

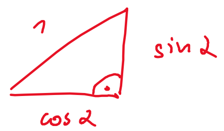


$x = \cos(\alpha)$, $y = \sin(\alpha)$

$0°$: $\cos(\alpha) = 1$, $\sin(\alpha) = 0$

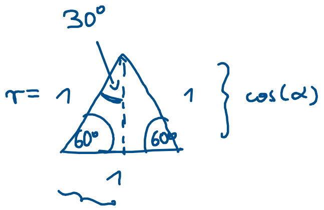
$90°$: $\cos(\alpha) = 0$, $\sin(\alpha) = 1$

$45°$: $\cos(\alpha) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} = \sin(\alpha)$



$\sin^2\alpha + \sin^2\alpha = 1$

$2\sin^2\alpha = 1 \Rightarrow \sin^2\alpha = \frac{1}{2}$



$30°$ $\sin(\alpha) = \frac{1}{2} \cancel{\cos(\alpha)} \Rightarrow \cancel{\cos(\alpha) = 2\sin(\alpha)}$

Satz Pyth.: $\sin^2\alpha + (2\sin(\alpha))^2 = 1$

$\Rightarrow \sin(\alpha) = \frac{1}{2}$, $\cos(\alpha) = \frac{\sqrt{3}}{2}$

$60°$ $\sin(\alpha) = \frac{\sqrt{3}}{2}$, $\cos(\alpha) = \frac{1}{2}$

Wertetabelle

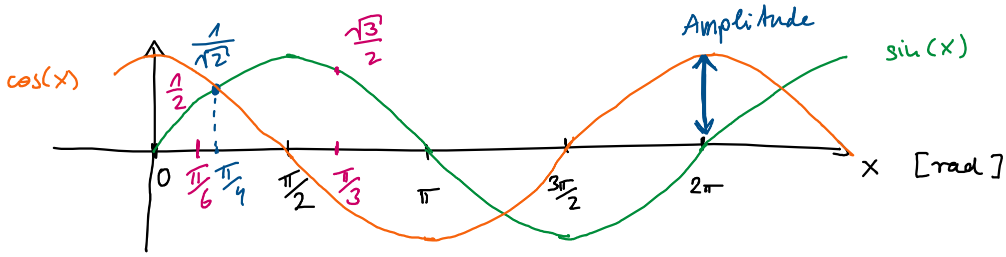
I $0 \le \alpha < 90°$

α	Gradmaß	0°	30°	45°	60°	90°
	Bogenmaß	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(\alpha)$		0	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{4}}{2}$
$\cos(\alpha)$		$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$

$\cos(\alpha)$	$\frac{\sqrt{4}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{1}}{2}$	$\frac{\sqrt{0}}{2}$
$\tan(\alpha) = \frac{\sin(\alpha)}{\cos(\alpha)}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$-$
$\cot(\alpha) = \frac{\cos(\alpha)}{\sin(\alpha)}$	$-$	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

II – IV : Symmetrie nutzen → Vorzeichen ändern

3.2 Graphen trigonometrischer Funktionen



Sinusfunktion:

Definitionsbereich $D = \mathbb{R}$, Wertemenge $W = [-1, 1]$

periodische Funktionen : $f(x+a) = f(x)$ es existiert ein solcher a

Periode $\sin(x + 2\pi) = \sin(x)$ „Periode 2π"

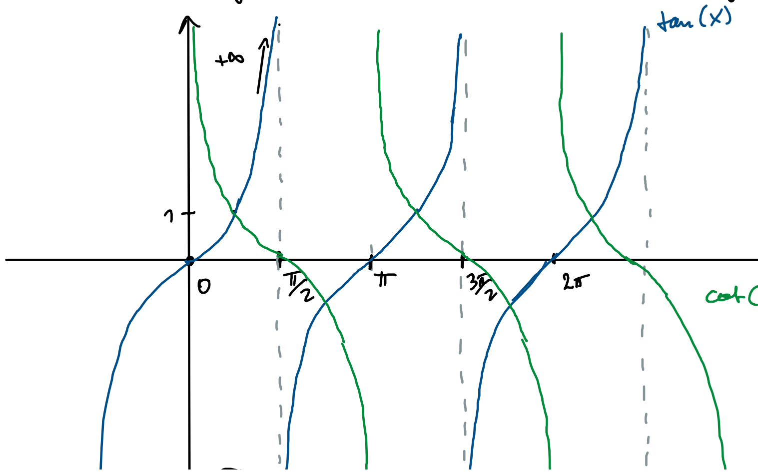
Amplitude $= 1$, $|\sin(x)| \leq 1$ beidseitig beschränkt

