Vorrechnen 10 — 16.10.2023

Nachtrag

ii) $f(x) = x^4 - 8x^2 + 9$

$f'(x) = 4x^3 - 16x \overset{!}{=} 0 \Rightarrow x \cdot (4x^2 - 16) = 0 \Rightarrow x \cdot (x^2 - 4) = 0$

$\Rightarrow x = 0 \lor x^2 - 4 = 0 \quad | +4 \quad | \sqrt{\ }$

$\Rightarrow x = 2 \lor x = -2$

$f'(x)$ Nullst.: $x \in \{-2, 0, 2\}$

$f''(x) = 12x^2 - 16$

$f''(-2) = 12 \cdot (-2)^2 - 16 = 32 > 0$ TP

$f''(0) = 12 \cdot 0 - 16 = -16 < 0$ HP

$f''(2) = 12 \cdot 2^2 - 16 = 32 > 0$ TP

Wendepunkte

$f''(x) \overset{!}{=} 0$

$\Rightarrow 12x^2 - 16 = 0 \quad | +16 \quad | :12$

$\Rightarrow x^2 = \frac{16}{12} = \frac{4}{3} \quad | \sqrt{\ } \quad \Rightarrow x = -\frac{2}{\sqrt{3}} \lor x = \frac{2}{\sqrt{3}}$

Nullstellen $f''(x)$ / Wendestellen $f(x)$: $x \in \{-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\}$

VZT: (für Krümmungsverhalten)

x	-2	$-\frac{2}{\sqrt{3}}$	0	$\frac{2}{\sqrt{3}}$	2
$f''(x)$	32	0	-16	0	32
	L.K.		R.K.		L.K

L.K. = Linkskrümmung

$f'(3) = 4 \cdot 3^3 - 16 \cdot 3$

$= 60 > 0$

VZT:

x		-2		0		2	3
$f'(x)$	↘	0	↗	0	↘	0	↗
		TP		HP		TP	

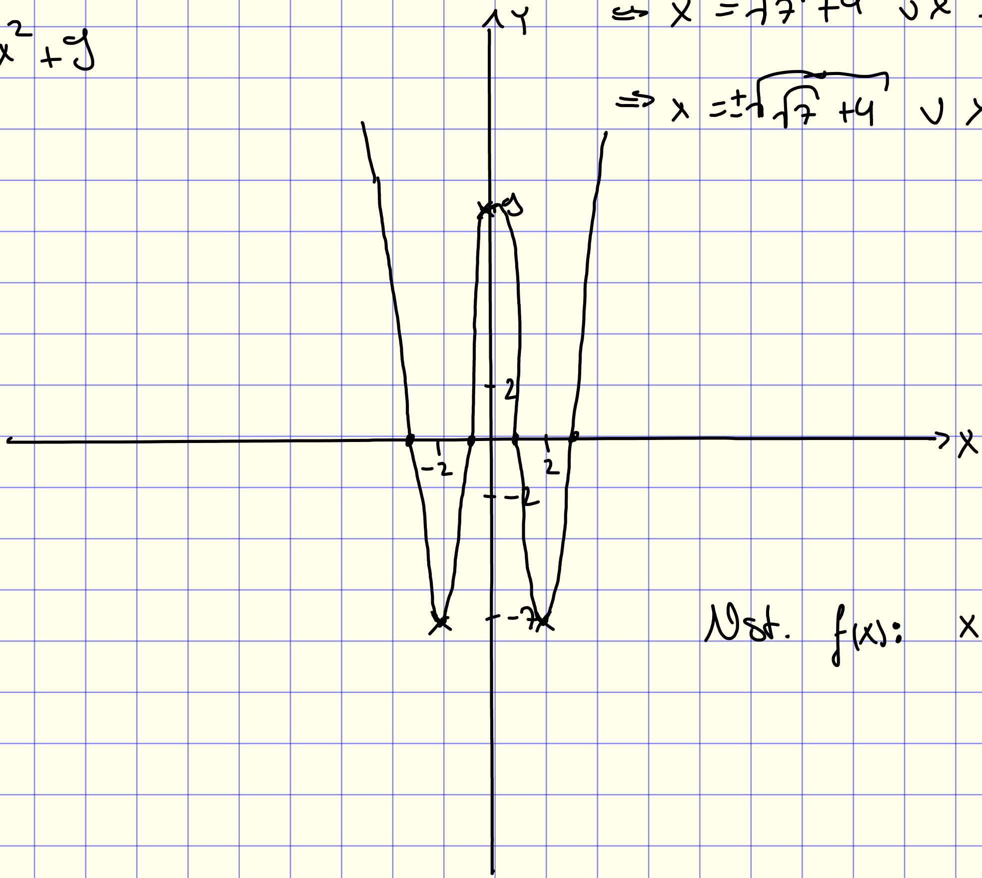
$f(x) = x^4 - 8x^2 + 9 \quad x^2 = y \quad \Rightarrow f(y) \overset{!}{=} 0 \Rightarrow y^2 - 8 \cdot y + 9 = 0$

