Komplexe Zahlen

Sinn?? : · sehr nützliche Rechenhilfsmittel
· unbedingt notwendig für Quantentheorie
in Quantenmechanik z.B. Schrödinger gl.

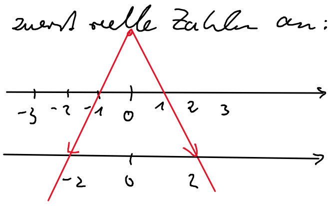
$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t)$$

Physik $|\psi|^2 \hat{=}$ Aufenthaltswahrscheinlichkeit

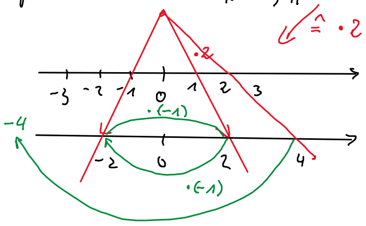
Motivation: Erweiterung der reellen Zahlen

$\mathbb{N} \in \mathbb{Z} \in \mathbb{Q} \in \mathbb{R} \in \mathbb{C}$

zuerst reelle Zahlen an: Multiplikation mit „2", „-2"

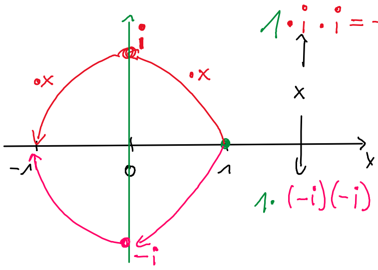


$\hat{=} \cdot 2 \cdot (-1)$



wollen was anderes machen, so dass

$1 \cdot x \cdot x = -1$



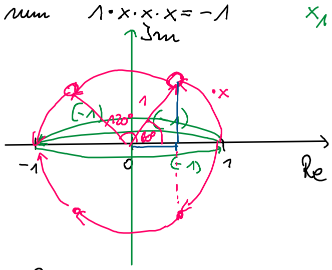
$1 \cdot i \cdot i = -1$; $i^2 = -1$

$x^2 = -1 \Leftrightarrow x = \pm i$

$i \cdot (-i) = -i \cdot i = +1$

$1 \cdot (-i)(-i) = -1$

nun $1 \cdot x \cdot x \cdot x = -1$ $\quad x_1 = -1$



$x_2 = \cos 60^\circ + i \sin 60^\circ$

$x_2 = \frac{1}{2} + i \cdot \frac{\sqrt{3}}{2}$

$x_3 = \frac{1}{2} - i \cdot \frac{\sqrt{3}}{2}$

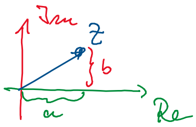
Gaußsche Zahlenebene ≙ Punkt in $\mathbb{R}^2$, eindeutig

Multiplizieren ≙ Drehung und/oder Streckung in Zahlenebene

Angabe von z: brauchen zwei Zahlen

1. kartesische Darstellung

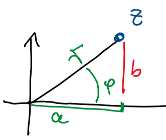
$\boxed{z = a + ib}$ a - Realteil, b - Imaginärteil



wenn $b = 0 \Rightarrow z \in \mathbb{R}$

2. polare Darstellung

$\boxed{z = r(\cos\varphi + i \sin\varphi)}$



$r = |z| \hat{=}$ Länge
$r = \sqrt{a^2 + b^2}$
$\tan\varphi = b/a$

$a = r\cos\varphi$; $b = r\sin\varphi$

3. exponentielle Darstellung (Beweis, später)

$\boxed{z = re^{i\varphi}}$ weil Euler-Formel $e^{i\varphi} = \cos\varphi + i\sin\varphi$

$\boxed{e^{i\pi} = -1}$ also $e^{i\pi}$ ...

$\boxed{z = r e^{i\varphi}}$ weil Euler-Formel $e^{i\varphi} = \cos\varphi + i\sin\varphi$

$\boxed{e^{i\pi} = -1}$ oder $e^{i\pi} + 1 = 0$

oder $e^{i(\pi + 2\pi)} = e^{i3\pi} = -1$

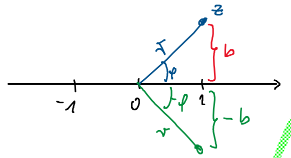
$e^{i(\varphi + 2\pi)} = e^{i\varphi}$

Rechnen: <u>Gleichheit</u>

$z_1 = z_2$ g.d.w. $a_1 = a_2$ & $b_1 = b_2$

oder $r_1 = r_2$ & $\varphi_1 = \varphi_2 + 2k\pi \quad k \in \mathbb{Z}$

Einschub Def.: konjugiert komplex $\hat{=}$ Spiegelung an x-Achse



$z = a + ib \Rightarrow \overline{z} = a - ib$

oder

$z = r e^{i\varphi} \Rightarrow \overline{z} = r e^{-i\varphi}$

wozu?

