Vorlesung 11 Ausblick Differential/Integralrechnung

Tuesday, 7 October 2025 09:05

wichtige Stammfunktionen

⋮

$\int \ln(x)dx = x\ln x - x + c$; $\int 1\cdot \ln x\, dx = \dots$

$\int \sin x\, dx = -\cos x + c$; ?! $\int f(x)g(x)dx \neq \int f(x)dx \cdot \int g(x)dx$

⋮

Uneigentliche Integrale

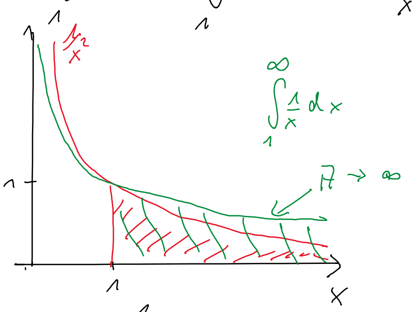
≙ wenn Grenzen nach ∞, oder Polstellen, ...

$\int_a^b f(x)dx = \lim_{c\to x_0^-} \int_a^c f(x)dx + \lim_{c\to x_0^+}\int_c^b f(x)dx$; mit Polstelle an x_0

analog in $(-\infty, a]$; $[b, \infty)$

$\int_{-\infty}^{a} f(x)dx = \lim_{c\to -\infty} \int_c^a f(x)dx$

Bsp. $\int_1^\infty \frac{1}{x^2}dx = \int_1^\infty x^{-2}dx = -\frac{1}{x}\Big|_1^\infty = \lim_{c\to\infty}\left(-\frac{1}{c}+\frac{1}{c}\right) = 1$



weitere Regeln:

① partielle Integration

$\int f(x)g(x)dx = f(x)G(x) - \int f'(x)G(x)dx$; $G(x) = \int g(x)dx$

woher $[u\cdot v]' = u'v + uv'$ $\quad |\int$

$\int [uv]' = \int u'v + \int uv'$

$uv = \int u'v + \int uv'$

$uv - \int uv' = \int u'v$; $u \hat{=} G$; $v \hat{=} f$

$u' \hat{=} g$

$\int 1\cdot \ln x\, dx = \dots$

② Substitution

$\int f(g(x))\cdot g'(x)dx = \int f(g(x))\cancel{\frac{dg}{dx}}\cdot \cancel{dx} = \int f(g(x))\, dg(x) = \int f(g)dg$

auch $\int bla\, d\, bla = \frac{1}{2} bla\, bla$

$\int ☺^2 d☺ = \frac{1}{3}☺^3$

$\int x dx = \frac{x^2}{2}$

$\int bla\, d\, bla = bla\, bla$ $\quad g(x) = x$

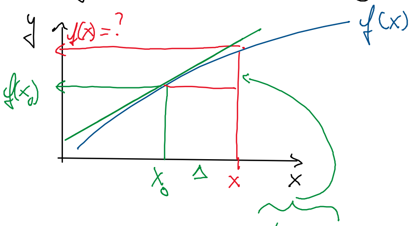
$\int 7dx = 7x$

Ausblick Differentialrechnung

a) Taylorentwicklung $\qquad$ Ziel: $f(x)$ als Polynom $p(x)$

Ausblick: Differentialrechnung

a) Taylorentwicklung



Ziel: $f(x)$ als Polynom $p(x)$

Idee: bauen Polynom $p(x)$ so dass,

$p(x_0) = f(x_0)$

$p'(x_0) = f'(x_0)$

$p''(x_0) = f''(x_0)$

$\vdots$

$$f(x) = f(x_0) + f'(x_0)\cdot\Delta + \dots$$

$$f(x) = \underbrace{\frac{f(x_0)}{0!}(x-x_0)^0 + \frac{f'(x_0)}{1!}\cdot(x-x_0)^1 + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots}_{p(x)}$$

einfaches Polynom

check: $p(x=x_0) = f(x_0)$

$$p'(x)\Big|_{x=x_0} = f'(x_0)$$

$$p''(x)\Big|_{x=x_0} = \frac{f''(x_0)}{2!}\cdot 2\cdot 1 \quad ; \quad \text{da } \frac{d^2(x-x_0)^2}{dx^2} = \frac{d}{dx}\left(2(x-x_0)\right) = 2$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{d^n f(x)}{dx^n}\Big|_{x=x_0} \cdot \frac{(x-x_0)^n}{n!}}$$

