Nachtrag Aufgabe 3 Blatt 8 13.10.2023

a) $1+3+5+7+9+\ldots = \sum_{k=0}^{\infty} (2k+1) = \sum_{k=1}^{\infty} (2k-1)$

b) $1+4+9+16+25+\ldots = \sum_{k=1}^{\infty} k^2$

c) $3+7+11+15+\ldots = \sum_{k=0}^{\infty} 3+4\cdot k = \sum_{k=1}^{\infty} 4k-1$

d) $\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\ldots = \sum_{k=0}^{\infty} \frac{1}{5^{k+1}}$

e) $2+3+2+3+2+3+\ldots = \sum_{k=1}^{\infty} 0{,}5\cdot(-1)^k + 2{,}5$

f) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8} = \sum_{k=0}^{\infty} \frac{1}{2^k}$

g) $1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots = \sum_{k=1}^{\infty} \frac{1}{k^2}$

b) $\sum_{k=1}^{n} k = \frac{n\cdot(n+1)}{2}$

$a_k = k^2$

$$\sum_{k=1}^{n} a_{k+1} - a_k = \cancel{a_2} - a_1 + \cancel{a_3} - \cancel{a_2} + \cancel{a_4} - \cancel{a_3} + a_5 - \cancel{a_4} + \ldots$$

$$= a_{n+1} - a_1$$

$$(k+1)^2 - k^2 = \cancel{k^2} + 2k+1 - \cancel{k^2} = 2k+1$$

$$\sum_{k=1}^{n} a_{k+1} - a_k = \sum_{k=1}^{n} (k+1)^2 - k^2 = \sum_{k=1}^{n} 2k+1$$

$$= a_{n+1} - a_1 = (n+1)^2 - 1^2 = n^2 + 2n$$

$$= n\cdot(n+2)$$

$$\Rightarrow \sum_{k=1}^{n} 2k+1 = n\cdot(n+2)$$

$$\Rightarrow \sum_{k=1}^{n} 2k + \sum_{k=1}^{n} 1 = n\cdot(n+2)$$

$$\Rightarrow 2\cdot\sum_{k=1}^{n} k + \underbrace{\sum_{k=1}^{n} 1}_{n\text{-mal } 1+1+1+1+\ldots+1 = n} = n\cdot(n+2) \quad |-n$$

$$2\cdot\sum_{k=1}^{n} k = n\cdot(n+2) - n \quad |:2$$

$$\Rightarrow \sum_{k=1}^{n} k = \frac{n\cdot(n+2)-n}{2} = \frac{n\cdot(n+1)}{2}$$

Aufgabe 1

$$DQ = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$DQ = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

i) a) $f(x) = x^2 + 1$

$$\lim_{\varepsilon \to 0} \frac{(x+\varepsilon)^2 + 1 - x^2 - 1}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{\cancel{x^2} + 2x\varepsilon + \varepsilon^2 \cancel{+1} \cancel{-x^2} \cancel{-1}}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{2x\varepsilon + \varepsilon^2}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{2x\cancel{\varepsilon}}{\cancel{\varepsilon}} + \frac{\varepsilon^{\cancel{2}}}{\cancel{\varepsilon}} = \lim_{\varepsilon \to 0} \underset{\substack{\downarrow \\ 2x}}{2x} + \underset{\substack{\downarrow \\ 0}}{\varepsilon} = 2x$$

b) $f(x) = x^3 - 5x^2 + 6x + 2$

$$f'(x) = \lim_{\varepsilon \to 0} \left[(x+\varepsilon)^3 - 5 \cdot (x+\varepsilon)^2 + 6 \cdot (x+\varepsilon) \cancel{+2} - x^3 + 5x^2 - 6x \cancel{-2} \right] \cdot \frac{1}{\varepsilon}$$

$$(x^2 + 2x\varepsilon + \varepsilon^2)(x+\varepsilon) = x^3 + 2x^2\varepsilon + x\varepsilon^2 + x^2\varepsilon + 2x\varepsilon^2 + \varepsilon^3$$

$$= x^3 + 3x^2\varepsilon + 3x\varepsilon^2 + \varepsilon^3$$

$$= \lim_{\varepsilon \to 0} \left[\cancel{x^3} + 3x^2\varepsilon + 3x\varepsilon^2 + \varepsilon^3 \cancel{-5x^2} - 10x\varepsilon - 5\varepsilon^2 \cancel{+6x} + 6\varepsilon \cancel{-x^3} \cancel{+5x^2} \cancel{-6x} \right] \frac{1}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \left[3x^2\cancel{\varepsilon} + 3x\varepsilon^{\cancel{2}} + \varepsilon^{\cancel{3}2} - 10x\cancel{\varepsilon} - 5\varepsilon^{\cancel{2}} + 6\varepsilon \right] \frac{1}{\cancel{\varepsilon}}$$