$z + \overline{z} = 2a = 2\mathrm{Re}\{z\}$

$z \cdot \overline{z} = (a+ib)(a-ib) = a^2 \underbrace{-iab + iab}_{0} + \underbrace{(ib)(-ib)}_{b^2} = a^2 + b^2 = r^2$

$z \cdot \overline{z} = |z|^2 = r^2$

Rechnen analog wie in $\mathbb{R}$ mit $i \cdot i = -1$

Summe: $z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$

Differenz: $z_1 - z_2 = (a_1 + ib_1) - (a_2 + ib_2) = (a_1 - a_2) + i(b_1 - b_2)$

Produkt: $z_1 \cdot z_2 = (a_1 + ib_1) \cdot (a_2 + ib_2) = a_1 \cdot a_2 + i a_1 b_2 + i b_1 a_2 - b_1 b_2$

$= (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$

oder

$z_1 \cdot z_2 = r_1 e^{i\varphi_1} \cdot r_2 e^{i\varphi_2} = r_1 r_2 e^{i(\varphi_1 + \varphi_2)}$

Quotient $\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} = \frac{a_1 + ib_1}{a_2 + ib_2} \underbrace{\frac{a_2 - ib_2}{a_2 - ib_2}}_{1} = \frac{(a_1 + ib_1)(a_2 - ib_2)}{a_2^2 - (ib_2)^2}$

$= \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2}$

oder

$\frac{z_1}{z_2} = \frac{r_1 e^{i\varphi_1}}{r_2 e^{i\varphi_2}} = \frac{r_1}{r_2} e^{i(\varphi_1 - \varphi_2)}$

logo, es gilt: $z_1 + z_2 = z_2 + z_1$; $z_1 \cdot z_2 = z_2 \cdot z_1$; $z_1(z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3$

<u>Bem:</u> größer und kleiner macht keinen Sinn in $\mathbb{C}$

Beweis: Annahme $i > 0$ dann $i > 0 \quad | \cdot i$

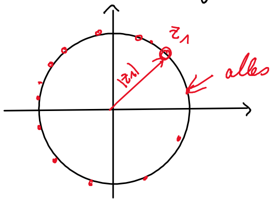
$-1 > 0$ ↯

$i < 0$ dann $i < 0 \quad | \cdot i$

$\Downarrow i < 0$

$-1 > 0$ ↯

aber man kann sagen dass $|z_1| > |z_2|$ da ja $\{|z_1|, |z_2|\} \in \mathbb{R}$



alles mögliche z_2 mit $|z_2| = |z_1|$

∞-Viele

Bew., dass $e^{i\varphi} = \cos\varphi + i \sin\varphi$ morgen, Donnerstag, 09.10.25 ab 9:15 Uhr.

Tip Taylorreihe

Aufgabe 1

(v) $x^3 + 27 = 0 \qquad X_n = a_n + ib_n$

$$(a_n + ib_n)^3 = -27 \quad | \sqrt[3]{}$$

$$a_n + ib_n = \sqrt[3]{-27} = 3 \cdot \sqrt[3]{-1} = -3$$

$\Rightarrow$ Finde: $(a+ib)^3 = -1 \quad 1 \cdot 27$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array} \qquad \begin{array}{ll} (\ )^1 & \\ (\ )^2 & a^2 + 2ab + b^2 \\ (\ )^3 & a^3 + 3a^2b + 3ab^2 + b^3 \end{array}$$

$$a^3 + 3ia^2b + 3a(ib)^2 + (ib)^3 = -1$$

$$a^3 + 3a^2bi - 3ab^2 - ib^3 = -1$$

$$\underbrace{a^3 - 3ab^2}_{=-1} + i\underbrace{(3a^2b - b^3)}_{\overset{!}{=}0} = \underbrace{-1}_{\mathbb{R}}$$