Taylor-Reihe

Bsp. oft ist $x_0 = 0$ gewählt.

$$\sin(x) = 0 + \frac{d}{dx}\sin(x)\Big|_{x=0} (x-\overset{0}{x_0})^1 + \frac{1}{2!}\frac{d^2}{dx^2}\sin(x)\cdot x^2$$

$$= 1\cdot x - \frac{1}{3!}\cdot x^3 + \frac{1}{5!}x^5 - \dots$$

analog $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$; so z.B. $e^1 = 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \dots$

$e = 2{,}718281828\dots$

so rechnet der Taschenrechner $\sin x$, $\cos x$, e^x, ...

$\Rightarrow$ morgen $e^{i\pi} = -1$; $\boxed{e^{i\pi} + 1 = 0}$

b) Differentialgleichungen

≙ Gleichungen wo Funktion in Ableitungen vorkommen, gesucht Funktion

z.B. $y'(x) = 2y(x)$, im Worten Änderung der Funktion ist proportional dem aktuellen Funktionswert

Lösung erraten $y(x) = e^{2x}$ ist okay weil $y'(x) = 2e^{2x} = 2y(x)$ ✓ okay

besser ist

$$y'(x) = \frac{dy}{dx} = 2y$$

$$\frac{dy}{2y} = dx \quad \Big| \int$$

$$\frac{1}{2}\int \frac{dy}{y} = \int dx$$

$2y$

$$\frac{1}{2}\int \frac{dy}{y} = \int dx$$

Konstante

$$\frac{1}{2}\ln y = x + C \quad \Rightarrow y(x) = e^{2x+c} = e^{2x}\cdot e^{c} = \tilde{c}\cdot e^{2x}$$

damit auch klar, dass auch $y(x) = 27{,}28\cdot e^{2x}$ funktioniert.

Dgl in Naturwissenschaften oft mit $x = t \mathrel{\hat=}$ Zeit $\Rightarrow$ dynamische Systeme

z.B. i) $F(t) = m\cdot a(t) = m\frac{d}{dt}v(t) = m\frac{d^2}{dt^2}x(t) = m\,\dot{v}(t) = m\,\ddot{x}(t)$

z.B. $a = $ const.

$$\cancel{m}\,a = \cancel{m}\frac{d}{dt}v(t)$$

$$a\,dt = dv \quad \Big| \int$$

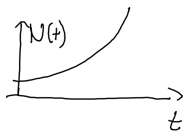
$$a\cdot t + v_0 = v(t)$$ ≙ gleichmäßig beschl. Bewegung

ii) $\dot{N}(t) = -\lambda N(t)$; $N(t)$ ≙ Anzahl der Atomkerne zum Zeitpunkt t

$$N(t) = N_0\, e^{-\lambda t}$$

iii) $\dot{N}(t) = k\,N(t)$; $k > 0$ ≙ Bakterien wachstum, PCR-Test

$$N(t) = N_0\, e^{kt}$$



c) partielle Ableitungen

wenn Funktion mehr als 1 Variable hat, z.B. $pV = nRT$; p - Druck, V - Volumen, T - Temperatur, n, R - const.

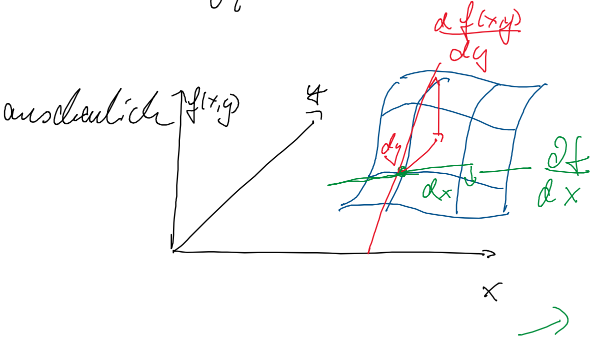
dann ist etwas $p'(V,T)$ sinnlos $\qquad p(V,T) = \frac{nRT}{V}$

daher Schreibweise

$$\frac{\partial p}{\partial T} = \frac{nR}{V} \ ; \text{oder} \ \frac{\partial p}{\partial V} = -\frac{nRT}{V^2}$$

↳ oder $\frac{\partial p}{\partial T} = \partial_T p$, analog $\partial_{TT}\, p = \frac{\partial^2}{\partial T^2} p$

anschaulich



Wozu?

