1.5 Gleichungen mit einer Variablen

1.5.1 Lineare Gleichungen

$$a \cdot x = b \quad \Rightarrow \quad x = \frac{b}{a} \qquad a \neq 0$$

1.5.2 Quadratische Gleichungen

$$ax^2 + bx + c = 0$$

$$(x - x_1)(x - x_2) = 0$$

Lösungen gegeben durch x_1 und x_2

Mitternachtsformel: $x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad a \neq 0$

p,q - Formel: $x_{1,2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$

1.5.3 Exponentialgleichungen

$$e^x = a \quad \Rightarrow \quad x = \ln(a)$$

Bsp: $3^{x^2-4} = 6^{-x}$

$$\Leftrightarrow \quad 3^{x^2-4} = 3^{-x} \, 2^{-x}$$

$$3^{x^2-4} \, 3^x = 2^{-x}$$

$$3^{x^2+x-4} = 2^{-x} \quad \Big| \log_3(\;)$$

$$x^2 + x - 4 = \log_3(2^{-x}) = -x \log_3(2)$$

$$x^2 + x - 4 + x \log_3(2) = 0$$

$$\Leftrightarrow \quad x^2 + x(1 + \log_3(2)) - 4 = 0$$

$$x_{1,2} = \frac{-(1 + \log_3 2) \pm \left((1 + \log_3(2))^2 + 16\right)^{1/2}}{2}$$

$$x_1 = -2{,}98 \qquad x_2 = 1{,}34$$

1.5.4 Faktorisieren

Bsp: $x^4 - x^2 = 0 \quad \Leftrightarrow \quad x^2(x^2 - 1) = 0$

Nullproduktsatz: $x^2 = 0 \quad \vee \quad x^2 - 1 = 0$

$$x \in \{-1, 0, 1\}$$

1.5.5 Wurzelgleichungen

Lösen oft mittels Quadrieren → keine Äquivalenzumformung

→ daher Probe der Lösung!

Bsp: $\sqrt{4x^2+\sqrt{x^2}+1} = x+1 \quad | \, (\ )^2$

$$4x^2+\sqrt{x^2}+1 = (x+1)^2 = x^2+2x+1$$

$$\Rightarrow \quad \sqrt{x^2} = -3x^2+2x \quad | \, (\ )^2$$

$$\underline{x^2} = (-3x^2+2x)^2 = 9x^4-12x^3+\underline{4x^2}$$

$$\Rightarrow \quad 9x^4-12x^3+3x^2=0 \quad \Rightarrow \quad x^2\cdot(9x^2-12x+3)=0$$

$$x^2=0 \quad \vee \quad 3x^2-4x+1=0$$

$$x=0 \quad \vee \quad x_{1,2} = \frac{4\pm\sqrt{16-12}}{6}$$

$$x_1 = 1/3 \, , \quad x_2 = 1$$

Probe: $x_0 = 0$ und $x_1 = 1/3$

$$\sqrt{4x^2+\sqrt{x^2}+1} = x+1$$

$$x_0=0 \quad \Rightarrow \quad 1=1$$

$$x_1 = \tfrac{1}{3} \quad \sqrt{\frac{4}{9}+\sqrt{\frac{1}{9}}+1} = \frac{1}{3}+1$$

$$\sqrt{\frac{4}{9}+\frac{1}{3}+1} = \frac{1}{3}+1 \quad \Rightarrow \quad \sqrt{\frac{16}{9}} = \frac{1}{3}+1$$

$$x_2=1 \quad \sqrt{4+\sqrt{1}+1} = 2 \quad \Rightarrow \quad \sqrt{6} = 2 \quad ↯$$

Achtung

$$\sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

diese Umformungen sind keine Äquivalenzumformungen,

dies zeigt die Lösung x_2

Lösungsmenge: $\mathbb{L} = \{0, 1/3\}$

1.5.6 Betragsgleichungen

Bsp: $|x-1| + |x-2| = 5$

$$|x-1| = \begin{cases} x-1, & \text{falls } x \geq 1 \\ -(x-1), & \text{falls } x < 1 \end{cases}$$

$$|x-2| = \begin{cases} x-2, & \text{falls } x \geq 2 \\ -(x-2), & \text{falls } x < 2 \end{cases}$$

