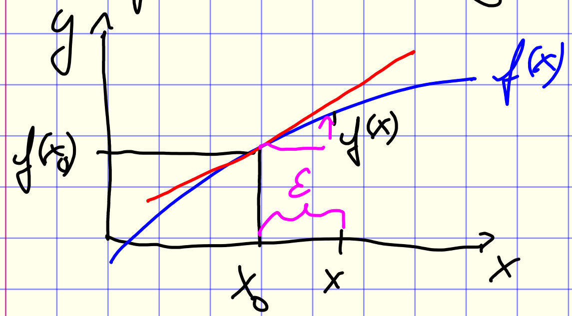
Ausblick Differential & Integralrechnung — 17.10.25

Differentialrechnung

a) Taylorentwicklung   Ziel: $f(x)$ durch Polynom darstellen $p(x)$



Idee: durch Polynome annähern sodass

$$p(x_0) = f(x_0)$$
$$p'(x_0) = f'(x_0)$$
$$p''(x_0) = f''(x_0)$$
$$\vdots$$

$$f(x) = \underbrace{\frac{f(x_0)}{0!}\cdot\overbrace{(x-x_0)^0}^{(x-x_0)^0=1} + \frac{f'(x_0)}{1!}\cdot(x-x_0)^1 + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots}_{p(x)}$$

$0! = 1$

Check $p(x=x_0) = f(x_0)$

$$p'(x)\Big|_{x=x_0} = 1\cdot f'(x_0)$$

$$p''(x)\Big|_{x=x_0} = \frac{f''(x_0)}{2!}\cdot 2 \quad \text{da } \frac{d^2(x-x_0)^2}{dx^2} = 2$$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{d^n f(x)}{dx^n}\Big|_{x=x_0} \cdot \frac{(x-x_0)^n}{n!}}$$

Bsp. oft für $x_0 = 0$ gewählt.

$$\sin(x) = 0 + \frac{d}{dx}\sin(x)\Big|_{x=0}\cdot(x-0)^1 + \frac{d^2\sin(x)}{dx^2}\Big|_{x=0}\frac{(x-0)^2}{2!} + \dots$$

$$= 0 + 1\cdot x + 0 - \frac{1}{3!}\cdot x^3 + \dots = x - \frac{1}{6}x^3 + \dots$$

analog $\cos(x) = 1 - \frac{x^2}{2!} + \dots$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \; ; \quad \text{no2.3}$$

$$e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots = 2{,}71828\,18\dots$$

Taschenrechner kann nur $+, -, \cdot, :$

aber mit Taylorreihe jede Fkt ausrechenbar.

siehe Wikipedia Taylorreihe

oft $x = x_0 + \varepsilon$; $\varepsilon \gtrless 0$; klein

$$f(x_0+\varepsilon) = \sum \left.\frac{d^n f(x)}{dx^n}\right|_{x_0} \cdot \frac{\varepsilon^n}{n!}$$ Ein Zeichen

5) Differentialgleichungen

$\hat{=}$ Gleichungen wo Funktion und Ableitung vorkommen
gesucht Funktion die die Gleichung erfüllt

z.B. $\boxed{y'(x) = 2y(x)}$, in Worten „Änderung an x proportional zu Wert der Funktion an x"

raten: $y(x) = e^{2x}$, check $2e^{2x} = 2 \cdot e^{2x}$ ✓ hat gepasst.

$$y'(x) = \frac{dy}{dx} = 2y(x)$$

$$\frac{1}{2}\frac{dy}{y(x)} = dx \quad \Big| \int$$

$$\frac{1}{2}\int \frac{dy}{y} = \int dx = \int 1 \cdot dx = x + C$$

$$\frac{1}{2}\ln y = x + C \qquad \text{ges } y(x)$$

$$\ln y = 2x + 2C \quad \Big| e^x$$

Konstante

$$y(x) = e^{2x+2C} = e^{2x} \cdot e^{2C} \; ; \; e^{2C} = C_1$$

$y(x) = C_1 \cdot e^{2x}$, daher auch z.B. $y(x) = 5e^{2x}$ eine Lsg von oben

$$(5e^{2x})' = 10e^{2x} = 2 \cdot 5e^{2x} \; ✓$$

Differentialgleichungen in Naturwissenschaften oft mit $x = t \hat{=}$ Zeit

$\Rightarrow$ dynamisches System

z.B. i) $F(t) = m\, a(t) = m \cdot \frac{d\, v(t)}{dt} = m \frac{d^2 x(t)}{dt^2}$