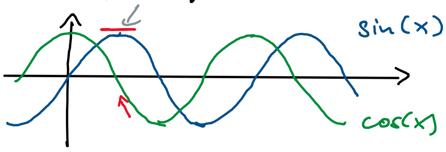
$\sin(-x) = -\sin(x)$ Punktsymmetrie

$\cos(-x) = \cos(x)$ Achsensymmetrie

Kosinusfunktion

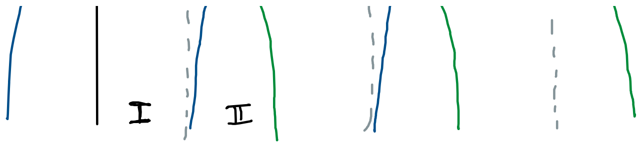
analog zur Sinusfunktion, alles gleich außer Symmetrie





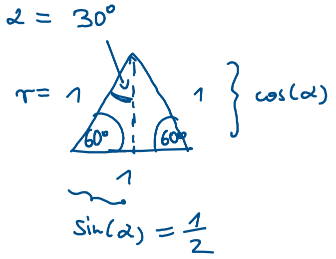
$\left.\begin{array}{lll} \sin(x) & \text{Periode} & 2\pi \\ \cos(x) & \text{Periode} & 2\pi \end{array}\right\} \Rightarrow \tan(x)$ Periode 2π

$\sin(x) \rightarrow \tan(x)$ hat



$\cos(x)$ Periode 2π) Periode 2π

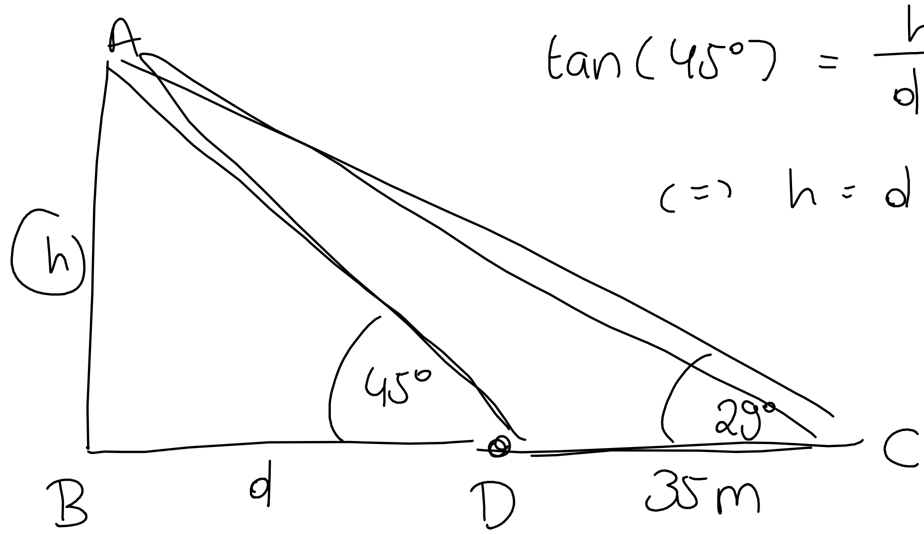
$\frac{\sin(x)}{\cos(x)}$ → $\tan(x)$ hat kleinste Periode π



$\sin(\alpha) = \frac{1}{2}$

Vorrechnen Blatt 5

4)



$\tan(45°) = \frac{h}{d} = 1$

$\Leftrightarrow h = d$

$\tan(29°) = \frac{h}{d + 35m} = \underline{\underline{\frac{d}{d + 35m}}}$

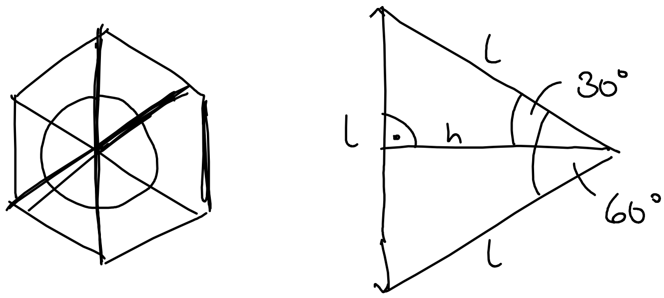
$\Leftrightarrow d = \tan(29°) \cdot (d + 35m)$