Strömung von Flüssigkeiten $\vec{v}(x,y,z)$

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right) = -\nabla p + \mu\,\Delta\vec{v}$$

Druck (∇p), Viskosität (μ)

$$\nabla = \text{Nabla} = \begin{pmatrix}\partial_x\\ \partial_y\\ \partial_z\end{pmatrix}$$

$$\Delta = \nabla\cdot\nabla = \begin{pmatrix}\partial_x\\ \partial_y\\ \partial_z\end{pmatrix}\cdot\begin{pmatrix}\partial_x\\ \partial_y\\ \partial_z\end{pmatrix} = \partial_{xx} + \partial_{yy} + \partial_{zz} = \text{Laplace}$$

Wer zeigen kann, mathematisch exakt, dass die Gl. eine eindeutige Lösung hat, bekommt \$ 1,000,000

Vorrechnen

1) i a) $\int \frac{1}{e^{5x}} + 5 \, dx$

Vorrechnen

1) i a) $\int \frac{1}{e^{3x}+5}\, dx$

$y = e^{3x} \qquad \frac{dy}{dx} = 3e^{3x} \quad \Rightarrow \quad dx = \frac{1}{3e^{3x}}\, dy$

$\int \frac{1}{(y+5)y}\frac{1}{3} dy$

$\frac{1}{(y+5)y} = \frac{A}{y+5} + \frac{B}{y}$

$1 = Ay + By + 5B$

$0y + 1 = y(A+B) + 5B$

$\Rightarrow A + B = 0$

$5B = 1 \quad \Rightarrow B = \frac{1}{5} \quad \Rightarrow A = -\frac{1}{5}$

$$\frac{1}{3} \int \frac{1}{(y+5)y} dy = \frac{1}{3} \cdot \frac{1}{5}\left(-\int \frac{1}{y+5}\right) + \frac{1}{y} dy$$
$$= \frac{1}{3} \cdot \frac{1}{5}\left(\int \frac{1}{y+5} dy + \int \frac{1}{y} dy\right)$$
$$= \frac{1}{15}\left(\ln y+5 + \ln y\right)$$
$$= \frac{1}{15} \ln \frac{y}{y+5}$$
$$= \frac{1}{15} \ln \frac{e^{3x}}{e^{3x}+5}$$

b) $\overbrace{\int \underset{u}{e^x} \underset{v'}{\cos(x)}\, dx}^{I} = e^x \sin x - \int \underset{u}{e^x} \underset{v'}{\sin x}\, dx \qquad \int f(x) f'(x)$

$= e^x \sin x + e^x \cos x - \underbrace{\int e^x \cos x\, dx}_{I} \qquad | + I$

$\Leftrightarrow \quad 2I = e^x \sin x + e^x \cos x \qquad | : 2$

$I = e^x \left(\frac{\sin x + \cos x}{2}\right)$

c) $\int \sqrt{\frac{1-x}{1+x}} \frac{1}{(1-x)^2} dx = \int \frac{1}{\sqrt{\frac{1+x}{1-x}}} \left(\frac{1}{1-x}\right)^2 dx$

$y = \frac{1+x}{1-x} = (1+x)(1-x)^{-1}$

$\frac{dy}{dx} = (1-x)^{-1} + \frac{(1+x)}{(1-x)^2}$

$= \frac{1-x+1+x}{(1-x)^2}$

$= \frac{2}{(1-x)^2}$

$dx = \frac{1}{2}(1-x)^2 dy$

$= \int \frac{1}{2} \frac{1}{\sqrt{y}} dy$

$= \sqrt{y}$

$= \sqrt{\frac{1+x}{1-x}}$

d) $\int \underset{u'}{x^2} \underset{v}{\ln x}\, dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \frac{1}{x} dx$

$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx$

d) $\int \underset{u'}{x^2} \underset{v}{\ln x}\, dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3 \frac{1}{x}\, dx$