Ableitung nach Zeit

$= m \cdot \dot{v}(t) = m\, \ddot{x}(t)$

z.B. $a(t) = \text{const.}$

$$\cancel{m} \cdot a = \cancel{m} \cdot \frac{d}{dt} v(t)$$

$$a \cdot dt = d\,v(t) \quad \Big| \int$$

$$a \cdot t + C = v(t)$$

, analog wenn z.B. $a(t) = \sin(t)$ dann $v(t)$ und $x(t)$ ausrechenbar

$$at + C = \frac{d}{dt} x(t) \quad \Big| \int$$

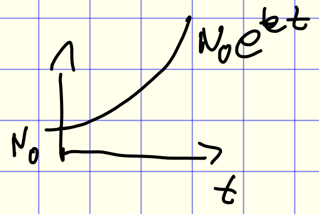
ii) $\dot{N}(t) = -\lambda N(t)$: $N(t) \hat{=}$ Anzahl Atomkerne zum Zeitpunkt t

$$dN(t) = -\lambda N(t)\, dt$$

$$\frac{1}{N(t)} dN(t) = -\lambda\, dt \quad \Big| \int \Rightarrow \boxed{N(t) = N(t_0) \cdot e^{-\lambda t}}$$

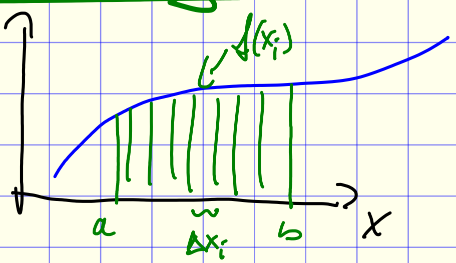
iii) $\dot{N}(t) = k\, N(t)$; $k > 0$

$N(t) = N(0) e^{kt}$ $\hat{=}$ Bakterienwachstum



nun **Integralrechnung**

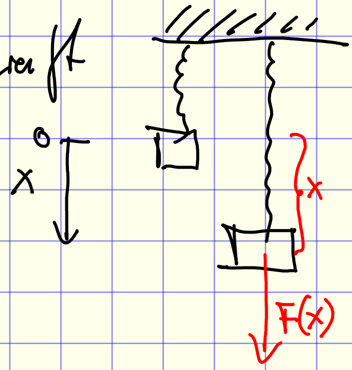
Anschauung

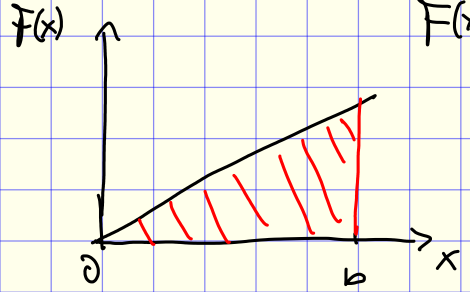


$A = \sum f(x_i) \cdot \Delta x_i$

$\Rightarrow$ $\Delta x \to dx$, $\sum \to \int$ $\Rightarrow A = \int_a^b f(x)\, dx$

Bem: muss nicht immer eine Fläche sein, Flächen, wenn x & $f(x)$ eine Länge anderes Bsp. i) Federkraft





$F(x) = k \cdot x$

$$\int_0^b F(x)\,dx = \frac{k}{2}x^2\Big|_0^b = \frac{k}{2}b^2$$

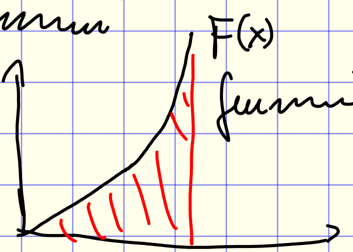
in Einheiten

$[x] = m$ ← Meter

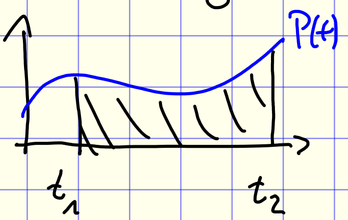
$[F] = N$ $\quad \left[\int F(x)\,dx\right] = Nm = J$