$= y^2 - 8 \cdot y + 9$

$\Rightarrow (y - 4)^2 + 9 - 16 = 0 \quad | +7 \quad | \sqrt{\ }$

$\Rightarrow y - 4 = \pm\sqrt{7} \quad | +4 \quad \Rightarrow y = \sqrt{7} + 4 \lor y = -\sqrt{7} + 4$

$f(x) = x^4 - 8x^2 + 9$



$\Rightarrow x^2 = \sqrt{7} + 4 \vee x^2 = -\sqrt{7} + 4 \mid \sqrt{\ }$

$\Rightarrow x = \pm\sqrt{\sqrt{7} + 4} \vee x = \pm\sqrt{-\sqrt{7} + 4}$

Nst. $f(x)$: $x \in \{2{,}57,\ -2{,}57,\ 1{,}16,\ -1{,}16\}$

Blatt 10:

1) a) $\frac{d}{dx}\left(\sqrt{a} - \sqrt{bx+c}\right)^2 \overset{KR}{=} \cancel{2}\cdot(\underbrace{\sqrt{a}}_{\text{const.}} - \sqrt{bx+c}) \cdot \left(-\frac{\frac{1}{\cancel{2}}}{\sqrt{bx+c}}\right) \cdot b$

$= -\frac{\sqrt{a} - \sqrt{bx+c}}{\sqrt{bx+c}} \cdot b = -\frac{\sqrt{a} \cdot b}{\sqrt{bx+c}} + b$

b) $\frac{d}{dx} \frac{(x+1)\sin(x+1)}{(x-1)^2} \overset{QR}{=}$

$\Rightarrow \frac{\left\{\frac{d}{dx}\left[(x+1)\sin(x+1)\right]\right\} \cdot (x-1)^2 - \left\{\frac{d}{dx}(x-1)^2\right\} \cdot (x+1)\sin(x+1)}{(x-1)^4}$

$\Rightarrow \frac{\left[\sin(x+1) + (x+1)\cos(x+1)\right] \cdot (x-1)^2 - 2(x+1)(x-1)\sin(x+1)}{(x-1)^4}$

$= \frac{\sin(x+1) + (x+1)\cos(x+1)}{(x-1)^2} - \frac{2 \cdot (x+1)\sin(x+1)}{(x-1)^3}$

c) $\frac{d}{dx} \ln\left(\sqrt{x^2 + \sin^2(x)}\right) \overset{KR\times 3}{=} \frac{1}{\sqrt{x^2+\sin^2(x)}} \cdot \frac{1}{\cancel{2} \cdot \sqrt{x^2+\sin^2(x)}} \cdot \overbrace{(\cancel{2}x + \cancel{2}\sin(x)\cos(x))}^{\frac{dy}{dx}}$