$$= \lim_{\varepsilon \to 0} \left[3x^2 + \underset{\substack{\downarrow \\ 0}}{3x\varepsilon} + \underset{\substack{\downarrow \\ 0}}{\varepsilon^2} - 10x \underset{\substack{\downarrow \\ 0}}{-5\varepsilon} + 6 \right] = 3x^2 - 10x + 6$$

c) $f(x) = \frac{1}{x}$

$$f'(x) = \lim_{\varepsilon \to 0} \left[\frac{1}{x+\varepsilon} - \frac{1}{x} \right] \cdot \frac{1}{\varepsilon} = \lim_{\varepsilon \to 0} \left[\frac{\cancel{x} - \cancel{x} - \cancel{\varepsilon}}{(x+\varepsilon)\cancel{x}} \right] \frac{1}{\cancel{\varepsilon}} = \lim_{\varepsilon \to 0} - \frac{1}{x^2 + \underset{\substack{\downarrow \\ 0}}{\varepsilon x}} = -\frac{1}{x^2}$$

d) $f(x) = \frac{x}{x-1}$

$$f'(x) = \lim_{\varepsilon \to 0} \left[\frac{x+\varepsilon}{x+\varepsilon-1} - \frac{x}{x-1} \right] \cdot \frac{1}{\varepsilon} = \lim_{\varepsilon \to 0} \left[\frac{(x+\varepsilon)(x-1) - x\cdot(x+\varepsilon-1)}{(x+\varepsilon-1)(x-1)} \right] \cdot \frac{1}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \left[\frac{\cancel{x^2} \cancel{-x} + \cancel{\varepsilon x} - \varepsilon \cancel{-x^2} - \cancel{\varepsilon x} \cancel{+x}}{x^2 + \varepsilon x - x - x - \varepsilon + 1} \right] \cdot \frac{1}{\cancel{\varepsilon}} = \lim_{\varepsilon \to 0} \left(- \frac{1}{x^2 - 2x + 1 + \underbrace{\cancel{\varepsilon\cdot(x-1)}}_{\to 0}} \right)$$

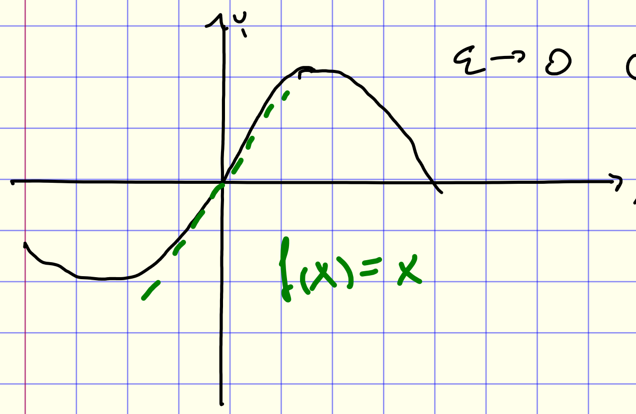
$$= - \frac{1}{x^2 - 2x + 1} = - \frac{1}{(x-1)^2}$$

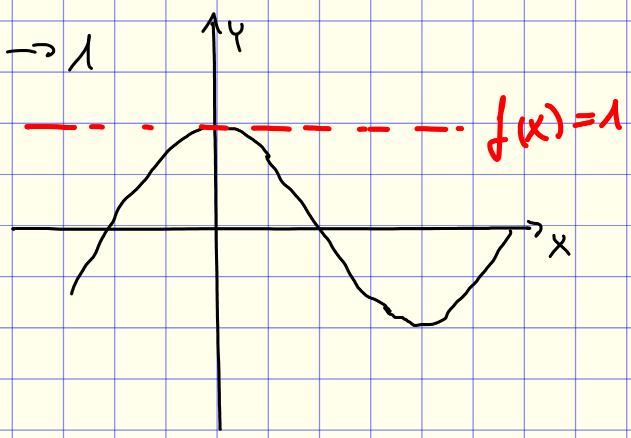
$$f(x) = \frac{x}{x-1} \qquad f'(x) \overset{Q.R.}{=} \frac{\cancel{x}-1 \cancel{-x}}{(x-1)^2} = - \frac{1}{(x-1)^2}$$

$$\begin{tikzcd} & \sin(x) \arrow[dr, "\frac{d}{dx}"] & \\ -\cos(x) \arrow[ur] & & \cos(x) \arrow[dl] \\ & -\sin(x) \arrow[ul] & \end{tikzcd}$$

e) $f(x) = \sin(x)$ $\qquad \varepsilon \to 0 \quad \sin(\varepsilon) \to \varepsilon \qquad$ „kleinwinkelnäherung"