≙ potentielle Energie in Feder ≙ Kraft · Weg

auch ausrechenbar für Gummi

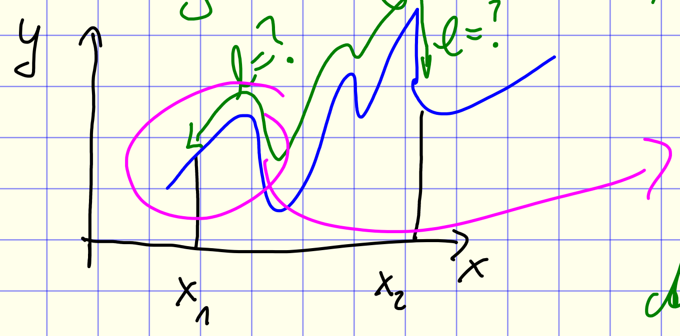


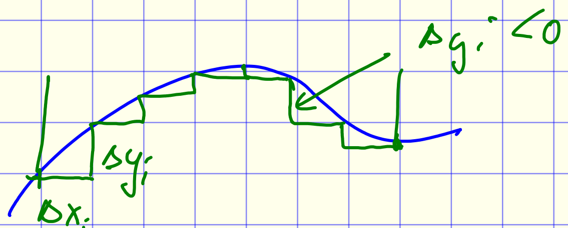
ii) Leistung Motor $P(t)$ in Watt



$$\left[\int_{t_1}^{t_2} P(t)\,dt\right] = Ws = J \text{ - Joule}$$

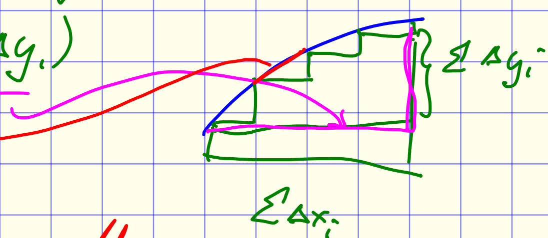
Anwendung: Länge Kurve, beliebig





Annahme Länge = Σ aller Δx_i & Δy_i

$l = \Sigma(\Delta x_i + \Delta y_i)$



Besser entlang Diagonalen:

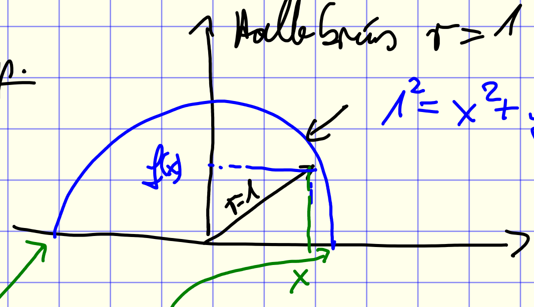
$$l = \Sigma \Delta l_i = \Sigma \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$$, wenn alle Δx_i gleich lang ≙ Δx

$$l = \Delta x \,\Sigma \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x}\right)^2}$$ wenn $\Delta \to d$, $\Sigma \to \int$ $\quad \frac{\Delta y(x)}{\Delta x} \Rightarrow \frac{dy(x)}{dx} = \frac{df}{dx}$

⇒ allgemeine Formel Bogenlänge

$$l = \int_a^b dx \sqrt{1+\left(\frac{df}{dx}\right)^2} \quad ; \quad \text{Bem.} \left(\frac{df}{dx}\right)^2 \neq \frac{d^2 f}{dx^2}$$