$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx$

$= \frac{1}{3}\left(x^3 \ln x - \frac{1}{3}x^3\right)$

$I = \frac{1}{3}x^3\left(\ln x - \frac{1}{3}\right)$

$f = \sin x \qquad f = \sin ax$

$F = -\cos x \qquad F = -\frac{1}{a}\cos ax$

e) $\overbrace{\int \underset{v'}{\sin(4x)}\underset{u}{\sin(6x)}\, dx}^{I} = -\frac{1}{4}\cos 4x \sin 6x + \frac{6}{4}\int \overset{u'}{\cos 4x}\, \overset{v}{\cos 6x}\, dx$

$= -\frac{1}{4}\cos 4x \sin 6x + \frac{6}{4}\left[\frac{1}{4}\sin 4x \cos 6x + \frac{6}{4}\int \sin 4x \sin 6x\, dx\right]$

$= -\frac{1}{4}\cos 4x \sin 6x + \frac{3}{8}\sin 4x \cos 6x + \frac{9}{4}\underbrace{\int \sin 4x \sin 6x\, dx}_{I} \quad \Big| -\frac{9}{4}I$

$-\frac{5}{4}I = -\frac{1}{4}\cos 4x \sin 6x + \frac{3}{8}\sin 4x \cos 6x \quad \Big| \left(-\frac{5}{4}\right)$

$I = \frac{1}{5}\cos 4x \sin 6x - \frac{3}{10}\sin 4x \cos 6x$

f) $\sin x\, e^{\cos x} = -e^{\cos x} \qquad \frac{d}{dx} e^{\cos x} = -\sin x\, e^{\cos x}$

g) $\int \frac{\sqrt{1+x}}{x}\, dx$

$y = \sqrt{1+x} \qquad \frac{dy}{dx} = \frac{1}{2\sqrt{1+x}}$

$dx = 2\sqrt{1+x}\, dy$

$\int \frac{\sqrt{1+x}}{x} 2\sqrt{1+x}\, dy = 2\int \frac{1+x}{x}\, dy$

$y = \sqrt{1+x} \quad |()^2$

$x = y^2 - 1$

einsetzen:

$\int \frac{y^2}{y^2-1} = \int \frac{y^2}{y^2-1}$

$y^2 : y^2 - 1 = 1 + \frac{1}{y^2-1} = \frac{y^2-1+1}{y^2-1} = \frac{y^2}{y^2-1}$

$-(y^2-1)$

1

$I = 2\int 1 dy + 2\int \frac{1}{y^2-1}\, dy$

PZB: $\frac{1}{y^2-1} = \frac{A}{y+1} + \frac{B}{y-1}$

$(A+B)y + B - A = 1$

$A = -B \qquad 2B = 1 \Rightarrow B = \frac{1}{2} \quad A = -\frac{1}{2}$

$\Rightarrow I = 2y + 2\int \frac{-\frac{1}{2}}{y+1}\, dy + 2\int \frac{\frac{1}{2}}{y-1}\, dy$

$= 2y - \ln|(y+1)| + \ln|y-1|$

$= 2y + \ln \left|\frac{y-1}{y+1}\right|$

$= 2\sqrt{1+x} + \ln \left|\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}\right|$

h) $\int \frac{1}{\sin x}\, dx$

$y = \cos x \quad \frac{dy}{dx} = -\sin x \quad \Rightarrow \quad dx = \frac{1}{-\sin x}\, dy$

$\Rightarrow \int \frac{1}{\sin x} \frac{1}{-\sin x}\, dy = -\int \frac{1}{\sin^2 x}\, dy$ $\qquad \sin^2 + \cos^2 = 1$

$= -\int \frac{1}{1-\cos^2 x}\, dy$ $\qquad \sin^2 = 1 - \cos^2$

$= \int \frac{1}{\cos^2 x - 1}\, dy$

$= \int \frac{1}{y^2-1}$

$= \frac{1}{2} \ln \left|\frac{y-1}{y+1}\right|$

$= \frac{1}{2} \ln \left|\frac{\cos x - 1}{\cos x + 1}\right|$

2 i) a) $\int_{-2}^{2} x^3 - x\, dx = \int_{-2}^{2} x^3 dx - \int_{-2}^{2} x\, dx$