$= \frac{1}{|x^2 + \sin^2(x)|} \cdot (x + \sin(x)\cos(x)) = \frac{x + \sin(x)\cos(x)}{x^2 + \sin^2(x)}$

$(x^2 + \sin^2(x) \geq 0)$

d) $f(x) = x^x$

$$= e^{\ln(x^x)} = e^{x\cdot\ln(x)} \overset{KR,\,PR}{=} e^{x\ln(x)} \cdot \left(\ln(x) + \frac{1}{\cancel{x}}\cdot \cancel{x}\right)$$

$$= x^x \cdot (\ln(x) + 1)$$

$$= e^{x\ln(x)} \cdot (\ln(x) + 1)$$

<u>Aufgabe 2:</u>

i) a) $\int x^2 dx = \frac{1}{3}x^3 + C$

b) $\int x^n dx$, $n \neq -1$ $= \frac{1}{n+1} x^{n+1} + C$

c) $\int \frac{1}{x} dx = \ln(|x|) + C$

d) $\int \frac{1}{x+23} dx = \ln(|x+23|) + C$

„KR rückwärts"

$u = x+23$

$\frac{du}{dx} = 1 \quad | \cdot dx \quad | :1$

$\Rightarrow dx = du$

$\int \frac{1}{x+23} dx$

$\Rightarrow \int \frac{1}{u} du$

$\Rightarrow \ln(|u|) + C \overset{RS}{=} \ln(|x+23|) + C$

e) $\int \frac{-5}{x^6} dx = -5 \int \frac{1}{x^6} dx$

$$= -5 \int x^{-6} dx = \cancel{-5} \cdot \left(-\frac{1}{\cancel{5}}\right) \cdot x^{-5} + C$$

$$= x^{-5} + C$$

f) $\int e^{-5x} dx = -\frac{1}{5} e^{-5x} + C$

g) $\int \sin(x)\cos(x)\, dx = \frac{1}{2}\sin^2(x) + C$

1. Methode: KR rückwärts

$f(z) = z^2 \quad g(x) = \sin(x) = z$

$f(g(x)) = \sin^2(x)$

$\frac{d}{dx} f(g(x)) = 2\sin(x)\cos(x)$

2. Methode: partielle Integration

$$\int \underset{\uparrow\, f(x)}{\sin(x)} \underset{\uparrow\, g(x)}{\cos(x)}\, dx = f(x)\,G(x) - \int f'(x)\,G(x)\,dx$$

$$= \sin(x)\sin(x) - \underbrace{\int \cos(x)\sin(x)\,dx}_{\int \sin(x)\cos(x)\,dx} \quad \Big| + \int \sin(x)\cos(x)\,dx$$

$$\Rightarrow 2\cdot \int \sin(x)\cos(x)\,dx = \sin^2(x) \quad |:2 \qquad C_1 + C_2 = C$$

$$\Rightarrow \int \sin(x)\cos(x)\,dx = \frac{1}{2}\sin^2(x) + C$$

3. Methode: Substitution

$$\int \sin(x)\,\underline{\cos(x)\,dx} = \qquad u = \sin(x)$$

$$\Rightarrow \frac{du}{dx} = \cos(x) \quad |\cdot dx \quad |:\cos(x)$$

$$\Rightarrow \int u\, \cancel{\cos(x)} \frac{1}{\cancel{\cos(x)}}\, du \qquad \Rightarrow dx = \frac{1}{\cos(x)}\, du$$

$$\Rightarrow \int u\, du = \frac{1}{2}u^2 + C \qquad (du = \cos(x)\,dx)$$

$$\overset{RS}{=} \frac{1}{2}\sin^2(x) + C$$

h) $I = \int (x\cos(x) + \sin(x))\,dx = \int x\cos(x)\,dx + \underbrace{\int \sin(x)\,dx}_{-\cos(x) + C_1}$