Bsp. Halbkreis $r=1$

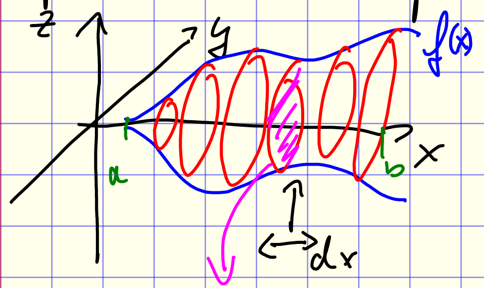


$$1^2 = x^2 + f(x)^2 \quad ; \quad f(x) = \sqrt{1-x^2}$$

$$\frac{df}{dx} = \frac{d}{dx}\left(1-x^2\right)^{\frac{1}{2}} = \frac{1}{2}\,\frac{-2x}{\sqrt{1-x^2}} = \frac{-x}{\sqrt{1-x^2}}$$

$$l = \int_{-1}^{1} dx \sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2} = \int_{-1}^{1} dx \frac{1}{\sqrt{1-x^2}} = \arcsin(x)\Big|_{-1}^{1} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \underline{\underline{\pi}}$$

Bsp: Rotationskörper

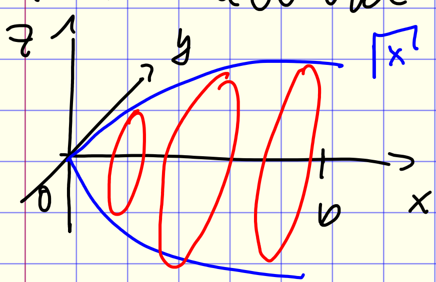


$$V = \int dV$$

$$V = \int_a^b dx \cdot \pi \cdot f(x)^2$$

$$dV = A(x) \cdot dx = \pi \cdot f(x)^2 \cdot dx$$

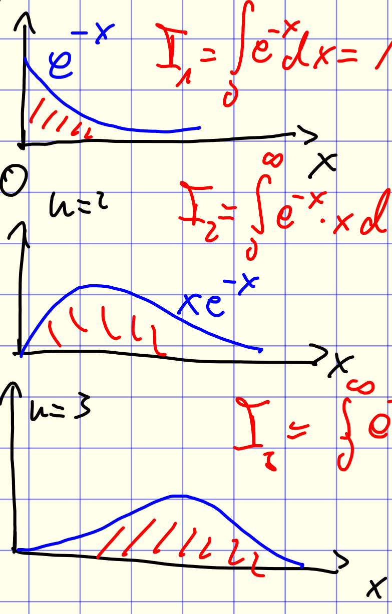
z.B. Paraboloid



$$V = \pi \int_0^b dx \left(\sqrt{x}\right)^2 = \pi \cdot \int_0^b dx \cdot x = \underline{\underline{\frac{\pi}{2} b^2}}$$

Beispiel part. Integration

Parameter $n=1$



$I_1 = \int_0^\infty e^{-x} dx = 1$

$I_2 = \int_0^\infty e^{-x} \cdot x \, dx = 1$

$I_3 = \int_0^\infty e^{-x} \cdot x^2 dx = 2$

Betrachten $I_n = \int_0^\infty \underbrace{e^{-x}}_{f'} \cdot \underbrace{x^{n-1}}_{g} dx \; ; \; n > 1$

$$I_n = [fg]_0^\infty - \int_0^\infty f g' dx$$

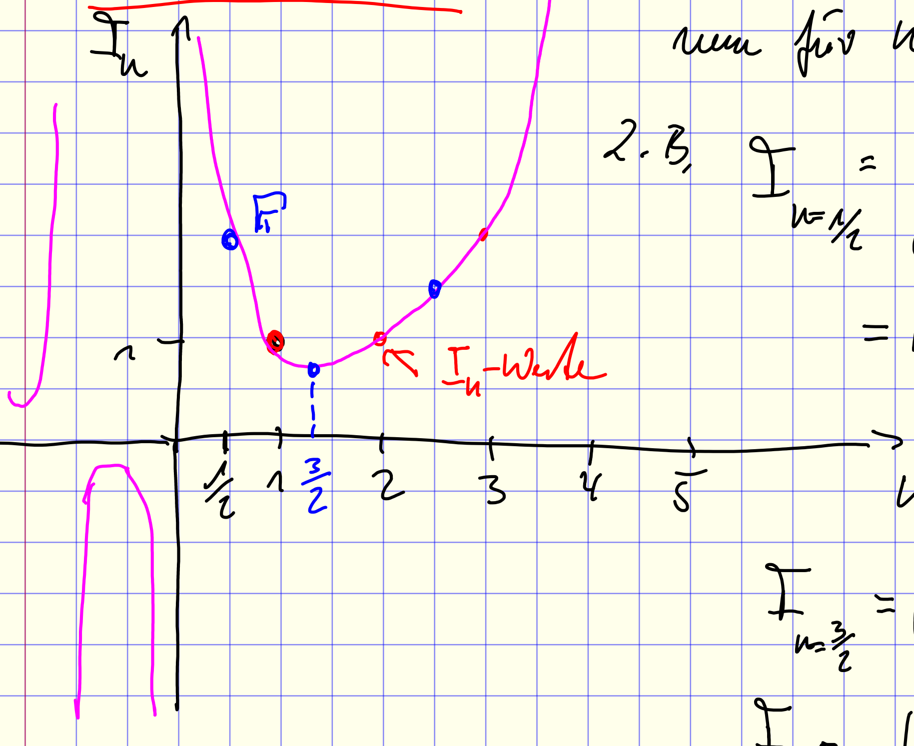
$$I_n = \left[-e^{-x} \cdot x^{n-1}\right]_0^\infty + \int_0^\infty e^{-x} x^{(n-2)} \cdot (n-1) \, dx$$

