Blatt 08 12.10.2023

Aufgabe 1:

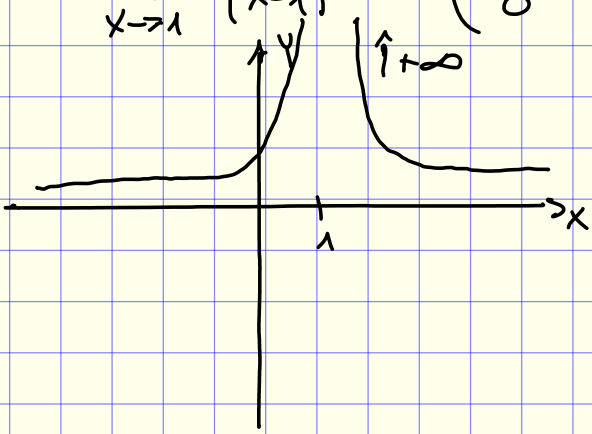
i) a) $\frac{3}{x-1}$, $x \to \infty$

$$\lim_{x\to\infty} \frac{3}{x-1} = 0^+ \qquad x \neq 0 \Rightarrow \lim_{x\to\infty} \frac{\overset{\to 0}{\frac{3}{x}}}{1-\underset{\to 0}{\frac{1}{x}}} = \frac{0}{1} = 0$$

b) $\frac{3}{\underbrace{(x-1)^2}_{\substack{\to 0 \\ >0}}}$, $x \to 1$

$$\lim_{x\to 1} \frac{3}{(x-1)^2} = \left(\frac{3}{0}\right) = +\infty$$

$\frac{1}{x^2}$



c) $\frac{2(x-2)}{x^2-4}$, $x \to 2$

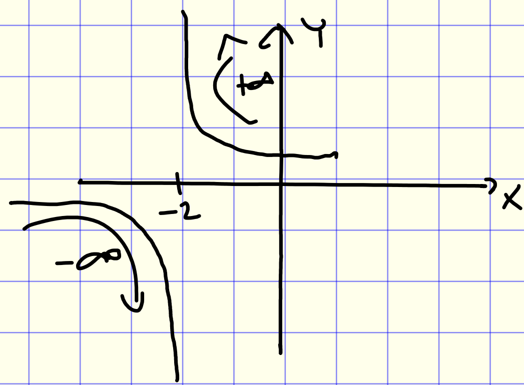
$$\lim_{x\to 2} \frac{2\cancel{(x-2)}}{\cancel{(x-2)}(x+2)} = \lim_{x\to 2} \frac{2}{x+2} = \frac{1}{2}$$

$x \neq 2$

d) $\frac{2(x-2)}{x^2-4}$, $x \to -2$

$$\lim_{x\to -2} \frac{2\cancel{(x-2)}}{\cancel{(x-2)}(x+2)} = \lim_{x\to -2^{\pm}} \frac{2}{\underset{\to 0}{x+2}} = \pm\infty$$

$x \neq 2$



e) $\lim_{x\to\infty} \frac{(x-2)^3 \,\}\to\infty}{x^2+1 \,\}\to\infty}$ → „aber schneller"

ausführlicher $\frac{(x-2)^3}{x^2+1} = \frac{x^3-6x^2+12x-8}{x^2+1}$

$$\lim_{x\to\infty} \frac{\cancel{x^3}\cdot\left(1-\underset{\to 0}{\frac{6}{x}}+\underset{\to 0}{\frac{12}{x^2}}-\underset{\to 0}{\frac{8}{x^3}}\right)}{\cancel{x^3}\cdot\left(\underset{\to 0}{\frac{1}{x}}+\underset{\to 0}{\frac{1}{x^3}}\right)} = \left(\lim_{x\to\infty} \frac{1}{0}\right) = \infty$$

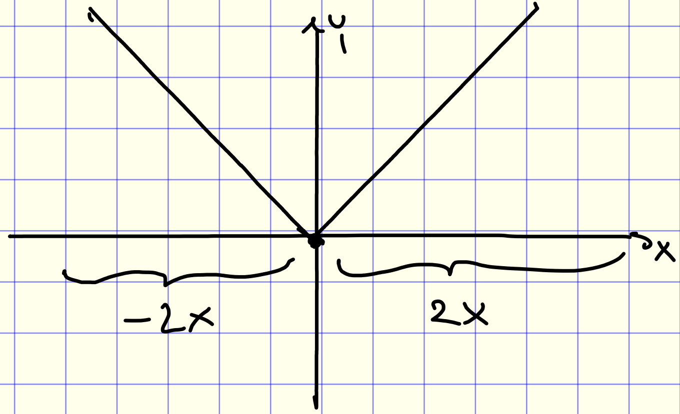
f) $\frac{x^2+1}{\underbrace{(x-2)^3}_{\hookrightarrow}}$, $x \to \infty$

