10.10.2023

1) a) $V = \pi R^2 \cdot h \qquad R = 2h$

$V = 4\pi h^3 \Rightarrow h = \sqrt[3]{\frac{V}{4\pi}}$

$1\,fL = 1\cdot 10^{-15}\,L = 1\cdot 10^{-18}\,m^3 \Rightarrow V = 9\cdot 10^{-17}\,m^3 \rightarrow h = 2\,\mu m \quad D = 8\,\mu m$

b) $x\,\frac{km}{h} = x\,\frac{1\,km}{1\,h} = x\,\frac{1000\,m}{3600\,s} = x\cdot\frac{1}{3{,}6}\,\frac{m}{s} \Longrightarrow y = \frac{x}{3{,}6}$

c) $1\,\frac{g}{cm^3} \qquad SI: \frac{kg}{m^3}$

$\Rightarrow 1\,\frac{g}{cm^3} = 1\,\frac{10^{-3}\,kg}{(10^{-2}\,m)^3} = 1\cdot 10^3\,\frac{kg}{m^3} = 1\,\frac{t}{m^3}$

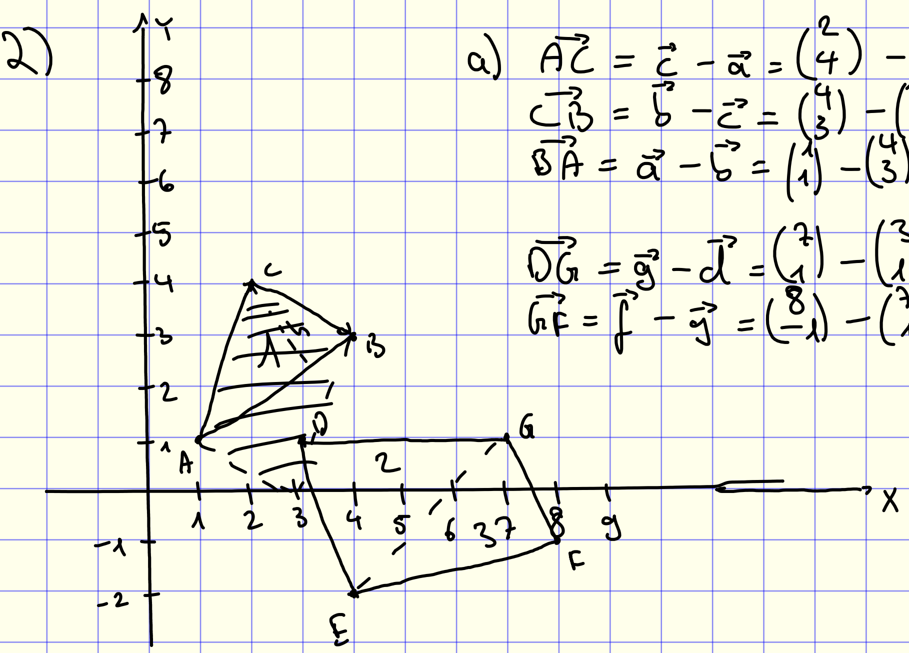
$1\,\frac{g}{cm^3} = 1\,\frac{\frac{1}{453{,}6}\,lb}{\left(\frac{1}{30{,}48}\,ft\right)^3} = 1\,\frac{30{,}48^3\,lb}{453{,}6\,ft^3} = 62{,}43\,\frac{lb}{ft^3}$

d) $P_{1PS} = 735{,}5\,W \qquad C_{Handy} = 3400\,mAh \qquad U_{Handy} = 4{,}4\,V$

$P = U\cdot I \qquad E = P\cdot t$

$E = U\cdot I\cdot t = 3400\,mAh\cdot 4{,}4\,V = 53{,}8\,kWs = 73\,PS\cdot s$

2)



a) $\vec{AC} = \vec{c} - \vec{a} = \begin{pmatrix}2\\4\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix}$

$\vec{CB} = \vec{b} - \vec{c} = \begin{pmatrix}4\\3\end{pmatrix} - \begin{pmatrix}2\\4\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}$