$$\int \underset{\uparrow\, f}{x}\,\underset{\uparrow\, g}{\cos(x)}\,dx \overset{P.I.}{=} x\cdot\sin(x) - \underbrace{\int 1\cdot\sin(x)\,dx}_{-\cos(x)+C} = x\cdot\sin(x) + \cos(x) + C_2$$

$$I = x\sin(x) + \cancel{\cos(x)} + C_2 - \cancel{\cos(x)} + C_1$$

$$= x\sin(x) + \overbrace{C_1 + C_2}^{C} = x\sin(x) + C$$

i) $\int \frac{x}{1+x^2}\,dx = \qquad |\underbrace{1+x^2}| = 1+x^2$

Substitution $u = 1+x^2$

$$\frac{du}{dx} = 2x \quad |\cdot dx \quad |:2x \Rightarrow dx = \frac{1}{2x}\,du$$

$$\Rightarrow \int \frac{\cancel{x}}{u} \frac{1}{2\cancel{x}} du = \frac{1}{2} \int \frac{1}{u} du = \frac{1}{2} \ln(|u|) + C$$

$$\overset{RS}{=} \frac{1}{2} \ln(|1+x^2|) + C \quad \text{da } 1+x^2 \geq 0 \ \forall x$$

$$= \frac{1}{2} \ln(1+x^2) + C$$

$$= \ln(\sqrt{1+x^2}) + C$$

3. Methode

$$\int \frac{x}{1+x^2} dx = \int \frac{x}{(x-i)(x+i)} dx$$

i = imaginary, imaginär

$$x^2+1=0 \quad |-1 \Rightarrow x^2=-1 \quad |\sqrt{\ } \Rightarrow x = \pm \underbrace{\sqrt{-1}}_{i}$$

$i \in \mathbb{C}$

$x^2+1 \neq 0 \ \forall x$

$$x^2+1 = (x-i)(x+i)$$

$$\frac{x}{(x-i)(x+i)} = \frac{A}{x-i} + \frac{B}{x+i} \Rightarrow \frac{(A+B)\cdot x + (A-B)\cdot i}{\cancel{(x-i)(x+i)}} = \frac{x}{\cancel{(x+i)(x-i)}}$$

$$= \frac{1}{2\cdot(x-i)} + \frac{1}{2\cdot(x+i)} \quad (1)\ A+B=1 \quad (2)\ A-B=0 \quad |+B \Rightarrow A=B \Rightarrow B=\frac{1}{2}$$

$$\Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$$

$$\Rightarrow \int \frac{1}{2\cdot(x-i)} dx + \int \frac{1}{2\cdot(x+i)} dx$$

$$\Rightarrow \frac{1}{2}\ln(x-i) + \frac{1}{2}\ln(x+i) = \frac{1}{2}\ln(\underbrace{(x-i)(x+i)}_{x^2-i^2 = x^2 - (\underbrace{\sqrt{-1}}_{-1})^2}) = \frac{1}{2}\ln(x^2+1) + C$$

$$= x^2+1$$

j) $\int (6\sin(4-3x) + 3e^{-2x} + 5)\, dx$

$$= 6\cdot\int \sin(4-3x)\, dx + 3\cdot\int e^{-2x} dx + \int 5\, dx$$

$$= 6\cos(4-3x)\cdot\frac{1}{3} + \left(-\frac{3}{2}\right)e^{-2x} + 5x + C$$

$$= 2\cdot\cos(4-3x) - \frac{3}{2}e^{-2x} + 5x + C$$

k) $\int \frac{x^3+2x^2-5x-6}{x^2-x-2}\,dx = I$ Grad Zähler > Grad Nenner $\rightarrow$ PD

$$(x^3+2x^2-5x-6):(x^2-x-2) = x+3$$
$$\underline{-(x^3-x^2-2x)}$$
$$3x^2-3x-6$$
$$\underline{-(3x^2-3x-6)}$$
$$0$$