$\Leftrightarrow d(1 - \tan(29°)) = 35m \tan(29°)$

$\Leftrightarrow d = \frac{35m \tan(29°)}{1 - \tan(29°)} = 43{,}53m = h$

$$d = \tan(29°) \cdot d + \tan(29°)\ 35m$$

3)

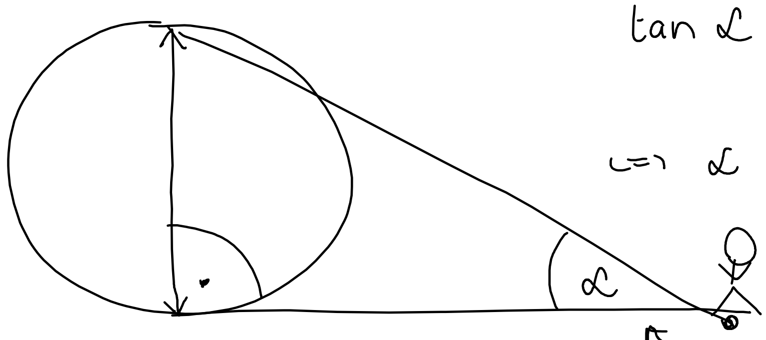


$$l^2 = h^2 + \left(\frac{l}{2}\right)^2 = h^2 + \frac{l^2}{4}$$

$$\Leftrightarrow \frac{3}{4} l^2 = h^2 \Leftrightarrow l = \frac{2}{\sqrt{3}} h$$

$$\frac{2}{\sqrt{3}} = 1{,}15 \rightarrow 15\%\ \text{größer}$$

2) b)



$$\tan\alpha = \frac{3480\,km}{384\,250\,km}$$

$$\Leftrightarrow \alpha = \tan^{-1}\left(\frac{3480}{384250}\right)$$

$$= 0{,}52°$$

$$\frac{363000km + 405500km}{2} = 384\,250\,km$$

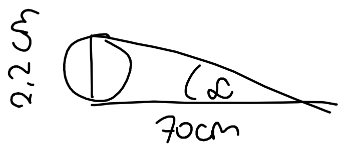
$$0{,}52° \quad 60 = 31{,}1'$$

a)

$$\tan(\alpha) = \frac{2{,}2}{70}$$

ξ . . 2.2 .

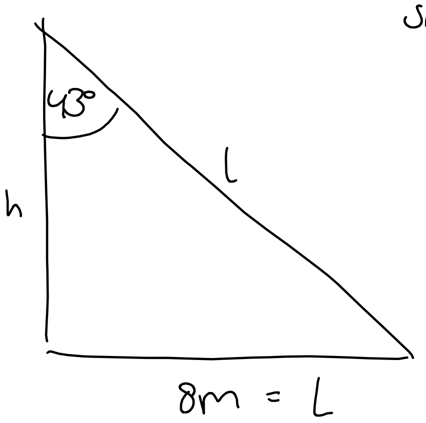
a)



$\tan(\alpha) = \frac{2,2}{70}$

$\alpha = \tan^{-1}\left(\frac{2,2}{70}\right) = 1,8°$

1) a)



$\sin(43°) = \frac{8m}{l}$

$\Leftrightarrow l = \frac{8}{\sin(43°)} = 11,23m$

b) $\tan(43°) = \frac{8}{h}$

$\Leftrightarrow h = \frac{8}{\tan(43°)}$

$= 8,58m$

$L^2 = h^2 + l^2 - 2hl\cos\alpha \Leftrightarrow \cos\alpha = \frac{L^2 - h^2 + l^2}{2hl}$