$= \frac{1}{4} x^4 \Big|_{-2}^{2} - \frac{1}{2} x^2 \Big|_{-2}^{2}$

$= 4 - 4 - 2 + 2$

$= 0$

b) $\int_0^1 \frac{x^2}{1+x^2}\, dx \qquad \frac{x^2}{1+x^2} = \frac{x^2+1-1}{x^2+1} = 1 - \frac{1}{x^2+1}$

$\int_0^1 1 - \frac{1}{1+x^2}\, dx = \underbrace{\int_0^1 1\, dx}_{=1} - \int_0^1 \frac{1}{1+x^2}\, dx$

$x = \tan y$

$y = \arctan x \qquad \frac{dy}{dx} = \frac{1}{1+x^2}$

$\left| f^{-1}(x) \right|' = \frac{1}{f'(y)} \Big|_{x=y}$

$= \frac{1}{\frac{d}{dy} \tan y} \Big|_{y = \tan^{-1}}$ $\qquad (\tan)' = \left(\frac{\sin}{\cos}\right)' = \frac{\cos\cos + \sin\sin}{\cos^2}$

$= \cos^2 \arctan x$ $\qquad = \frac{1}{\cos^2}$

$= \frac{1}{1+x^2}$

$\int \frac{x^2}{x^2+1}\, dx = 1 - \arctan x \Big|_0^1$

$\arctan 1 = \frac{\pi}{4} \quad \arctan 0 = 0$

$= 1 - \frac{\pi}{4}$

c) $\int_2^4 \frac{|x-3|}{x^2}\, dx \qquad |x| : \begin{cases} x & x > 0 \\ -x & x < 0 \end{cases} \qquad a < c < b$

c) $\int_2^4 \frac{|x-3|}{x^2}\,dx$ $\qquad |x| : \begin{cases} x & x>0 \\ -x & x<0 \end{cases}$ $\qquad a<c<b$

$-\int_2^3 \frac{x-3}{x^2}\,dx + \int_3^4 \frac{x-3}{x^2}\,dx$ $\qquad \int_a^b x\,dx = \int_a^c x\,dx + \int_c^b x\,dx$

$-\int_2^3 \frac{1}{x}\,dx + \int_2^3 \frac{3}{x^2}\,dx + \int_3^4 \frac{1}{x}\,dx - \int_3^4 \frac{3}{x^2}\,dx$ $\qquad \int_{-\infty}^{\infty} e^{\frac{|x|+|y|+|z-z'|}{R}}$

$= \ln\left(\frac{8}{9}\right) - \frac{3}{4}$

13 geg: $d_R = 4\,mm \Rightarrow r_R = 2\,mm = 2\cdot 10^{-3}\,m$

$d_N = 10\,\mu m \Rightarrow r_N = 5\,\mu m = 5\cdot 10^{-6}\,m$

$\rho_W = 1\,\frac{g}{cm^3} = 1000\,\frac{kg}{m^3}$

$\rho_L = 1{,}2\,\frac{kg}{m^3}$

$\eta = 0{,}02\,mPas = 0{,}02\cdot 10^{-3}\,Pas$

$F_G = m\cdot g$ $\qquad \rho = \frac{m}{V} \Leftrightarrow m = \rho\cdot V$

$V = \frac{4}{3}\pi r^3$

$m = \frac{4}{3}\pi r^3 \cdot \rho$

$F_G = F_N$

$m\cdot g = \frac{1}{2}\rho_L v^2 c_W A$ $\qquad A = \pi r^2$

$v = \sqrt{\frac{2mg}{\rho_L c_W \pi r^2}}$

$v = \sqrt{\frac{\frac{8}{3} r \rho_W g}{\rho_L c_W}}$

Regen: $v = 9{,}843\,\frac{m}{s}$

Nebel: $v = 0{,}492\,\frac{m}{s}$

Stokes $F_g = F_S$

$mg = 6\pi \eta_L r v$

$v = \frac{\frac{2}{9} r^2 \rho_W g}{\eta_L}$

Regentropfen: $436\,\frac{m}{s}$

Nebel: $2{,}725\cdot 10^{-3}\,\frac{m}{s}$ ca $10\,\frac{m}{h}$