$\vec{BA} = \vec{a} - \vec{b} = \begin{pmatrix}1\\1\end{pmatrix} - \begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}-3\\-2\end{pmatrix}$

$\vec{DG} = \vec{g} - \vec{d} = \begin{pmatrix}7\\1\end{pmatrix} - \begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}4\\0\end{pmatrix}$

$\vec{GF} = \vec{f} - \vec{g} = \begin{pmatrix}8\\-1\end{pmatrix} - \begin{pmatrix}7\\1\end{pmatrix} = \begin{pmatrix}1\\-2\end{pmatrix}$

$\vec{FE} = \vec{e} - \vec{f} = \begin{pmatrix}4\\-2\end{pmatrix} - \begin{pmatrix}8\\-1\end{pmatrix} = \begin{pmatrix}-4\\-1\end{pmatrix}$

$\vec{ED} = \vec{d} - \vec{e} = \begin{pmatrix}3\\1\end{pmatrix} - \begin{pmatrix}4\\-2\end{pmatrix} = \begin{pmatrix}-1\\3\end{pmatrix}$

b) $\vec{AC} = \begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}1\\3\\0\end{pmatrix} \qquad \vec{AB} = \vec{b} - \vec{a} = \begin{pmatrix}4\\3\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix} = \begin{pmatrix}3\\2\\0\end{pmatrix}$

$\cancel{|\vec{AC} \times \vec{AB}|} \Rightarrow \frac{1}{2}|\vec{AC}\times\vec{AB}| = \frac{1}{2}\left|\begin{pmatrix}1\\3\\0\end{pmatrix}\times\begin{pmatrix}3\\2\\0\end{pmatrix}\right| = \frac{1}{2}\left|\begin{pmatrix}0\\0\\-7\end{pmatrix}\right| = \frac{1}{2}\cdot\sqrt{7^2} = \frac{1}{2}\cdot 7 = \frac{7}{2}$

$\vec{DG} = \begin{pmatrix}4\\0\\0\end{pmatrix}$ $\quad \vec{DE} = \vec{e} - \vec{d} = \begin{pmatrix}4\\-2\end{pmatrix} - \begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}1\\-3\\0\end{pmatrix} \Rightarrow A_2 = \frac{1}{2}\left|\begin{pmatrix}4\\0\\0\end{pmatrix} \times \begin{pmatrix}1\\-3\\0\end{pmatrix}\right| = \frac{1}{2}\left|\begin{pmatrix}0\\0\\-12\end{pmatrix}\right|$

$= \frac{1}{2} \cdot 12 = 6$

$\vec{EG} = \vec{g} - \vec{e} = \begin{pmatrix}7\\1\end{pmatrix} - \begin{pmatrix}4\\-2\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix} = \begin{pmatrix}3\\3\\0\end{pmatrix} \Rightarrow A_3 = \frac{1}{2}\left|\begin{pmatrix}3\\3\\0\end{pmatrix} \times \begin{pmatrix}4\\1\\0\end{pmatrix}\right| = \frac{1}{2}\left|\begin{pmatrix}0\\0\\-9\end{pmatrix}\right| = \frac{1}{2} \cdot 9 = \frac{9}{2}$

$\vec{EF} = \vec{f} - \vec{e} = \begin{pmatrix}8\\-1\end{pmatrix} - \begin{pmatrix}4\\-2\end{pmatrix} = \begin{pmatrix}4\\1\end{pmatrix}$

$= \begin{pmatrix}4\\1\\0\end{pmatrix} \qquad \Rightarrow A_{ges} = A_2 + A_3 = 6 + \frac{9}{2} = \frac{21}{2}$

3) i) $\mathbb{R}^2 \quad \vec{r} = \begin{pmatrix}-1\\2\end{pmatrix} = -\hat{e}_x + 2\hat{e}_y$

<u>ges:</u> $r, \varphi \qquad r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$

$x = r\cos(\varphi) \quad |:r \quad |\arccos$

$\Rightarrow \varphi = \arccos\left(\frac{x}{r}\right) = \arccos\left(\frac{-1}{\sqrt{5}}\right) = 116°$