$\alpha = \cos^{-1}\left(\frac{L^2 - h^2 + l^2}{2hl}\right) = 54°$

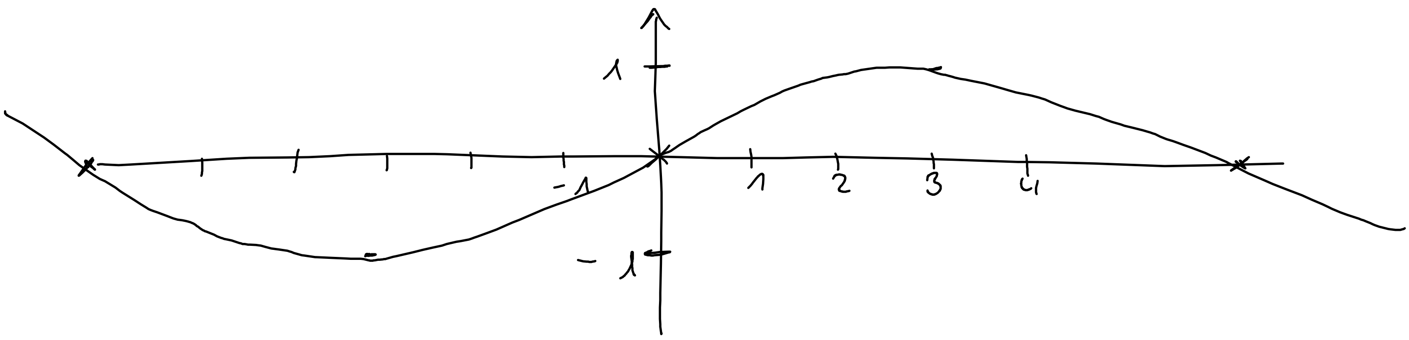
5) a) $\sin(x/2)$

$\sin(x)$: Nullstellen $x = z \cdot \pi \quad z \in \mathbb{Z}$

Maxima $x = \frac{(4z+1)}{2}\pi$

$\sin\left(\frac{x}{2}\right)$ Nullstellen $x = 2z\pi$

Maxima $x = (4z+1)\pi$



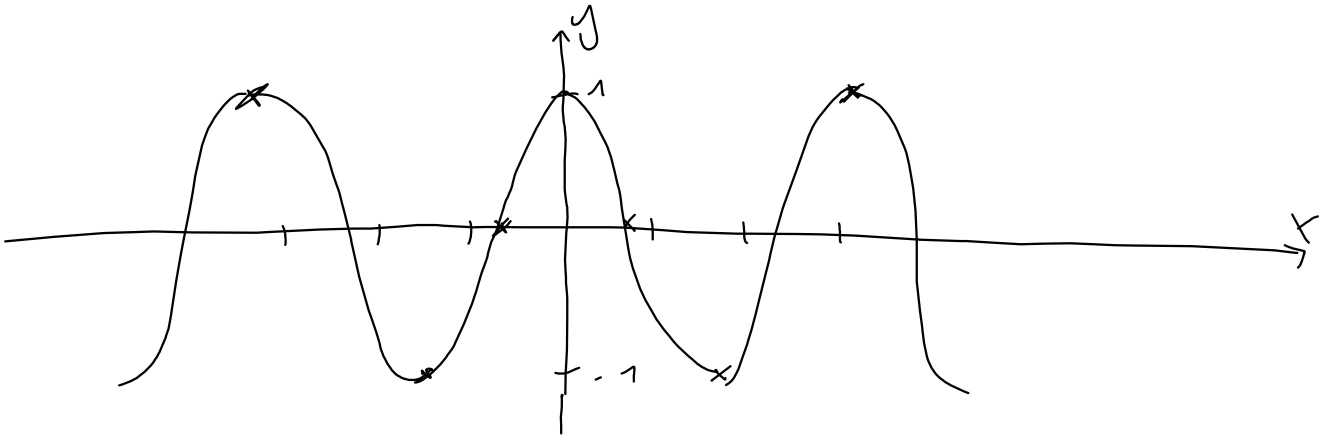
b) $\cos(2x)$

$\cos(x)$ Nullstellen $x = 2z\pi$ $\quad z \in \mathbb{Z}$

Maxima $x = \left(\frac{2z+1}{2}\right)\pi$

$\cos(2x)$ Nulls $x = z\pi$

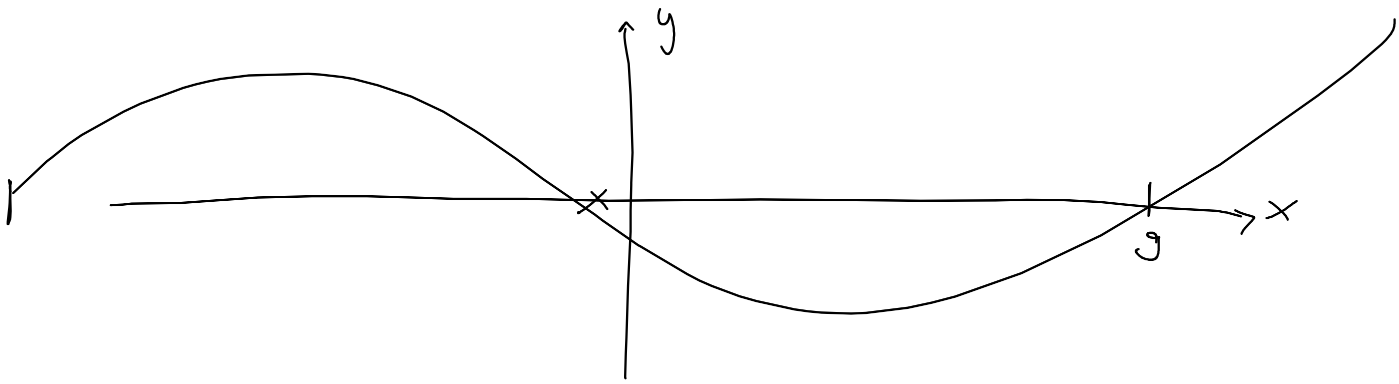
Max. $x = \left(\frac{z}{2} + \frac{1}{4}\right)\pi$