① $x \in (-\infty, 1)$

$|x-1| + |x-2| = 5 \Leftrightarrow (1-x) + (2-x) = 5 \Rightarrow -2x = 2$

$\Rightarrow x = -1$

② $x \in [1, 2)$

$|x-1| + |x-2| = 5 \Leftrightarrow (x-1) + (2-x) = 5 \Leftrightarrow 1 = 5$ ↯

kein Beitrag zur Lösung

③ $x \in [2, \infty)$

$|x-1| + |x-2| = 5 \Leftrightarrow (x-1) + (x-2) = 5$

$2x = 8 \Rightarrow x = 4$

insgesamt: $\mathbb{L} = \{-1, 4\}$

1.5.7 Substitution

Ersetzen eines Teilterms durch eine neue Variable

<u>Bsp</u> $(\ln(x))^2 - 4\ln(x) + 4 = 0$ $\quad \ln(x) = y$

$y^2 - 4y + 4 = 0$

$(y-2)^2 = 0 \Rightarrow y_0 = 2$ ist Lösung

Rücksubstitution: $\ln(x_0) = y_0$

→ Umkehrfunktion

$x_0 = e^{y_0} = e^2$ $\quad \mathbb{L} = \{e^2\}$

1.5.8 Ungleichungen mit Bruchtermen

<u>Bsp</u> $\frac{x-1}{x+3} \geq 1$ $\quad D = \mathbb{R} \setminus \{-3\}$ ↯

Fallunterscheidung: ① $x + 3 > 0$

$x - 1 \geq x + 3 \Rightarrow -1 \geq 3$

② $x + 3 < 0 \Rightarrow x < -3$

$x - 1 \leq x + 3 \Rightarrow -1 \leq 3$

$(-1 \leq 3) \Leftrightarrow (-1 < 3) \vee (-1 = 3)$

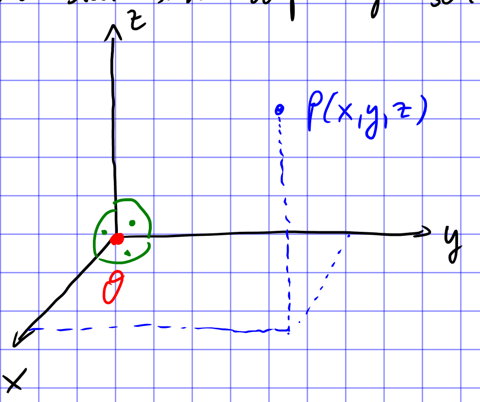
Gesamtlösungsmenge : $\mathbb{L} = (-\infty, -3)$

2. Elementare Geometrie

2.1 Koordinatensysteme

- kartesische Koordinatensystem

  3 paarweise senkrecht aufeinander stehende Geraden, die sich im Ursprung schneiden



