Übung 1

1. Vereinfachen

a) $20x^2 \cdot \frac{3a}{5x} - \frac{a(x+6)}{3} = 12ax - \frac{6a}{3} - \frac{a}{3}x$

$= a \cdot \left(\frac{35}{3}x - 2\right)$

b) $\frac{a^2 - b^2}{a+b} = \frac{(a+b)(a-b)}{a+b} = a - b$

c) $x^{3k+2} \cdot 3x^{4k+7} \cdot 7x^{n-9-7k}$

$= x^{3k+2+4k+7+n-9-7k} \cdot 21$

$= 21x^{2+7+n-9} = x^n \cdot 21$

d) $\left(\frac{x^2 y}{u^2 v^2}\right)^4 : \left(\frac{xy^3}{u^2 v}\right)^2$

$= \frac{x^8 y^4}{u^8 v^8} : \left(\frac{x^2 y^6}{u^4 v^2}\right) = \frac{x^8 y^4}{u^8 v^8} \cdot \frac{u^4 v^2}{x^2 y^6}$

$= \frac{x^6}{u^4 v^6 y^2}$

e) $(-a)^{-2} \cdot a = \frac{1}{(-a)^2} \cdot a = \frac{a}{a^2} = \frac{1}{a}$

f) $-a^{-2} a = -\frac{1}{a}$

g) $\sqrt[5]{32 y^{10}} = \sqrt[5]{32} \cdot \sqrt[5]{y^{10}} = \sqrt[5]{32}\, y^{10/5} = \sqrt[5]{32}\, y^2$

$= 2y^2$

h) $\sqrt[3]{\sqrt[4]{x^{24}}} = x^{24/12} = x^2$

i) $\frac{x-y}{y-x} = \frac{x-y}{-(x-y)} = -1$

j) $\frac{2-x}{4-x^2} + \frac{x+1}{x} - \frac{x+4}{x+2} - \frac{2}{x^2+2x}$

$\frac{2-x}{(2-x)(2+x)} + 1 + \frac{1}{x} - \frac{x+4}{x+2} - \frac{2}{x} \cdot \frac{1}{2+x}$

$= \frac{1}{2+x} + 1 + \frac{1}{x} - \frac{x+4}{x+2} - \frac{2/x}{2+x}$

$= \frac{1-x-4-\frac{2}{x}}{2+x} + \underbrace{1 + \frac{1}{x}}_{\frac{x+1}{x}}$

$$= \frac{x - x^2 - 4x - 2 + (2+x)(x+1)}{(2+x)x}$$

$$= \frac{-3x - x^2 - 2 + 2x + 2 + x^2 + x}{(2+x)x} = 0$$

k) $$\frac{(x^2)^4 - x^{2^4}}{x^8} + x^8 = \frac{x^8 - x^{16}}{x^8} + x^8$$

$$= 1 - x^8 + x^8 = 1$$

l) $$\frac{x^{6n+2} \cdot x^{3-n}}{(x^2)^n (x^{n+3})^2} = \frac{x^{6n+2+3-n}}{x^{2n+2n+6}} = \frac{x^{5n+5}}{x^{4n+6}}$$

$$= x^{n-1}$$

Aufgabe 2 Gleichungen und Ungleichungen

a) $$\frac{2x-1}{2-x} = \frac{7}{3x+4} \qquad D = \mathbb{R} \setminus \{2, -\tfrac{4}{3}\}$$

$$\Rightarrow (2x-1)(3x+4) = 14 - 7x$$

$$6x^2 + 8x - 3x - 4 = 14 - 7x$$

$$\Rightarrow 6x^2 + 12x - 18 = 0 \quad | : 6$$

$$x^2 + 2x - 3 = 0$$

$$x = -1 \pm \sqrt{1+3} = -1 \pm 2$$

$$x = 1 \quad \vee \quad x = -3 \quad , \quad x \in \{1, -3\}$$

b) $$\frac{x+1}{2x-4} = \frac{x+2}{x-2} \qquad D = \mathbb{R} \setminus \{2\}$$

$$(x+1)(x-2) = (x+2)(x-2) \cdot 2$$

gelöst durch $x = 2 \quad \vee \quad (x+1) = (x+2) \cdot 2 \Rightarrow x = -3$

$x \neq 2$: $\quad (x+1)(x-2) = (x+2)(x-2) \cdot 2 \quad | : (x-2)$

$x = 2$ nicht in Definitionsbereich!

c) $$2 - 3(7 - 4x) = 5x - 7 + 2(4x + 3)$$

$$2 - 21 + 12x = 5x - 7 + 8x + 6$$

$$\Rightarrow -4 - 21 + 4x - 5x + 7 = 0$$

$$\Rightarrow -18 = x$$

d) $x(x-15)(x+23)=0$ → Linearfaktorzerlegung / Nullstellenform

$x \in \{0, -23, 15\}$

äquivalent zu $x^3 + 8x^2 - 345x = 0$

$x = 0 \quad \vee \quad x^2 + 8x - 345 = 0$

$x = -4 \pm \sqrt{16 + 345}$

$= -4 \pm \sqrt{361} = -4 \pm 19$

e) $\frac{6x-1}{3x+2} = \frac{2x}{x-1}$ $\quad D = \mathbb{R} \setminus \{-\frac{2}{3}, 1\}$