c) $\sin\left(\frac{x}{3} - 3\right) = \sin\left(\frac{1}{3}(x-9)\right)$

Nullstellen $x = 3z\pi + 9$

Max $x = \left(6z + \frac{3}{2}\right)\pi + 9$

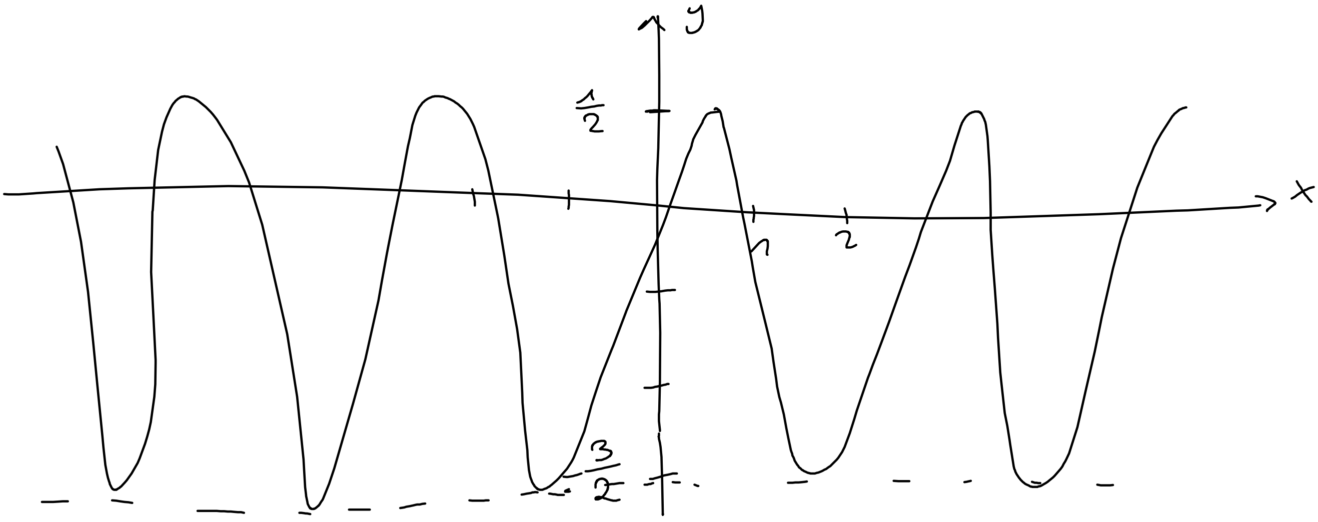


d) $\sin(3x) - \frac{1}{2}$

Schnittpunkte mit $y = -\frac{1}{2}$: $x = \frac{k\pi}{3}$

Max $x = \frac{2k\pi}{3} + \frac{\pi}{6}$

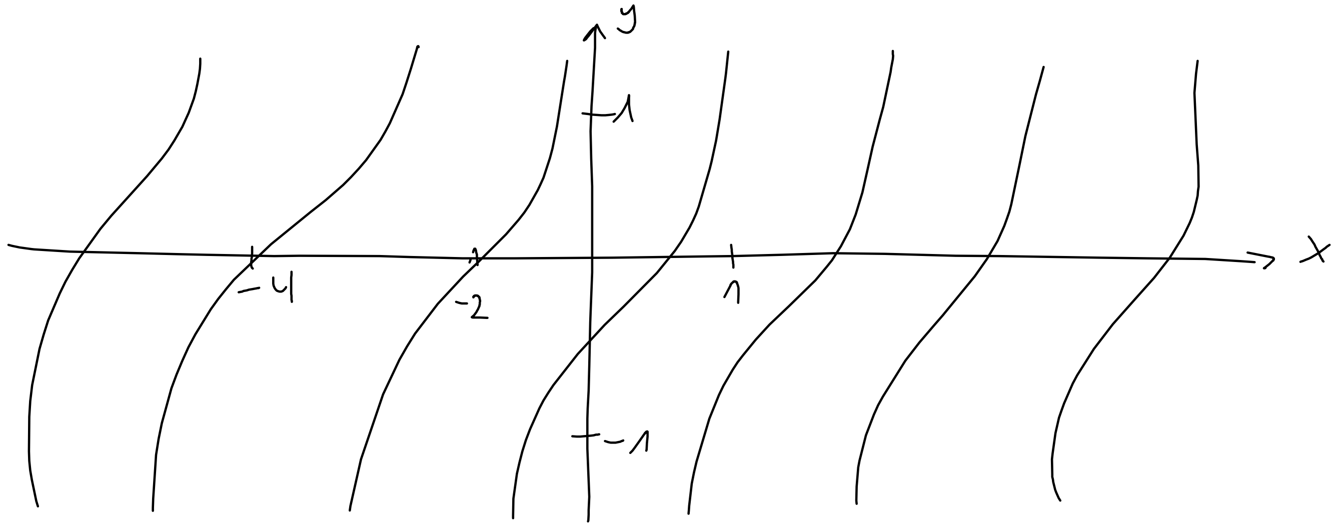
Max $\quad x = \frac{2k\pi}{3} + \frac{\pi}{6}$



$k \in \mathbb{Z}$

e) $\tan(x)$: $k\pi$ Nullst, Wendestellen $\frac{\pi}{2} + k\pi$

$\tan(1{,}5x - 1) = \tan(1{,}5(x - \frac{2}{3}))$



Nullstellen $x = (\frac{2}{3}\pi z + \frac{2}{3})$, Polstellen $\frac{2}{3} + (\frac{2}{3}z + \frac{1}{3})\pi$

f) $\cos^2(2x) = \frac{1}{2} + \frac{1}{2}\cos(4x)$ $\qquad k \in \mathbb{Z}$

Nullst $\quad x = \frac{k}{4}\pi + \frac{1}{8}\pi$ $\qquad$ Max $\quad x = \frac{1}{2}k\pi$

6) 1) a) $\sin(\frac{\pi}{4})\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} = \frac{1}{2}$

b) $\cos(2a) = \cos(a + a) = \cos^2(a) - \sin^2(a)$