„schneller gegen ∞"

$$\lim_{x\to\infty} \frac{x^2+1}{(x-2)^3} = 0$$

ii) a) $f(x) = 2 \cdot |x|$

$f(x) = |x|$ $x = -2$ $f(-2) = |-2| = 2$



$$= \begin{cases} -2x, & x<0 \\ 0, & x=0 \\ 2x, & x>0 \end{cases}$$

Stetigkeit am Punkt x_0

$\Rightarrow \lim_{x\to x_0^+} f(x) = \lim_{x\to x_0^-} f(x) = f(x)$

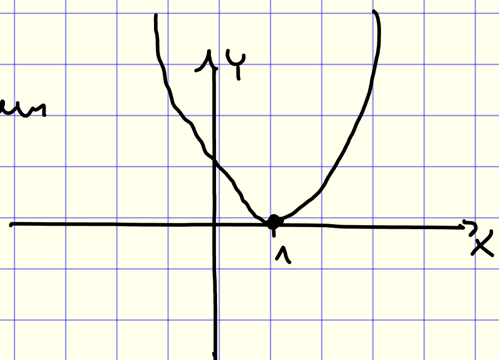
$\lim_{x\to 0^+} 2x = 0$

$=$ $f(0) = 0$

$\lim_{x\to 0^-} -2x = 0$

$\Rightarrow f(x)$ ist stetig :D

ε-δ-Kriterium



b) $f(x) = |(x-1)^2|$

$= (x-1)^2$

$\rightarrow$ stetig auf $\mathbb{R}$

$\lim_{x\to 1^+} (x-1)^2 = (1-1)^2 = 0^2 = 0$

$=$ $f(1) =$

$\lim_{x\to 1^-} (x-1)^2 = (1-1)^2 = 0^2 = 0$ $(1-1)^2 = 0$

c) $f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$\lim_{x\to 0^+} \frac{1}{x} = +\infty$

$\neq$ $f(0) = 0$

$\lim_{x\to 0^-} \frac{1}{x} = -\infty$

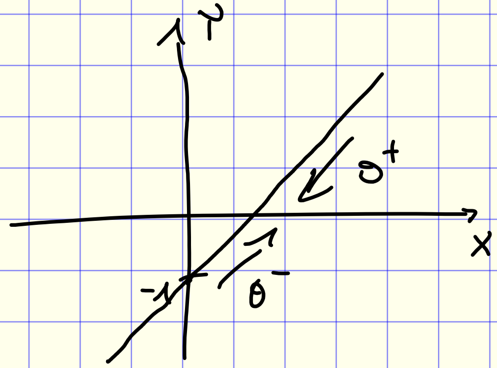
$\Rightarrow$ nicht stetig :(

d) $f(x) = \begin{cases} \frac{x^2-2x+1}{x-1}, & x \neq 1 \\ 0, & x = 1 \end{cases}$

$\frac{x^2-2x+1}{x-1} = \frac{(x-1)^2}{\cancel{x-1}} \quad (x \neq 1)$

$= x-1$

$= \begin{cases} x-1, & x \neq 1 \\ 0, & x = 1 \end{cases}$



$\lim_{\substack{x \to 1^\pm \\ x \neq 1}} f(x) = \lim_{x \to 1^\pm} x-1 = 0^\pm$

$=$

$x = 1 \Rightarrow f(1) = 0$

$\Rightarrow$ stetig :D

e) $f(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ \sqrt{1-x}, & x < 0 \end{cases}$

$\lim_{x \to 0^+} \sqrt{x} = 0^+ \quad \neq \quad \lim_{x \to 0^-} \sqrt{1-x} = 1^+$