$$\Rightarrow I = \int x+3\,dx = \int x\,dx + \int 3\,dx = \frac{1}{2}x^2 + 3x + c$$

l) $\int \frac{x+29}{x^2+3x-28}\,dx$ Grad Zähler < Grad Nenner $\Rightarrow$ PBZ

Nullstellen $\rightarrow$ Linearfaktoren

$$x^2+3x-28 \overset{!}{=} 0 \Rightarrow x^2+3x+\frac{9}{4}-\frac{9}{4}-28=0$$
$$\Rightarrow \left(x+\frac{3}{2}\right)^2 - \frac{121}{4} = 0 \quad \Big| + \frac{121}{4} \quad \Big|\sqrt{}$$
$$\Rightarrow x+\frac{3}{2} = \frac{11}{2} \vee x+\frac{3}{2} = -\frac{11}{2} \quad \Big| - \frac{3}{2}$$
$$\Rightarrow x = -\frac{14}{2} = -7 \vee x = \frac{8}{2} = 4$$

$$x^2+3x-28 = (x+7)(x-4)$$
$$\frac{x+29}{(x+7)(x-4)} = \frac{A}{x+7} + \frac{B}{x-4} = -\frac{2}{x+7} + \frac{3}{x-4}$$
$$= \frac{(x-4)A + (x+7)B}{(x+7)(x-4)} \quad \Big| \cdot (x+7)(x-4)$$

$$\Rightarrow x+29 = Ax - 4A + Bx + 7B$$
$$= x\cdot(A+B) + 7B - 4A$$

① $A+B = 1 \quad | -B \Rightarrow A = 1-B = -2$

② $7B - 4A = 29$

$\Rightarrow 7\cdot B - 4\cdot(1-B) = 29$

$\Rightarrow 7B - 4 + 4B = 29 \quad | +4 \quad | :11$

$\Rightarrow B = 3$

$$\int \frac{x+29}{x^2+3x-28}\,dx = \int \frac{-2}{x+7} + \int \frac{3}{x-4}\,dx$$

$$= -2\ln(|x+7|) + 3\cdot\ln(|x-4|) + C$$

$$= \ln\left(\frac{|x-4|^3}{|x+7|^2}\right) + C$$

ii) a) $f(x) = 4x - x^2$ Nullstellen: $f(x) \overset{!}{=} 0 \Rightarrow 4x - x^2 = 0$

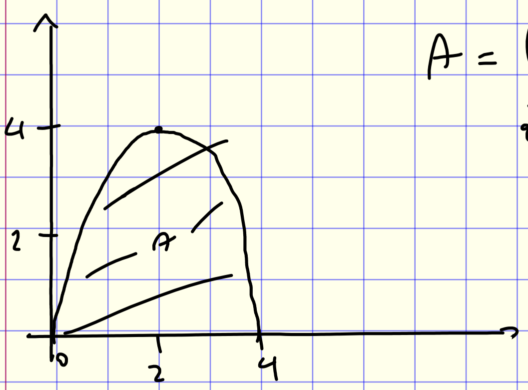
$\Rightarrow x \cdot (4 - x) = 0$

$\Rightarrow$ Nullst.: $x \in \{0, 4\}$

$f'(x) = 4 - 2x$

$f'(x) \overset{!}{=} 0 \Rightarrow 4 - 2x = 0 \quad |+2x|:2$

$\Rightarrow x = 2 \quad f(2) = 4 \cdot 2 - 2^2 = 4$



$$A = \int_0^4 4x - x^2\,dx = \left[\tfrac{1}{2}\cdot 4 \cdot x^2 - \tfrac{1}{3}x^3\right]_0^4$$

$$= \left[2x^2 - \tfrac{1}{3}x^3\right] = 32 - \frac{64}{3}$$

$$= 10.\overline{6} = \frac{32}{3}$$