$$= \cos^2 a - (1 - \cos^2 a) = 2\cos^2 a - 1$$

$$\Leftrightarrow \underline{\cos^2 a} = \frac{1 + \cos(2a)}{2}$$

c) $$\cos^2(\underbrace{\tfrac{3}{8}\pi}_{a}) = \frac{1 + \cos(\frac{3}{4}\pi)}{2}$$

$$\cos(\tfrac{3}{4}\pi) = \cos(\tfrac{\pi}{2} + \tfrac{\pi}{4}) = \cos\tfrac{\pi}{2}\cos\tfrac{\pi}{4} - \sin\tfrac{\pi}{2}\sin\tfrac{\pi}{4}$$

$$= 0\,\frac{\sqrt{2}}{2} - 1\,\frac{\sqrt{2}}{2} = -\frac{\sqrt{2}}{2}$$

$$\Rightarrow \cos^2(\tfrac{3}{8}\pi) = \frac{1 - \frac{\sqrt{2}}{2}}{2} = \frac{1}{2} - \frac{\sqrt{2}}{4} = \frac{1}{2} - \frac{1}{\sqrt{8}}$$

d) (*) $$\cos^2 a = \frac{1 + \cos(2a)}{2} = 1 - \sin^2 a$$

$$\Leftrightarrow \sin^2 a = \frac{1 - \cos(2a)}{2}$$

$$\Leftrightarrow 1 - 2\sin^2 a = \cos(2a)$$

$$1 - 2\sin^2(\tfrac{\pi}{8}) = \cos(\tfrac{\pi}{4}) = \frac{1}{\sqrt{2}}$$

e) $$\tan^2(\tfrac{\pi}{8}) + 1 = \frac{\sin^2(\frac{\pi}{8})}{\cos^2(\frac{\pi}{8})} + 1$$

$$= \frac{\sin^2(\frac{\pi}{8})}{\cos^2(\frac{\pi}{8})} + \underbrace{\frac{\cos^2(\frac{\pi}{8})}{\cos^2(\frac{\pi}{8})}}$$