$=$ $\quad\quad \neq$

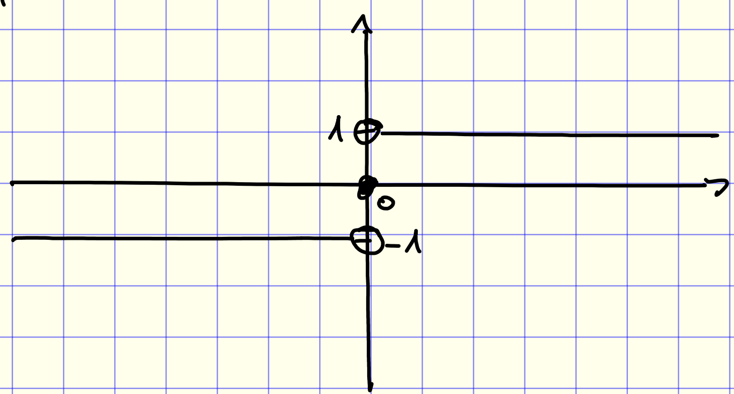
$f(0) = 0$

$\Rightarrow$ f nennt man in diesen Fall rechtsseitig stetig

Sei $x_0 > 0$:

$\lim_{x \to x_0^\pm} f(x) = \lim_{x \to x_0^\pm} \sqrt{x} = \sqrt{x_0}^\pm$

f) $\operatorname{sign}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$



$\lim_{x \to 0^+} \operatorname{sign}(x) = 1, \quad \lim_{x \to 0^-} \operatorname{sign}(x) = -1, \quad \operatorname{sign}(0) = 0$

$\rightarrow$ nicht stetig

:(

g) $f(x) = \begin{cases} \frac{x+2}{x^2+x-2}, & x \neq -2 \\ \frac{1}{3}, & x = 2 \end{cases}$

Nullstellen: $x^2 + x - 2 \overset{!}{=} 0 \Rightarrow x = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 2}$

$= -\frac{1}{2} \pm \sqrt{\frac{9}{4}} = -\frac{1}{2} \pm \frac{3}{2}$

$\Rightarrow x = 1 \vee x = -2$

$x^2 + x - 2 = (x-1)(x+2)$

$\frac{\cancel{x+2}}{(x-1)\cancel{(x+2)}} = \frac{1}{x-1}$

$x \neq 2$

$\Rightarrow f(x) = \begin{cases} \frac{1}{x-1}, & x \neq -2 \\ \frac{1}{3}, & x = -2 \end{cases}$

$\lim_{x \to -2^\pm} f(x) = \lim_{x \to 2^\pm} \frac{1}{x-1} = -\frac{1}{3}^\pm \qquad f(-2) = \frac{1}{3}$

$\to$ Unstetigkeit

$\Rightarrow$ Polstelle bei $x = 1$

h) $\underline{f(x) = \frac{2x^2+4x-6}{x^2+x-2}}$

$2x^2 + 4x - 6 = 0 \quad | :2 \Rightarrow x^2 + 2x - 3 = 0$

$\Rightarrow x = 1 \vee x = -3$

$x^2 + x - 2 = 0 \Rightarrow x = 1 \vee x = -2$

$\Rightarrow f(x) = \frac{2 \cdot \cancel{(x-1)}(x+3)}{\cancel{(x-1)}(x+2)}$

hebbare Definitionslücke

$x \neq 1$

Polstelle bei $x = -2$

$\hookrightarrow \lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{2(x+3)}{x+2} = \frac{2 \cdot 4}{3} = \frac{8}{3}$

$f(x) = \begin{cases} \frac{2(x+3)}{x+2}, & x \neq 1 \\ \frac{8}{3}, & x = 1 \end{cases}$

$\overset{!}{\hookrightarrow}$ damit Definitionslücke behoben

$f(x)$ stetig auf $\mathbb{R} \setminus \{-2\}$

iii) wird ausgelassen

Aufgabe 2:

i) a) $\frac{1}{x^2-25}$ Nennergrad $\geq$ Zählergrad

$x^2-25 \Rightarrow x^2=25 \mid \sqrt{\,} \Rightarrow x=\pm 5$

$x^2-25=(x+5)(x-5)$

$\frac{1}{(x+5)(x-5)} \overset{!}{=} \frac{A}{x+5}+\frac{B}{x-5} = \frac{-\frac{1}{10}}{x+5}+\frac{\frac{1}{10}}{x-5} = -\frac{1}{10(x+5)}+\frac{1}{10(x-5)}$

$= \frac{Ax-5A+Bx+5B}{(x+5)(x-5)} \quad \mid \cdot (x+5)(x-5)$

$\Rightarrow 1 = Ax-5A+Bx+5B$

$= x\cdot(A+B)+5B-5A$ Koeffizientenvergleich

$0\cdot x+1 = x\cdot(A+B)+5B-5A$

① $A+B=0 \Rightarrow B=-A$ in ② $\Rightarrow B=\frac{1}{10}$

② $5B-5A=1 \Rightarrow -5A-5A=1 \Rightarrow -10A=1 \mid :(-10) \Rightarrow A=-\frac{1}{10}$ in ①

b) $\frac{x+29}{x^2+3x-28}$ $x^2+3x-28=0 \Rightarrow x=-\frac{3}{2}\pm\sqrt{\frac{9}{4}+28}$