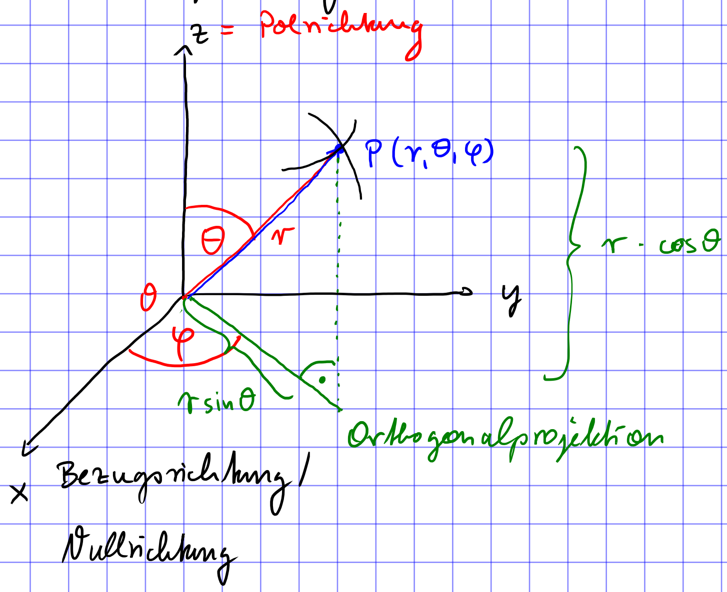
- Kugelkoordinaten

  Pol O zwei senkrecht aufeinander stehende Strahlen

  r : Radius, Abstand OP

  θ : Polarwinkel zwischen Polrichtung und OP

  φ : Azimutwinkel zwischen Orthogonalprojektion von OP und Bezugsrichtung



Transformationsgleichungen :

$$x = r \cdot \sin\theta \cdot \cos\varphi$$

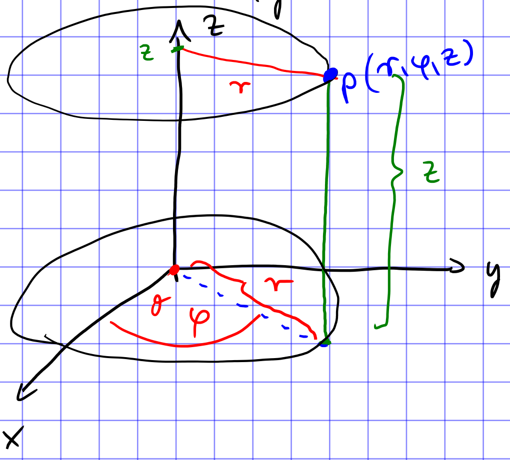
$$y = r \cdot \sin\theta \cdot \sin\varphi$$

$$z = r \cdot \cos\theta$$

- Zylinderkoordinaten

  r, φ ebene Polarkoordinaten → r: Abstand zur z-Achse, → φ: Azimuth

  z Abstand zu x,y-Ebene



- Transformationsgleichungen

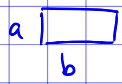
$$x = r\cos\varphi$$

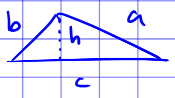
$$y = r\sin\varphi$$

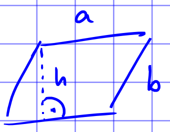
$$z = z$$

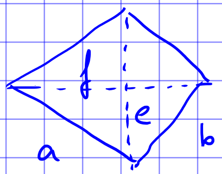
2.2 Elementare geometrische Objekte

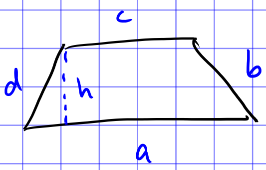
2D: Flächeninhalt und Umfang

- Kreis  $A = \pi \cdot r^2$ $U = 2\pi r$
- Rechteck 

 $A = a \cdot b$ $U = 2(a+b)$
- Dreieck 

 $A = \frac{1}{2} c h$ $U = a+b+c$
- Parallelogram 

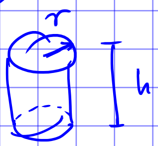
 $A = a \cdot h$ $U = 2(a+b)$
- Drachen 

 $A = \frac{1}{2} e \cdot f$ $U = 2(a+b)$
- Trapez 

 $A = \frac{a+c}{2} \cdot h$ $U = a+b+c+d$

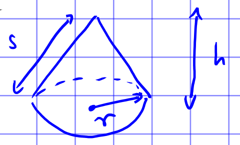
3D: Volumen und Oberfläche

- Kugel 

 $V = \frac{4}{3}\pi r^3$, $O = 4\pi r^2$
- Zylinder 

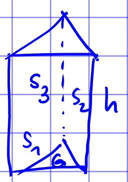
 $V = \pi r^2 \cdot h$ $O = 2\pi r h + 2\pi r^2$