$$= \frac{\overbrace{\sin^2(\frac{\pi}{8}) + \cos^2(\frac{\pi}{8})}^{=1}}{\cos^2(\frac{\pi}{8})} = \frac{1}{\cos^2(\frac{\pi}{8})}$$

$$\frac{1}{\tan^2(\frac{\pi}{8} + 1)} = \cos^2(\tfrac{\pi}{8}) \overset{(*)\,b)}{=} \frac{1 + \cos(\frac{\pi}{4})}{2}$$

$$= \frac{1 + \frac{\sqrt{2}}{2}}{2} = \frac{1}{2} + \frac{1}{\sqrt{8}}$$

f) $$\cot^2(\tfrac{3\pi}{8}) - 1 = \frac{\cos^2(\;)}{\sin^2(.\;)} - 1$$

$$= \cos^2(.) - \sin^2(\;)$$

c) $= -\frac{\sqrt{2}}{2}$

$$\cot\left(\tfrac{\cdot}{8}\right) - 1 \qquad - \qquad \sin^2(\,.\,) \qquad - 1$$

$$= \frac{\cos^2(\,.\,) - \sin^2(\,\,)}{\sin^2(\,\,)}$$

$$= \frac{\cos(2\,..\,)}{\sin^2(\,..\,)} = \frac{\overbrace{\cos\left(\frac{3\pi}{4}\right)}^{c) \;=\; -\frac{\sqrt{2}}{2}}}{\underbrace{\sin^2\left(\frac{3\pi}{8}\right)}}$$

b) $\dfrac{1 - \cos\left(\frac{3\pi}{4}\right)}{2} = \dfrac{1 + \frac{\sqrt{2}}{2}}{2}$

$$\frac{-\frac{\sqrt{2}}{2}}{\frac{1 + \frac{\sqrt{2}}{2}}{2}} = \frac{-\sqrt{2}}{1 + \frac{\sqrt{2}}{2}} = \frac{-2}{\sqrt{2} + 1}$$

11) a) $\tan(x \pm y) = \dfrac{\sin(x \pm y)}{\cos(x \pm y)}$

$$= \frac{\sin x \cos y \pm \cos x \sin y}{\cos x \cos y \mp \sin x \sin y}$$

$$\overset{: \cos x \cos y}{=} \frac{\frac{\sin x}{\cos x} \pm \frac{\sin y}{\cos(y)}}{1 \mp \frac{\sin x \,\sin y}{\cos x \,\cos y}} = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

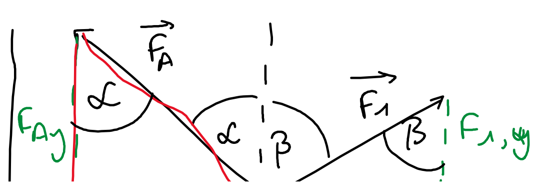
$$\frac{\frac{\sin x \cancel{\cos y}}{\cos x \cancel{\cos y}} \pm \frac{\cancel{\cos x} \sin y}{\cancel{\cos x} \cos y}}{\frac{\cancel{\cos x \cos y}}{\cancel{\cos x \cos y}} \mp \frac{\sin x \sin y}{\cos x \cos y}}$$

b) $\cot(x \pm y) = \dfrac{\cos(x \pm y)}{\sin(x \pm y)} = \dfrac{\cos x \cos y \mp \sin x \sin y}{\sin x \cos y \pm \cos x \sin y}$

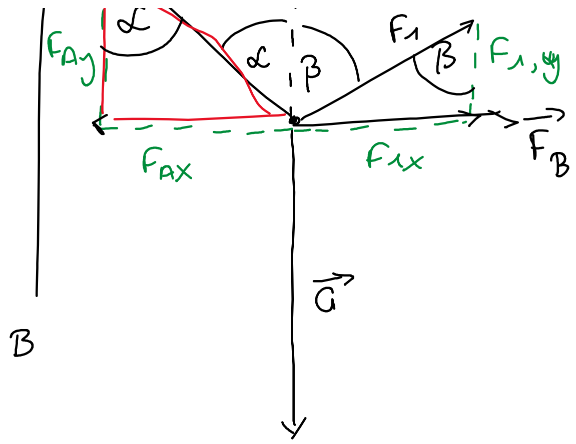
$$\overset{: \sin x \sin y}{=} \frac{\frac{\cos x}{\sin x}\,\frac{\cos y}{\sin y} \mp 1}{\frac{\cos y}{\sin y} \pm \frac{\cos x}{\sin x}} = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x}$$