$\vec{r} = \sqrt{5}\begin{pmatrix}\cos(\varphi)\\ \sin(\varphi)\end{pmatrix}$

ii)

$A = \begin{pmatrix}-1\\12\\5\end{pmatrix},\ B = \begin{pmatrix}1\\2\\5\end{pmatrix},\ C = \begin{pmatrix}6\\12\\5\end{pmatrix} \qquad \vec{AB} = \vec{b} - \vec{a} = \begin{pmatrix}1\\2\\5\end{pmatrix} - \begin{pmatrix}-1\\12\\5\end{pmatrix} = \begin{pmatrix}2\\-10\\0\end{pmatrix}$

$\vec{AC} = \vec{c} - \vec{a}$

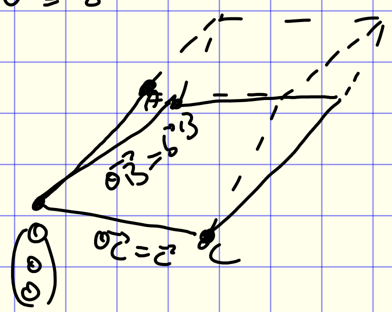
$= \begin{pmatrix}6\\12\\5\end{pmatrix} - \begin{pmatrix}-1\\12\\5\end{pmatrix} = \begin{pmatrix}7\\0\\0\end{pmatrix}$

$\frac{1}{2}\left|\begin{pmatrix}2\\-10\\0\end{pmatrix} \times \begin{pmatrix}7\\0\\0\end{pmatrix}\right| = \frac{1}{2}\left|\begin{pmatrix}0\\0\\70\end{pmatrix}\right| = \frac{1}{2} \cdot 70 = 35$

$\vec{OC} = \vec{c} - \vec{o} = \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} - \begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}$

iii)

$A = \begin{pmatrix}-1\\2\\1\end{pmatrix},\ B = \begin{pmatrix}1\\0\\3\end{pmatrix},\ C = \begin{pmatrix}2\\2\\1\end{pmatrix}$



$|(\vec{A} \times \vec{B}) \cdot \vec{C}| = \left|\left(\begin{pmatrix}-1\\2\\1\end{pmatrix} \times \begin{pmatrix}1\\0\\3\end{pmatrix}\right) \cdot \begin{pmatrix}2\\2\\1\end{pmatrix}\right| = \left|\begin{pmatrix}6\\4\\-2\end{pmatrix} \cdot \begin{pmatrix}2\\2\\1\end{pmatrix}\right| = |6 \cdot 2 + 4 \cdot 2 - 2 \cdot 1| = |18| = 18$

iv) a) $g_1: \vec{r} = \begin{pmatrix}x\\3x-7\end{pmatrix} = \begin{pmatrix}0\\-7\end{pmatrix} + x \cdot \begin{pmatrix}1\\3\end{pmatrix}$

$g_2: \vec{r} = \begin{pmatrix}x\\7x-3\end{pmatrix} = \begin{pmatrix}0\\-3\end{pmatrix} + x \cdot \begin{pmatrix}1\\7\end{pmatrix}$

$\cos(\varphi) = \frac{\begin{pmatrix}1\\3\end{pmatrix} \cdot \begin{pmatrix}1\\7\end{pmatrix}}{\left|\begin{pmatrix}1\\3\end{pmatrix}\right| \cdot \left|\begin{pmatrix}1\\7\end{pmatrix}\right|} = \frac{1 + 21}{\sqrt{1+9} \cdot \sqrt{1+49}}$

$= \frac{22}{\sqrt{10} \cdot \sqrt{50}} = \frac{22}{\sqrt{500}}$

$\varphi = 10{,}3°$

$y = 3x - 7$

$x = 0 \quad , \; x = 1$

$\Rightarrow y = -7 \qquad y = 3 \cdot 1 - 7 = -4$

$\Rightarrow \vec{b} = \begin{pmatrix} 0 \\ -7 \end{pmatrix} \qquad \Rightarrow \vec{a} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \qquad \vec{r} = \vec{a} + \lambda \cdot \begin{pmatrix} -1 \\ -3 \end{pmatrix}$