$(6x-1)(x-1) = 2x(3x+2)$

$\Rightarrow \quad 10x + x = 1 \quad \Rightarrow x = \frac{1}{11}$

f) $\log_{10}(3x+4) = 3 \quad | \; 10^{\dots}$

$10^{\log_{10}(3x+4)} = 10^3$

$3x + 4 = 1000 \quad \Rightarrow \quad 3x = 996 \Rightarrow x = 332$

g) $|x-1| \leq 1$

$|x-1| = \begin{cases} x-1, & x-1 \geq 0 \\ 1-x, & x-1 < 0 \end{cases} = \begin{cases} x-1, & x \geq 1 \\ 1-x, & x < 1 \end{cases}$

1. Fall $x \geq 1$: $x - 1 \leq 1 \Rightarrow x \leq 2 \Rightarrow x \geq 1 \wedge x \leq 2$

2. Fall $x < 1$: $1 - x \leq 1 \Rightarrow x \geq 0 \Rightarrow x < 1 \wedge x \geq 0$

und

es gilt 1. Fall $\vee$ 2. Fall

oder

$x \in [1, 2] \qquad x \in [0, 1) = [0, 1[$

insgesamt: $x \in [1, 2] \cup [0, 1) = [0, 2]$

h) $\frac{4}{x-3} \leq 1 \qquad D = \mathbb{R} \setminus \{3\}$

$3 < 4 \quad | \cdot (-1) \not\Rightarrow \quad -3 < -4$

① $4 \leq x - 3$, falls $x - 3 > 0 \Rightarrow x > 3$

$\Rightarrow x \geq 7$

hier gilt: $x > 3 \wedge x \geq 7 \Rightarrow x \in [7, \infty)$

② $4 \geq x - 3$, falls $x - 3 < 0 \Rightarrow x < 3$

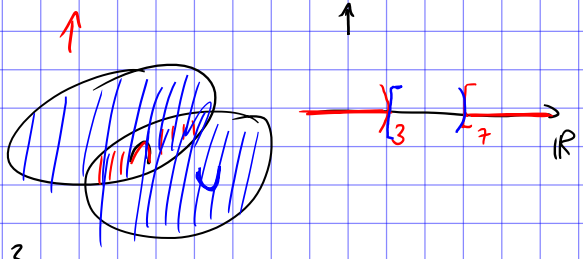
$\Rightarrow x \leq 7$

hier gilt: $x < 3 \wedge x \leq 7 \Rightarrow x \in (-\infty, 3)$

gesamte Lösung: ① ∨ ②

$x \in [7, \infty) \vee x \in (-\infty, 3)$

$\Leftrightarrow x \in [7, \infty) \cup (-\infty, 3) = \mathbb{R} \setminus [3, 7)$



(i) $\frac{x}{2x+1} < 2$ $\quad \mathbb{D} = \mathbb{R} \setminus \{-\frac{1}{2}\}$

① $2x + 1 > 0 \Rightarrow x > -\frac{1}{2}$

$x < 4x + 2 \Rightarrow x > -\frac{2}{3}$

$\Rightarrow x > -\frac{1}{2} \wedge x > -\frac{2}{3} \Rightarrow x \in (-\frac{1}{2}, \infty)$

② $2x + 1 < 0 \Rightarrow x < -\frac{1}{2}$

$x > 4x + 2 \Rightarrow x < -\frac{2}{3}$

$x < -\frac{2}{3} \wedge x < -\frac{1}{2} \Rightarrow x \in (-\infty, -\frac{2}{3})$

insgesamt ① ∨ ② $\Rightarrow x \in (-\infty, -\frac{2}{3}) \cup (-\frac{1}{2}, +\infty)$

$\Rightarrow x \in \mathbb{R} \setminus [-\frac{2}{3}, -\frac{1}{2}]$

j) $6 + \frac{1}{x+3} < 1$ $\quad \mathbb{D} = \mathbb{R} \setminus \{-3\}$

$\Leftrightarrow \frac{1}{x+3} < -5$ ohne Einschränkung

① $x + 3 > 0 \Rightarrow x > -3$

$1 < -5x - 15 \Rightarrow 5x < -16 \Rightarrow x < -\frac{16}{5}$

$x > -3 \wedge x < -\frac{16}{5} = -3{,}2 \Rightarrow x \in \{\}$

② $x + 3 < 0 \Rightarrow x < -3 \Rightarrow 1 > -5x - 15 \Rightarrow x > -\frac{16}{5}$