$\cot(1 + 2)$

$$\vec{F} = \begin{pmatrix} F_x \\ F_y \end{pmatrix}$$

7) 

$$\sum F_x = 0$$



$$\Sigma F_x = 0$$

$$F_{1x} + F_{Ax} + \vec{F}_B = 0$$

$$\Sigma F_y = 0$$

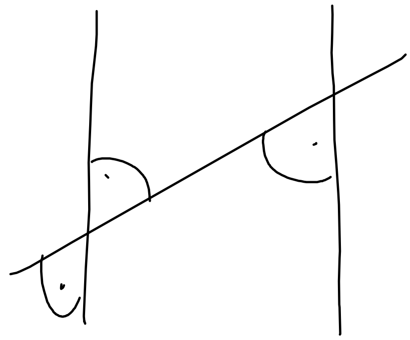
$$F_{1y} + F_{Ay} + \underbrace{\vec{G}}_{-G} = 0$$

$$\sin\beta = \frac{F_{1x}}{F_1} \iff F_{1x} = F_1 \sin\beta$$

$$\sin\alpha = \frac{F_{Ax}}{F_A} \iff F_{Ax} = -\sin\alpha \; F_A$$

$$\cos\beta = \frac{F_{1y}}{F_1} \iff F_{1y} = F_1 \cos\beta$$

$$\cos\alpha = \frac{F_{Ay}}{F_A} \iff F_{Ay} = \cos\alpha \; F_A$$



$$\text{I} \quad F_1 \sin\beta - F_A \sin\alpha + F_B = 0$$

$$\text{II} \quad F_1 \cos\beta + F_A \cos\alpha - G = 0$$

$$F_A = \frac{G - F_1 \cos\beta}{\cos\alpha} \qquad F_B = F_A \sin\alpha - F_1 \sin\beta$$

$F_1 = 0 \Rightarrow F_A = 777{,}9\,N, \quad F_B = 595{,}9\,N$

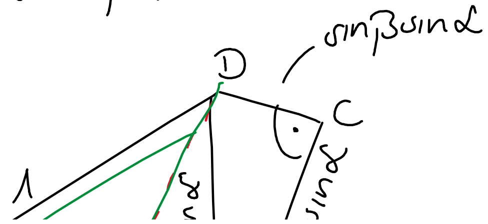
d) $F_1 = 200N \Rightarrow F_A = 622\,N, \quad F_B = 303\,N$

c) $\vec{F}_B = 0$

$$\text{I} \quad F_A = \frac{F_1 \sin\beta}{\sin\alpha}$$

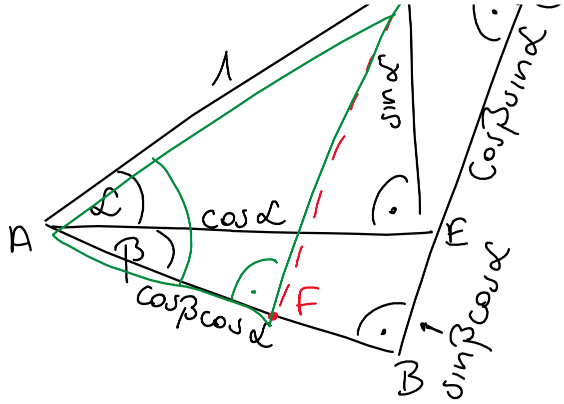
$$\text{II} \Rightarrow F_1 = G \frac{\sin\alpha}{\sin(\alpha+\beta)} = 407{,}6\,N$$

g)



$$\cos\alpha = \frac{\overline{AE}}{1}$$

8)



$$\cos\alpha = \frac{\overline{AE}}{1}$$

$$\Leftrightarrow \overline{AE} = \cos\alpha$$

$$\cos\beta = \frac{\overline{AB}}{\cos\alpha}$$

$$\Leftrightarrow \overline{AB} = \cos\beta\cos\alpha$$

$$\overline{AB} - \overline{CD} = \cos\beta\cos\alpha - \sin\beta\sin\alpha = \overline{AF} = \cos(\alpha+\beta)$$

$$\cos(\alpha+\beta) = \frac{\overline{AF}}{1} = \overline{AF}$$

$$\overline{BE} + \overline{EC} = \sin\beta\cos\alpha + \cos\beta\sin\alpha = \overline{BC} = \sin(\alpha+\beta)$$

$$\sin(\alpha+\beta) = \frac{\overline{BC}}{1}$$