$\vec{b} - \vec{a} = \begin{pmatrix} 0 \\ -7 \end{pmatrix} - \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \end{pmatrix} \qquad = \begin{pmatrix} 0 \\ -7 \end{pmatrix} + \lambda \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

b) $g_1: \vec{r} = \begin{pmatrix} 0 \\ -7 \end{pmatrix} + x \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$g_2: \vec{r} = \begin{pmatrix} 0 \\ 14 \end{pmatrix} + x \cdot \begin{pmatrix} 1 \\ 7 \end{pmatrix} \qquad \Rightarrow \varphi = 10{,}3^\circ$

c) $g_1: \vec{r} = \begin{pmatrix} 4 \\ -1 \\ 4 \end{pmatrix} + a \cdot \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \qquad \Rightarrow \begin{pmatrix} 4 \\ -1 \\ 4 \end{pmatrix} + a \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} + b \cdot \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}$

$g_2: \vec{r} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} + b \cdot \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 4 + a \\ -1 - 2a \\ 4 + 3a \end{pmatrix} = \begin{pmatrix} 3b \\ b \\ 6 - 5b \end{pmatrix} \begin{matrix} \mathrm{I} \\ \mathrm{II} \\ \mathrm{III} \end{matrix}$

$(\mathrm{II}): -1 - 2a = b \overset{\text{in I}}{\Rightarrow} 4 + a = 3 \cdot (-1 - 2a)$

$\Rightarrow 4 + a = -3 - 6a \quad | + 6a | - 4$

$\Rightarrow 7a = -7 \quad | : 7$

$\Rightarrow a = -1$

in II: $-1 + 2 = b \Rightarrow b = 1$

$\vec{r} = \begin{pmatrix} 4 \\ -1 \\ 4 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \qquad \vec{r} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} + 1 \cdot \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ $\Rightarrow$ Geraden schneiden sich im Punkt $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$

$$\cos(\varphi) = \frac{\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}}{\left|\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}\right| \cdot \left|\begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}\right|} = \frac{3 - 2 - 15}{\sqrt{1 + 4 + 9} \cdot \sqrt{9 + 1 + 25}} = \frac{-14}{\sqrt{14} \cdot \sqrt{35}} = -\sqrt{\frac{14}{35}} = -\sqrt{\frac{2}{5}}$$

$\varphi = \arccos\left(-\sqrt{\frac{2}{5}}\right) \approx 50{,}768^\circ \; (129{,}232^\circ)$

vi)
$$\vec{x} = \begin{pmatrix}0\\1\\1\end{pmatrix} + \alpha\begin{pmatrix}-1\\3\\1\end{pmatrix} + \beta\begin{pmatrix}-4\\1\\-7\end{pmatrix} \qquad \begin{pmatrix}-4\\3\\2\end{pmatrix}\cdot\vec{x} = 5$$

E_1 in Normalenform:

$$\begin{pmatrix}-1\\3\\1\end{pmatrix} \times \begin{pmatrix}-4\\1\\-7\end{pmatrix} = \begin{pmatrix}-22\\-11\\11\end{pmatrix} \qquad \vec{n} = \frac{\begin{pmatrix}-22\\-11\\11\end{pmatrix}}{\left\|\begin{pmatrix}-22\\-11\\11\end{pmatrix}\right\|} = \frac{\cancel{11}\cdot\begin{pmatrix}-2\\-1\\1\end{pmatrix}}{\cancel{11}\cdot\left\|\begin{pmatrix}-2\\-1\\1\end{pmatrix}\right\|} = \begin{pmatrix}-2\\-1\\1\end{pmatrix}\cdot\frac{1}{\sqrt{4+1+1}}$$

$$= \begin{pmatrix}-2\\-1\\1\end{pmatrix}\cdot\frac{1}{\sqrt{6}} = \vec{n}$$

$$\Rightarrow \frac{1}{\sqrt{6}}\begin{pmatrix}2\\1\\-1\end{pmatrix}\cdot\vec{x} = 0$$