$\varepsilon \to 0 \quad \cos(\varepsilon) \to 1$





$$f'(x) = \lim_{\varepsilon \to 0} \frac{\sin(x+\varepsilon) - \sin(x)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\sin(x) \underbrace{\cancel{\cos(\varepsilon)}}_{\to 1} + \underbrace{\cancel{\sin(\varepsilon)}}_{\varepsilon} \cos(x) - \sin(x) \right)$$

$$= \lim_{\varepsilon \to 0} \left(\cancel{\sin(x)} + \cancel{\varepsilon} \cos(x) - \cancel{\sin(x)} \right) \frac{1}{\cancel{\varepsilon}} = \cos(x)$$

f) $f(x) = \cos(x)$

$$f'(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\cos(x+\varepsilon) - \cos(x) \right) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\cos(x) \underbrace{\cancel{\cos(\varepsilon)}}_{\to 1} - \sin(x) \underbrace{\cancel{\sin(\varepsilon)}}_{\to \varepsilon} - \cos(x) \right)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\cancel{\varepsilon}} \left(\cancel{\cos(x)} - \cancel{\varepsilon} \sin(x) - \cancel{\cos(x)} \right)$$

$$= - \sin(x)$$

ii) a) $2x$

c) $\frac{1}{3\sqrt[3]{(x+4)^2}}$

b) $8x^3 - 9x^2 + 7$

d) $\frac{1}{2} \cdot \frac{1}{\sqrt{x+1}} \cdot (x^2+1) + (x+1)^{\frac{1}{2}} \cdot 2x$

e) $3x^2 + 4x - 4$

g) $\frac{x^2+5}{(x^2+5)^2}$

f) $5x^4 + \frac{4}{x^3}$

h) $(4 - 6\sin(x)\cos(x)) \cdot 5 \cdot (4x + 3\cos^2(x))^4$

i) $\frac{d}{dx}(6^x x^6 \sin(x)) = 6^x \ln(6) x^6 \sin(x) + 6^x\, 6 x^5 \sin(x) + 6^x x^6 \cos(x)$

$$\left(\frac{d}{dx}(g(x)\cdot f(x)\cdot h(x)\cdot \psi(x)) = g'(x)\cdot f(x)\cdot h(x)\cdot \psi(x) + g(x)\cdot f'(x)\cdot h(x)\cdot \psi(x) + g(x)\cdot f(x)\cdot h'(x)\cdot \psi(x) + g(x)\cdot f(x)\cdot h(x)\cdot \psi'(x)\right)$$

j) $\frac{d}{dx} \ln(\underbrace{\sqrt{e^x + x^4}}) = \frac{e^x + 4x^3}{2e^x + 2x^4}$

k) $\frac{d}{dx}\sqrt{\frac{2x-3}{4x^2+5}} = \frac{d}{dx}\left(\frac{2x-3}{4x^2+5}\right)^{\frac{1}{2}} = \frac{1}{2}\cdot\left(\frac{2x-3}{4x^2+5}\right)^{-\frac{1}{2}} \cdot \frac{d}{dx}\left(\frac{2x-3}{4x^2+5}\right)$

$\underbrace{\frac{d}{dx}\left(\frac{2x-3}{4x^2+5}\right)}_{QR} = \frac{2\cdot(4x^2+5) - 8x\cdot(2x-3)}{(4x^2+5)^2}$

$\left(\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)\cdot g(x) + f(x)\cdot g'(x)}{(g(x))^2}\right)$

$= \frac{-8x^2+24x+10}{(4x^2+5)^2}$

$\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

$\Rightarrow \frac{1}{2}\cdot\frac{(2x-3)^{-\frac{1}{2}}}{(4x^2+5)^{-\frac{1}{2}}} \cdot \left(\frac{-8x^2+24x+10}{(4x^2+5)^2}\right)$

$= \frac{(4x^2+5)^{-\frac{3}{2}}}{2\cdot\sqrt{2x-3}} \cdot (-8x^2 + 24x + 10)$

l) $\frac{d}{dx}(e^{2x+3} + 4x + 5)^6 = 6\cdot(e^{2x+3} + 4x + 5)^5 \cdot (e^{2x+3}\cdot 2 + 4)$

m) $\frac{d}{dx}(\sin^2(x)+1)(\ln(x)+2) = 2\sin(x)\cos(x)\cdot(\ln(x)+2) + (\sin^2(x)+1)\cdot\frac{1}{x}$

$$h)\quad e^{\frac{x^2+3}{x^2+1}} = \exp\left(\frac{x^2+3}{x^2+1}\right)$$

$$= \exp\left(\frac{x^2+3}{x^2+1}\right) \cdot \underbrace{\frac{d}{dx}\left(\frac{x^2+3}{x^2+1}\right)}_{QR:\ = \frac{2x(x^2+1)-2x(x^2+3)}{(x^2+1)^2}} = -\frac{4x}{(x^2+1)^2}$$