$x < -3 \wedge x > -\frac{16}{5} \Rightarrow x \in (-3{,}2, -3)$

① ∨ ② $\Rightarrow x \in (-\frac{16}{5}, -3)$

k) $\frac{|1-x|}{x+3} \geq -2$

① $|1-x| \geq -2(x+3)$, falls $x+3>0 \Rightarrow x>-3$

Ⓐ $|1-x| = 1-x$, falls $1-x \geq 0 \Rightarrow x \leq 1$

$1-x \geq -2(x+3) \Rightarrow x \geq -7$

Ⓑ $x-1 \geq -2(x+3)$, falls $x>1$

$\Rightarrow x \geq -\frac{5}{3}$

② $|1-x| \leq -2(x+3)$, falls $x+3<0 \Rightarrow x<-3$

Ⓐ $1-x \leq -2(x+3)$, falls $1-x \geq 0$, $x \leq 1$

$\Rightarrow x \leq -7$

Ⓑ $x-1 \leq -2(x+3)$, falls $1-x<0$, $x>1$

$\Rightarrow x \leq -\frac{5}{3}$

insgesamt: $((① \wedge Ⓐ) \vee (① \wedge Ⓑ)) \vee ((② \wedge Ⓐ) \vee (② \wedge Ⓑ))$

① Ⓐ $x>-3 \wedge x \geq -7 \wedge x \leq 1 \Rightarrow x \in (-3,1]$

Ⓑ $x>-3 \wedge x>1 \wedge x \geq -\frac{5}{3} \Rightarrow x \in (1,\infty)$

② Ⓐ $x<-3 \wedge x \leq 1 \wedge x \leq -7 \Rightarrow x \in (-\infty,-7]$

Ⓑ $x<-3 \wedge x>1 \wedge x \leq -\frac{5}{3} \Rightarrow x \in \{\} = \emptyset$

zusammen

$x \in (-\infty,-7] \cup (-3,1] \cup (1,\infty)$

$= (-\infty,-7] \cup (-3,\infty) = \mathbb{R} \setminus (-7,-3]$

l) $\frac{2|x|}{x+3} \leq 1 \quad D = \mathbb{R} \setminus \{-3\}$

① $x+3>0 \Rightarrow x>-3$

$2|x| \leq x+3$

Ⓐ $2x \leq x+3$, $x \geq 0$

$\Rightarrow x \leq 3$

Ⓑ $-2x \leq x+3$, $x<0$

$\Rightarrow -3x \leq 3 \Rightarrow x \geq -1$

② $x+3<0 \Rightarrow x<-3$

$2|x| \geq x+3$

Ⓐ $2x \geq x+3, \quad x \geq 0$

$x \geq 3$

Ⓑ $-2x \geq x+3, \quad x<0$

$\Rightarrow -3x \geq 3 \Rightarrow x \leq -1$

① Ⓐ $x>-3 \wedge x \geq 0 \wedge x \leq 3 \Rightarrow x \in [0,3]$

Ⓑ $x>-3 \wedge x<0 \wedge x \geq -1 \Rightarrow x \in [-1,0)$

② Ⓐ $x<-3 \wedge x \geq 0 \wedge x \geq 3 \Rightarrow x \in \{\}$

Ⓑ $x<-3, \wedge x<0 \wedge x \leq -1 \Rightarrow x \in (-\infty,-3)$

insgesamt: $x \in (-\infty,-3) \cup [-1,0) \cup [0,3]$

$x \in (-\infty,-3) \cup [-1,3]$

m) $|2x+4| \leq x+5$

① $2x+4 \leq x+5, \quad 2x+4 \geq 0 \Rightarrow x \geq -2$

$\Rightarrow x \leq 1$

$\rightarrow x \in [-2,1]$

② $-2x-4 \leq x+5, \quad 2x+4 \leq 0, \quad x<-2$

$-3x \leq 9 \Rightarrow -x \leq 3 \quad x \geq -3$

$\rightarrow x \in [-3,-2)$

$x \in [-3,-2) \cup [-2,1] = [-3,1]$

n) $\dfrac{x^2+2x-12}{x^2+8x+15} \geq 1$

Vorzeichen Nenner? → Nullstellen $x^2+8x+15=0$

$x \in \{-3,-5\}$

$x \to -\infty \Rightarrow x^2+8x+15 \to \infty$

$x \to +\infty \Rightarrow x^2+8x+15 \to \infty$

① $x^2+8x+15 \geq 0$ falls $x \in (-\infty,-5] \cup [-3,\infty)$

$x^2+8x+15 \leq x^2+2x-12 \Rightarrow 6x \leq -27$

$\Rightarrow x \leq -\frac{9}{2}$

$x \in (-\infty,-5] \cup [-3,\infty) \wedge x \leq -\frac{9}{2}$

$\Rightarrow x \in (-\infty,-5]$

② $\cancel{x^2} + 8x + 15 \geq \cancel{x^2} + 2x - 12$

$x \geq -9/2$

$x \in (-5, -3) \cup [-9/2, \infty) \Rightarrow x \in [-9/2, -3)$

Zusammen: ① ∨ ② $x \in [-9/2, -3) \cup (-\infty, -5]$