$\Rightarrow$ ① $3a^2b - b^3 = 0 \qquad = b \cdot (3a^2 - b^2) = 0$

② $a^3 - 3ab^2 = -1 \qquad \overset{①}{\Rightarrow} b = 0 \ \vee \ 3a^2 - b^2 = 0$

1. Fall $b = 0 \overset{②}{\Rightarrow} a^3 = -1 \Rightarrow a = -1$

$\hookrightarrow x_1 = -1 + i$

2. Fall $3a^2 - b^2 = 0 \Rightarrow 3a^2 = b^2 \Rightarrow a = \pm \frac{b}{\sqrt{3}}$

a) $a = \frac{b}{\sqrt{3}} \overset{②}{\Rightarrow} \left(\frac{b}{\sqrt{3}}\right)^3 - 3\frac{b}{\sqrt{3}} b^2 = -1$

$$\frac{b^3}{\sqrt{3}^3} - \frac{3b^3}{\sqrt{3}} = -1$$

$$b^3\left(\frac{1}{\sqrt{27}} - \frac{3}{\sqrt{3}}\right) = -1$$

$$b = \sqrt[3]{\frac{-1}{\frac{1}{\sqrt{27}} - \frac{3}{\sqrt{3}}}} \approx 0{,}866$$

$$\Rightarrow a = \frac{0{,}866}{\sqrt{3}} = \frac{1}{2}$$

b) $a = \frac{-b}{\sqrt{3}}$

b) $a = \frac{-b}{\sqrt{3}}$

$x^3 + 27 = 0$

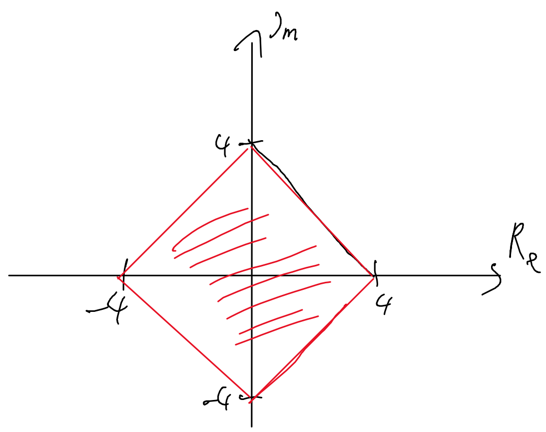
$x^3 = -27$

$x = 3 i^{\frac{2}{3}}$ $\qquad (3 i^{\frac{2}{3}})^3 = 3^3 \cdot i^{\frac{2}{3} \cdot 3} = -27$

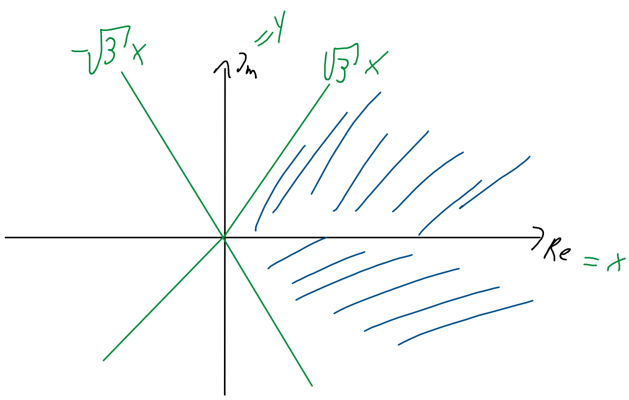
ii) a) $|\underbrace{Re(z)}_{x}| + |\underbrace{Im(z)}_{y}| \leq 4 \iff |x| + |y| \leq 4$

1. $x \geq 0 \Rightarrow |x| = x \Rightarrow x + |y| \leq 4$
   a) $y \geq 0 \Rightarrow |y| = y \Rightarrow x + y \leq 4$ $\leftarrow$
   b) $y \leq 0 \Rightarrow |y| = -y \Rightarrow x - y \leq 4$

2. $x \leq 0 \Rightarrow |x| = -x \Rightarrow -x + |y| \leq 4$
   a) $y \geq 0 \Rightarrow y - x \leq 4$
   $y \leq \Rightarrow -y - x \leq 4$



b)



$$|z| \leq 2\,Re(z) \qquad Re(z) = x$$
$$\sqrt{x^2+y^2} \leq 2x \quad | \; ()^2 \qquad Im(z) = y$$
$$x^2 + y^2 \leq 4x^2$$
$$\Rightarrow y^2 \leq 3x^2 \quad | \sqrt{\;}$$
$$y_{1,2} \leq \pm \sqrt{3}\, x$$