$$\Rightarrow \begin{pmatrix}2\\1\\-1\end{pmatrix}\cdot\vec{x} = 0 \qquad \vec{a} = \begin{pmatrix}0\\1\\1\end{pmatrix}$$

$$\begin{pmatrix}2\\1\\-1\end{pmatrix}\cdot(\vec{x}-\vec{a}) = 0$$

$$\Rightarrow \begin{pmatrix}2\\1\\-1\end{pmatrix}\cdot\vec{x} - \begin{pmatrix}2\\1\\-1\end{pmatrix}\cdot\vec{a} = 0$$

$$\begin{pmatrix}2\\1\\-1\end{pmatrix}\cdot\begin{pmatrix}0\\1\\1\end{pmatrix} = 2\cdot 0 + 1\cdot 1 - 1\cdot 1 = 0$$

$$\Rightarrow \begin{pmatrix}2\\1\\-1\end{pmatrix}\cdot\vec{x} - 0 = 0$$

$$\Rightarrow \begin{pmatrix}2\\1\\-1\end{pmatrix}\cdot\vec{x} = 0$$

E_1: $2x_1 + x_2 - x_3 = 0$ (I)

E_2: $-4x_1 + 3x_2 + 2x_3 = 5$ (II)

aus I $\Rightarrow x_3 = 2x_1 + x_2$ $*$ in II

$\Rightarrow -4x_1 + 3x_2 + 4x_1 + 2x_2 = 5 \Rightarrow 5x_2 = 5 \Rightarrow x_2 = 1$

$x_2 = 1$ in $*$: $x_3 = 2x_1 + 1$

$$\vec{x} = \begin{pmatrix}x_1\\1\\2x_1+1\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix} + x_1\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}$$

$\Rightarrow$ Lösung $\vec{x} = \begin{pmatrix}0\\1\\1\end{pmatrix} + \xi\cdot\begin{pmatrix}1\\0\\2\end{pmatrix}$

4)

a) $\vec{a} \times \vec{b} \perp \vec{a}/\vec{b}$

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} \times \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{pmatrix}a_2b_3 - a_3b_2\\a_3b_1 - a_1b_3\\a_1b_2 - a_2b_1\end{pmatrix}$$

$$\begin{pmatrix}a_2b_3 - a_3b_2\\a_3b_1 - a_1b_3\\a_1b_2 - a_2b_1\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} = \cancel{a_1a_2b_3} - \cancel{a_1a_3b_2} + \cancel{a_2a_3b_1} - \cancel{a_2a_1b_3} + \cancel{a_3b_2a_1} - \cancel{a_3a_2b_1}$$

$$= 0 \quad \blacksquare$$

b) $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\vec{c}) - \vec{c}(\vec{a}\vec{b})$ (BAC-CAB-Regel)

$$\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\left(\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}\times\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}\right) = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_2c_3-b_3c_2\\b_3c_1-b_1c_3\\b_1c_2-b_2c_1\end{pmatrix}$$

$$=\begin{pmatrix}a_2b_1c_2-a_2b_2c_1-a_3b_3c_1+a_3b_1c_3\\a_3b_2c_3-a_3b_3c_2-a_1b_1c_2+a_1b_2c_1\\a_1b_3c_1-a_1b_1c_3-a_2b_2c_3+a_2b_3c_2\end{pmatrix}$$

$\vec{b}(\vec{a}\vec{c}) - \vec{c}(\vec{a}\vec{b})$:

$$\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}(a_1c_1+a_2c_2+a_3c_3) - \begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}(a_1b_1+a_2b_2+a_3c_3)$$

$$=\begin{pmatrix}a_1c_1b_1+a_2c_2b_1+a_3c_3b_1-c_1a_1b_1-c_1a_2b_2-c_1a_3b_3\\b_2a_1c_1+b_2a_2c_2+b_2a_3c_3-c_2a_1b_1-c_2a_2b_2-c_2a_3b_3\\b_3a_1c_1+b_3a_2c_2+b_3a_3c_3-c_3a_1b_1-c_3a_2b_2-c_3a_3b_3\end{pmatrix}$$