$=-\frac{3}{2}\pm\sqrt{\frac{9+112}{4}}$

$=-\frac{3}{2}\pm\sqrt{\frac{121}{4}}=-\frac{3}{2}\pm\frac{11}{2}$

$\Rightarrow x=4 \lor x=-7$

$x^2+3x-28=(x-4)(x+7)$

$\frac{x+29}{(x-4)(x+7)} = \frac{A}{(x-4)}+\frac{B}{(x+7)} = \frac{3}{x-4}-\frac{2}{x+7}$

$= \frac{Ax+7A+Bx-4B}{(x-4)(x+7)} \quad \mid \cdot (x-4)(x+7)$

$$x+29 = Ax+7A+Bx-4B$$

$$1\cdot x+29 = x\cdot(A+B)+7A-4B$$

① $A+B=1 \Rightarrow B=1-A$ in ② $\Rightarrow B=1-3=-2$

② $7A-4B=29 \Rightarrow 7A-4\cdot(1-A)=29 \Rightarrow 7A-4+4A=29 \quad |+4$

$\Rightarrow 11A=33 \quad |:11 \Rightarrow A=3$

c) $$\frac{3x^2+5x-2}{\underbrace{(x^2+1)}_{\text{doppelte Nullstelle}}(3x+1)} \overset{!}{=} \frac{Ax+B}{x^2+1}+\frac{C}{3x+1} = \frac{-\frac{5}{2}x+\frac{5}{2}}{x^2+1}+\frac{\frac{21}{2}}{3x+1}$$ in ①

$$=(Ax+B)(3x+1)+C\cdot(x^2+1)$$

$$=3Ax^2+Ax+3Bx+B+Cx^2+C$$

$$=x^2\cdot(3A+C)+x\cdot(A+3B)+B+C$$

① $3A+C=3 \quad |-3A \Rightarrow C=3-3A$ in ③ $\Rightarrow C=3-3\cdot 2=-3$

② $A+3B=5 \Rightarrow A+3\cdot(-5+3A)=5 \Rightarrow A-15+9A=5 \quad |+15$ (*)

③ $B+C=-2 \Rightarrow B+3-3A=-2 \quad |+3A \quad |-3 \Rightarrow B=-5+3A \Rightarrow$ in ②

(*) $10A=20 \quad |:10$ $\quad\Rightarrow B=-5+3\cdot 2$

$\Rightarrow A=2$ $\quad = 1$

ii) a) $$\frac{2x^3-x^2-10x+19}{x^2+x-6}$$ Nennergrad > Zählergrad

$$(2x^3-x^2-10x+19):(x^2+x-6)=2x-3+\frac{5x+1}{x^2+x-6}$$
$$-(2x^3+2x^2-12x)$$
$$-3x^2+2x+19$$
$$-(-3x^2-3x+18)$$
$$5x+1$$

$$\frac{5x+1}{x^2+x-6} \qquad x^2+x-6=0 \Rightarrow x=2 \vee x=-3$$

$$x^2+x-6 = (x-2)(x+3)$$

$$\frac{5x+1}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3} = \frac{11}{5\cdot(x-2)} + \frac{14}{5(x+3)}$$

$$= \frac{Ax+3A+Bx-2B}{(x-2)(x+3)} \quad | \cdot (x-2)(x+3)$$

$$\Rightarrow 5x+1 = Ax+3A+Bx-2B$$

$$= x\cdot(A+B)+3A-2B$$

① $A+B=5 \quad | -B \quad \Rightarrow A = 5-B$ in ② $\Rightarrow A = 5-\frac{14}{5} = \frac{11}{5}$

② $3A-2B=1 \qquad \Rightarrow 3(5-B)-2B=1$

$$\Rightarrow 15-3B-2B=1 \quad | +5B \quad | -1$$

$$\Rightarrow 5B = 14 \quad | :5 \quad \Rightarrow B = \frac{14}{5}$$

$$\Rightarrow \frac{2x^3-x^2-10x+19}{x^2+x-6} = 2x-3+\frac{11}{5(x-2)} + \frac{14}{5\cdot(x+3)}$$