2 a)
$$\int \frac{\underbrace{\frac{\sin^2(x)}{\cos^2(x)}}_{= \tan^2(x)} \frac{1}{\cos^2(x)} - \frac{4}{\cos^2(x)}}{\tan^3(x) - \tan^2(x) - 4\tan(x) + 4}\, dx$$

$$u = \tan(x)$$
$$\frac{du}{dx} = \frac{1}{\cos^2(x)}$$
$$dx = du \; \cos^2(x)$$

$$\Rightarrow \int \frac{\left(u^2 \frac{1}{\cancel{\cos^2(x)}} - \frac{4}{\cancel{\cos^2(x)}}\right) \cancel{\cos^2(x)}}{u^3 - u^2 - 4u + 4}\, du$$

$$\Rightarrow \int \frac{u^2 - 4}{u^3 - u^2 - 4u + 4}\, du$$

$$(u^3 - u^2 - 4u + 4) : (u-2) = u^2 + u - 2$$
$$\underline{-(u^3 - 2u^2)}$$
$$u^2 - 4u + 4$$
$$-(u^2 - 2u)$$
$$-2u + 4$$
$$\underline{-(-2u + 4)}$$
$$0$$

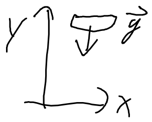
$$u^2 + u - 2 \overset{!}{=} 0$$
$$(u+2) \cdot (u-1) = 0$$

$$(u^2 - 4) = (u+2) \cdot (u-2) \quad \Leftarrow \text{3. Binomisch}$$

$$\Rightarrow \int \frac{\cancel{(u+2)}\cdot\cancel{(u-2)}}{\cancel{(u+2)}\cdot\cancel{(u-2)}\,(u-1)}\,du = \int \frac{1}{u-1}\,du = \ln(u-1) \overset{RS}{=} = \ln(\tan(x)-1)+C$$

Differentialgleichungen (DGL)

Gleichungen bei den eine differenzierte Funktion vorkommt.



$$F_g = F = m\cdot a$$

2mal abgeleitet

$$\Leftrightarrow -m\cdot g = m\cdot \ddot{s}(t) \qquad \mid :m$$

$$-g = \ddot{s}(t) \qquad \mid \int dt$$

1. Methode:

Direktes Integrieren

$$\int -g\,dt = \int \ddot{s}(t)\,dt$$

$$-gt + c_2 = \dot{s}(t) + c_1 \qquad \mid v_0 = c_2 - c_1$$

$$-gt + v_0 = \dot{s}(t) \qquad \mid \int dt$$

$$\int -gt + v_0\,dt = \int \dot{s}(t)\,dt$$

$$-\frac{gt^2}{2} + v_0\cdot t + s_0 = s(t)$$

$$s(0) = 1{,}7\,m = s_0$$

$$\dot{s}(0) = 0\,\frac{m}{s} = v_0$$

$$s(t) = \frac{-gt^2}{2} + 0\cdot t + 1{,}7\,m$$

Radioaktiver-Zerfall: $\quad f'(t) = -3f(t)$

$$\frac{d\,f(t)}{dt} = -3\,f(t) \qquad \mid \text{"}\cdot dt\text{"} \qquad \mid : f(t)$$

$$\int \frac{1}{f(t)}\,df(t) = \int -3\,dt$$

$$\ln(f(t)) = -3t + c \quad | \, e^{c)}$$

$$f(t) = e^{-3t+c} = e^{-3t} \cdot e^{c} = e^{-3t} \cdot \tilde{c}$$

$$f(0) = 4$$

$$f(0) = e^{-3\cdot 0} \cdot \tilde{c} \overset{!}{=} 4$$

$$\Rightarrow \tilde{c} = 4$$

Methode 2:

$\Rightarrow$ Trennung der Variablen

$$y'(t) = g(y(t)) \cdot h(t)$$

$$\frac{dy(t)}{dt} = g(y(t)) \cdot h(t) \qquad | : g(y(t)) \quad | \cdot dt$$

Trennung $\Rightarrow \int \frac{1}{g(y(t))} \, dy(t) = \int h(t) \, dt$

$$y''(t) = +\omega^2 y(t)$$

Raten

$$y(t) = e^{\lambda t}$$

$$y'(t) = \lambda e^{\lambda t}$$

$$y''(t) = \lambda^2 e^{\lambda t}$$

$$\lambda^2 e^{\lambda t} = \omega^2 e^{\lambda t} \quad | : e^{\lambda t}$$

$$\lambda^2 = \omega^2$$

$$\lambda_{1,2} = \pm \omega$$

$$y_1(t) = e^{\omega t} \qquad y_2(t) = e^{-\omega t}$$

2 Lösungen

partikulare Lösungen

$$y(t) = A e^{\omega t} + B e^{-\omega t}$$

A, B kann aus Randbedingungen Bestimmt werden

$$y''(t) = -\omega^2 y(t) \qquad y(t) = e^{\lambda t}$$

$$\lambda^2 e^{\lambda t} = -\omega^2 e^{\lambda t} \quad | : e^{\lambda t}$$

$$\lambda^2 = -\omega^2$$

$$\lambda_{1,2} = \pm\sqrt{-\omega^2} = \pm i w$$

$\leftarrow$ Schwierig

Stattdessen nutze $y(t) = \cos(\lambda t)$ $(\sin(\omega t))$

$$y'(t) = -\lambda \sin(\lambda t)$$
$$y''(t) = -\lambda^2 \cos(\lambda t)$$

$$\Rightarrow \quad -\lambda^2 \cos(\lambda t) = -\omega^2 \cos(\lambda t) \quad | : \cos(\lambda t)$$
$$-\lambda^2 = -\omega^2 \quad | \cdot (-1)$$
$$\lambda_{1,2} = \pm\omega$$

$\Rightarrow$ Partikulare Lsg $\quad y_1(t) = A\cos(\omega t)$

$$y_2(t) = B\cos(-\omega t)$$

$\Rightarrow$ allg. $\quad y(t) = A\cos(\omega t) + B\cos(-\omega t)$

Methode 3: Ansatz Raten

Homogene DGL vs. Inhomogene DGL

$\hookrightarrow y'(t) + 3y(t) = 0 \qquad \hookrightarrow y'(t) + 3y(t) = At$

$\hookrightarrow$ Inhomogenität

Bsp. $\quad y'(t) - y(t)t = 9t$ $\rightarrow$ Inhomogenität

$$y'(t) = t\,y(t) + 9t$$

homogene Lsg $\Rightarrow$

$$\frac{dy}{dt} = t \quad | \; dt \quad | \cdot \frac{1}{y}$$
$$\int \frac{1}{y}\,dy = \int t\,dt$$
$$\ln(y) = \frac{1}{2}t^2 + c \quad | e^{()}$$
$$y(t) = e^{\frac{1}{2}t^2} \cdot \tilde{C}$$

Methode 4:

Variation der Konstante

$\tilde{C} \Rightarrow \tilde{C}(t)$

$$y(t) = e^{\frac{1}{2}t^2} \cdot \tilde{C}(t)$$
$$y'(t) = t e^{\frac{1}{2}t^2} \cdot \tilde{C}(t) + e^{\frac{1}{2}t^2} \cdot \tilde{C}'(t)$$

$$y'(t) = y(t) \cdot t + 9t$$

$$y'(t) = y(t)\cdot t + gt$$

$$\underline{t e^{\frac{1}{2}t^2} \tilde{C}(t)} + e^{\frac{1}{2}t^2} \tilde{C}'(t) = \underline{t e^{\frac{1}{2}t^2} \cdot \tilde{C}(t)} + gt \qquad \Big| - t e^{\frac{1}{2}t^2} \tilde{C}(t)$$

$$e^{\frac{1}{2}t^2} \tilde{C}'(t) = gt$$

$$\tilde{C}'(t) = gt \cdot e^{-\frac{1}{2}t^2}$$

$u = t^2$

$dt = du \cdot \frac{1}{2t}$

$$\tilde{C}(t) = \int gt \cdot e^{-\frac{1}{2}t^2}\, dt$$

$$\tilde{C}(t) = \int \frac{gt}{2t} \cdot e^{-\frac{1}{2}u}\, du$$

$$= -\frac{g}{\cancel{2}} \cancel{2}\, e^{-\frac{1}{2}u} = -g\, e^{-\frac{1}{2}u} \overset{RS}{=} -g e^{-\frac{1}{2}t^2} + \alpha = \tilde{C}(t)$$

$$y(t) = e^{\frac{1}{2}t^2} \cdot \left(-g\, e^{-\frac{1}{2}t^2} + \alpha\right)$$