$$\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=0$$

$$\Rightarrow \vec{b}(\vec{a}\vec{c}) - \vec{c}(\vec{a}\vec{b}) + \vec{c}(\vec{b}\vec{a}) - \vec{a}(\vec{b}\vec{c}) + \vec{a}(\vec{c}\vec{b}) - \vec{b}(\vec{c}\vec{a}) = 0$$

$$\Rightarrow \vec{a}\cdot\vec{c} = \vec{c}\cdot\vec{a}$$

c) $\underbrace{(\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})} + \underbrace{(\vec{a}\vec{b})(\vec{a}\vec{b})} = (\vec{a}\vec{a})(\vec{b}\vec{b})$

$$(\vec{a}\times\vec{b})(\vec{a}\times\vec{b}) = \left(\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}\right)\cdot\left(\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}\times\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}\right) = \begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix}\cdot\begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix}$$

$$= (a_2b_3-a_3b_2)(a_2b_3-a_3b_2)+(a_3b_1-a_1b_3)\cdot(a_3b_1-a_1b_3)+(a_1b_2-a_2b_1)\cdot(a_1b_2-a_2b_1)$$

$$= a_2^2b_3^2-2a_2a_3b_3b_2+a_3^2b_2^2$$

$$+a_3^2b_1^2-2a_3a_1b_1b_3+a_1^2b_3^2$$

$$+a_1^2b_2^2-2a_2b_1a_1b_2+a_2^2b_1^2$$

$$(\vec a \vec b)(\vec a \vec b) = (a_1 b_1 + a_2 b_2 + a_3 b_3)(a_1 b_1 + a_2 b_2 + a_3 b_3)$$

$$= a_1^2 b_1^2 + a_2^2 b_2^2 + a_3^2 b_3^2 + 2a_1 b_1 a_2 b_2 + 2a_1 b_1 a_3 b_3 + 2 a_2 b_2 a_3 b_3$$

$$(\vec a \times \vec b)(\vec a \times \vec b) + (\vec a \vec b)(\vec a \vec b)$$

$$= a_1^2 b_1^2 + a_2^2 b_2^2 + a_3^2 b_3^2 + a_1^2 b_2^2 + a_2^2 b_1^2 + a_1^2 b_3^2 + a_3^2 b_1^2 + a_2^2 b_3^2 + a_3^2 b_2^2$$

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2)$$

$$= a_1^2 b_1^2 + a_1^2 b_2^2 + a_1^2 b_3^2 + a_2^2 b_1^2 + a_2^2 b_2^2 + a_2^2 b_3^2 + a_3^2 b_1^2 + a_3^2 b_2^2 + a_3^2 b_3^2$$

$$= (\vec a \vec a)(\vec b \vec b)$$

$|\vec a \times \vec b| = A$ $\qquad\qquad$ $|\vec a \times \vec b| \overset{!}{=} A$

$$A = |\vec a|\,|\vec b|\sin(\varphi)$$

$$\Rightarrow (\vec a \times \vec b)(\vec a \times \vec b) = |\vec a \times \vec b|^2 = A^2 = |\vec a|^2 |\vec b|^2 \underbrace{\sin^2(\varphi)}_{1-\cos^2(\varphi)}$$

Winkelformel: $\cos^2(\varphi) = \left(\dfrac{\vec a \cdot b}{|\vec a| \cdot |\vec b|}\right)^2$

$$\Rightarrow A^2 = |\vec a|^2 |\vec b|^2 \cdot \left(1 - \frac{(\vec a \cdot \vec b)^2}{|\vec a|^2 |\vec b|^2}\right) = \underbrace{|\vec a|^2 |\vec b|^2 - (\vec a \cdot \vec b)^2 = |\vec a \times \vec b|^2}$$

$$\Rightarrow (\vec a \times \vec b)(\vec a \times \vec b) = |\vec a|^2 |\vec b|^2 - (\vec a \cdot \vec b)^2$$