$$\Rightarrow \frac{d}{dx} e^{\frac{x^2+3}{x^2+1}} = e^{\frac{x^2+3}{x^2+1}} \cdot \left(-\frac{4x}{(x^2+1)^2}\right)$$

$$e)\quad f(x) = \ln\left(\sqrt{x^3 e^{2x} \ln(x)}\right) = \frac{1}{2} \cdot \frac{1}{x^3 e^{2x} \ln(x)} \cdot \left(3x^2 \cdot e^{2x} \ln(x) + x^3\, 2e^{2x} \ln(x) + x^2 e^{2x}\right)$$

$$\frac{1}{\sqrt{x^3 e^{2x} \ln(x)}} \cdot \frac{d}{dx} \underbrace{\left(\sqrt{x^3 e^{2x} \ln(x)}\right)}_{\frac{1}{2\sqrt{x^3 e^{2x} \ln(x)}} \cdot \frac{d}{dx} \underbrace{\left(x^3 e^{2x} \ln(x)\right)}_{3x^2 e^{2x} \ln(x) + x^3 \cdot 2e^{2x} \ln(x) + x^{\cancel{3}\,2} e^{2x} \frac{1}{\cancel{x}}}}$$

$$p)\quad \frac{d}{dx}\left[\sin((3-x^2)^2) + \cos((3-x^2)^2) + \underbrace{\sin^2(3-x^2) + \cos^2(3-x^2)}_{1}\right]$$

$$= \cos((3-x^2)^2) \cdot [2 \cdot (3-x^2) - 2x]$$

$$- \sin((3-x^2)^2) \cdot [2 \cdot (3-x^2) - 2x]$$

$$= -4x(3-x^2)\left[\underbrace{\cos((3-x^2)^2) - \sin((3-x^2)^2)}_{\sqrt{2} \cdot \sin\left(\frac{\pi}{4} - (3-x^2)^2\right)}\right]$$

$$q)\quad \frac{d}{dx}\left(\frac{x \cdot \sin(ax+b)}{x^2+3}\right)$$

$$\overset{QR,\,PR}{=} \frac{(\sin(ax+b) + ax\cos(ax+b))(x^2+3) - 2x\, x \cdot \sin(ax+b)}{(x^2+3)^2}$$

$$= \frac{\sin(ax+b)(3-x^2) + \cos(ax+b)(ax^3+3ax)}{(x^2+3)^2}$$

c) $\left(\frac{x^2+1}{x^2+3}\right)^{\sin(2x)}$ Erinnerung: b^x

$$= e^{\ln\left(\left(\frac{x^2+1}{x^2+3}\right)^{\sin(2x)}\right)} = e^{\sin(2x)\cdot\ln\left(\frac{x^2+1}{x^2+3}\right)}$$

$$= e^{\sin(2x)\cdot(\ln(x^2+1)-\ln(x^2+3))} = e^{\sin(2x)\ln(x^2+1)}\cdot e^{-\sin(2x)\ln(x^2+3)}$$

$$\Rightarrow \frac{d}{dx}\left(\left(\frac{x^2+1}{x^2+3}\right)^{\sin(2x)}\right)$$

→ innere Funktion

$$= \frac{d}{dx}\left(e^{\sin(2x)\ln(x^2+1)-\sin(2x)\ln(x^2+3)}\right)$$

$$= e^{\sin(2x)\ln(x^2+1)-\sin(2x)\ln(x^2+3)}\cdot\frac{d}{dx}\left(\sin(2x)\ln(x^2+1)-\sin(2x)\ln(x^2+3)\right)$$

$$= e^{\sin(2x)\ln(x^2+1)-\sin(2x)\ln(x^2+3)}\cdot\Big[2\cos(2x)\ln(x^2+1)+\sin(2x)\frac{1}{x^2+1}\cdot 2x - 2\sin(2x)\ln(x^2+3)-\sin(2x)\frac{1}{x^2+3}\cdot 2x\Big]$$

$$= \left(\frac{x^2+1}{x^2+3}\right)^{\sin(2x)}\cdot\left[2\cos(2x)\cdot\ln\left(\frac{x^2+1}{x^2+3}\right)+\sin(2x)\cdot\frac{2x}{(x^2+1)(x^2+3)}\right]$$

Aufgabe 2:

„für alle" „für einen einzigen"

i) a) $f(x) = 8-7x$ $f'(x) = -7 < 0 \quad \forall x$ $\exists$

→ streng monoton fallend

$f'(x) \neq 0 \quad \forall x \Rightarrow$ keine lokalen Extrema

$W = \mathbb{R}$