- Kegel



$V = \frac{1}{3}\pi r^2 h \qquad O = \pi r^2 + \pi r s$

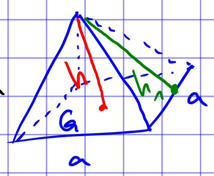
- senkrechte Prisma



$V = G h \qquad O = 2 \cdot G + s_1 + s_2 + s_3 + s_4 + \dots$

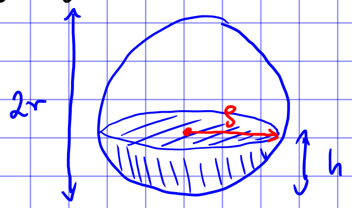
- senkrechte Pyramide
(n Seiten)

Bsp: 4 Seiten



$V = \frac{1}{3} G \cdot h \qquad O = G + \frac{1}{2} h_1 \cdot a \cdot n$

- Kugelsegment



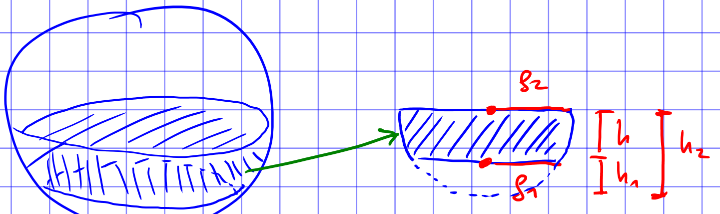
$V = \frac{\pi}{6} h (3\rho^2 + h^2) = \frac{\pi}{3} h^2 (3r - h)$

$O = 2\pi r h + \pi \rho^2 = \pi (2rh + \rho^2)$

Radius des Schnittkreises:

$\rho = \sqrt{h(2r-h)}$

- Kugelschicht

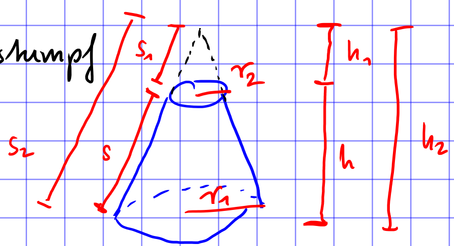


$\rho_1 = \sqrt{h_1(2r - h_1)} \qquad \rho_2 = \sqrt{(h+h_1)(2r-h-h_1)}$

$V = \frac{\pi}{2} h (3\rho_1^2 + 3\rho_2^2 + h^2)$

$O = \pi (2rh + \rho_1^2 + \rho_2^2)$

- Kegelstumpf



$V = \frac{\pi}{3} h (r_2^2 + r_2 r_1 + r_1^2)$

$O = \pi r_2 (s_2 + s) + \pi r_1 (s_1 + s)$

Aufgabe:

Berechnen Sie das Gewicht des Körpers (Dichte $\rho = 8{,}8 \frac{g}{cm^3}$)

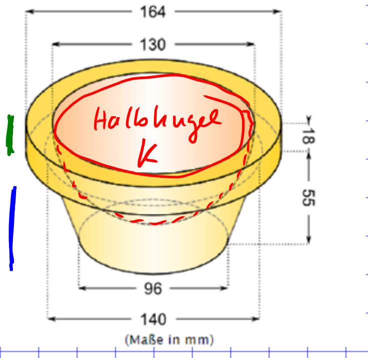
$V_{ges} = V_Z + V_{KS} - V_K$

$= \pi r_z^2 h_z - \frac{\pi}{3} \cdot 2 r_k^3$

$+ \frac{\pi}{3} h_{KS} (r_2^2 + r_2 r_1 + r_1^2)$

Zylinder Z

Kegelstumpf KS



$h_z = 18\ mm,\ r_z = 82\ mm$

$r_k = 65\ mm,\ r_2 = 70\ mm,\ r_1 = 48\ mm$

$h_{KS} = 55\ mm$

$\longrightarrow V_{ges} = 413502\ mm^3$

$\rho = 8{,}8 \ g/cm^3$ $m = \rho \cdot V$

V_{ges} (in cm^3) $= 413{,}502 \ cm^3$

$m = 8{,}8 \cdot 413{,}502 \ \frac{g}{\cancel{cm^3}} \cdot \cancel{cm^3} = 3639 \ g = 3{,}639 \ kg$