$$I_n = (n-1) \int_0^\infty e^{-x} x^{n-2} dx = (n-1) \cdot I_{n-1}$$

z.B. $I_3 = (3-1) \cdot I_2 = 2 \cdot I_2 = 2 \cdot 1$

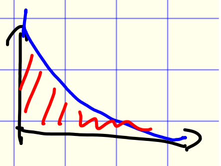
$I_4 = (4-1) I_3 = 3 \cdot 2 \cdot 1 = 6$

$$\boxed{I_n = (n-1) I_{n-1}}$$ Rekursionsvorschrift



nun für $n \in \mathbb{N}$, nun für $n \in \mathbb{R}$

z.B. $I_{n=1/2} = \int_0^\infty e^{-x} \cdot x^{\frac{1}{2}-1} dx = \int_0^\infty \frac{e^{-x}}{\sqrt{x}} dx$



$= \sqrt{\pi} \approx 1,77$

$I_{n=\frac{3}{2}} = \left(\frac{3}{2} - 1\right) I_{1/2} = \frac{\sqrt{\pi}}{2} = 0,886\ldots$

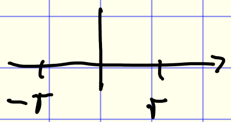
$I_{n=\frac{5}{2}} = \left(\frac{5}{2} - 1\right) I_{3/2} = \ldots$

$\Rightarrow$ Def. $I_x = (x-1)! = \Gamma(x)$ – Gamma von x

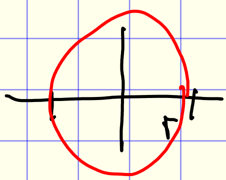
Wozu ist dies gut?

z.B. für das allgemeine Volumen einer Kugel in n-Dimensionen

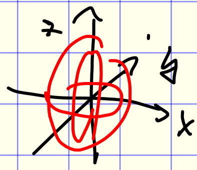
$n = 0 \triangleq$ Punkt • $V = 0$

$n = 1 \triangleq$ Linie 

$V = 2r$

$n = 2 \triangleq$ Kreis 

$V = \pi r^2$

$n = 3 \triangleq$ Kugel 

$V = \frac{4}{3}\pi r^3$

$$V_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} \cdot r^n$$

$$V_1 = \frac{\pi^{1/2}}{\Gamma\left(\frac{1}{2}+1\right)} r^1 = 2r$$

$$V_2 = \frac{\pi}{\Gamma(2)} r^2 = \pi r^2$$

$$V_3 = \frac{\pi^{3/2}}{\Gamma\left(\frac{5}{2}\right)} r^3 = \frac{4}{3}\pi r^3$$

jetzt alles Einheitskugeln $\Rightarrow r = 1$

$V_0 = 0$

$\Rightarrow V_1 = 2$

$V_2 = \pi = 3.14$

$V_3 = \frac{4}{3}\pi = 4.19$

$V_4 = 4{,}93$

$V_5 = 5{,}26 \triangleq$ Maximum

$V_6 = 5{,}17$

$V_7 < 5{,}17$

auch für Oberfläche

dort ist Maximum am $n = 7$