Vorrechnen 11 17.10.2023

Nachtrag (Blatt 10)

Aufgabe 2

ii) b) $f(x) = x^3 - 4x^2 + 4x$

Nullstellen: $x=0$, $x=2$ ← doppelte Nullstelle

$$x \cdot (x^2 - 4x + 4) = 0 \Rightarrow x = 0 \lor \underbrace{x^2 - 4x + 4}_{(x-2)^2} = 0$$

$$f(x) = x \cdot (x-2)^2$$

$$f'(x) = 3x^2 - 8x + 4 \overset{!}{=} 0$$

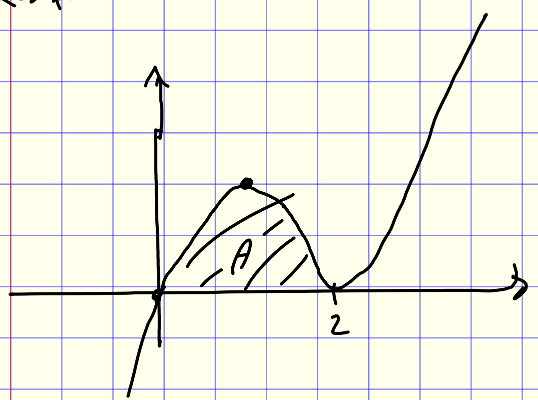
$$\Leftrightarrow x^2 - \frac{8}{3}x + \frac{4}{3} = 0 \Rightarrow x = \frac{4}{3} \pm \sqrt{\frac{16}{9} - \frac{4}{3}}$$

$$= \ldots$$

$$= \underbrace{\frac{6}{3}}_{2} \lor \frac{2}{3}$$

$$\lim_{x \to -\infty} f(x) = -\infty$$

$$\lim_{x \to +\infty} f(x) = +\infty$$



$$\int_0^2 (x^3 - 4x^2 + 4x)\,dx$$

$$= \left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + 2x^2\right]_0^2$$

$$= \frac{4}{3}$$

iii) a) $\int \frac{\cos(x)}{\sqrt{\sin(x)+3}}\,dx = I$

$u = \sin(x) + 3$

$\frac{du}{dx} = \cos(x) \quad | \cdot dx \mid : \cos(x)$

$\Rightarrow dx = \frac{1}{\cos(x)}\,du$

$$\Rightarrow \int \frac{\cancel{\cos(x)}}{\sqrt{u}} \frac{1}{\cancel{\cos(x)}}\,du \Rightarrow \int \frac{1}{\sqrt{u}}\,du = \int u^{-\frac{1}{2}}\,du = 2u^{\frac{1}{2}} = 2\sqrt{u} = 2 \cdot \sqrt{\sin(x)+3}$$

b) $\int \frac{(\ln(x^3))^2}{x}\,dx = \int \frac{(\ln(x)\,3)^2}{x}\,dx = 9 \cdot \int \frac{\ln^2(x)}{x}\,dx$

$u = \ln(x)$

$\frac{du}{dx} = \frac{1}{x} \quad |\cdot dx| \cdot x \qquad \Rightarrow 9\int \frac{u^2}{\cancel{x}} \cdot \cancel{x}\,du \Rightarrow 9\int u^2\,du = \frac{1}{3}u^3 \cdot 9$

$\Rightarrow dx = x \cdot du \qquad\qquad \overset{RS}{=} \frac{1}{3}(\ln(x))^3 \cdot 9$

$= 3\ln(x)^3$

c) $I = \int \frac{e^{2x}+e^x}{e^{2x}-e^x-2}\,dx \quad , \qquad x > \ln(2)$

$u = e^x$

$\frac{du}{dx} = e^x \quad |\cdot dx| : e^x \qquad \Rightarrow \int \frac{(u+1)\cdot \cancel{u}}{u^2-u-2} \cdot \underbrace{\frac{1}{e^x}}_{\frac{1}{\cancel{u}}}\,du = \int \frac{u+1}{u^2-u-2}\,du$

$\Rightarrow dx = \frac{1}{e^x}\,du$

PBZ: $u^2-u-2=0 \Rightarrow u = \frac{1}{2} \pm \sqrt{\frac{1}{4}+2} = 2 \vee -1$

$u^2-u-2 = (u-2)(u+1)$

$I = \int \frac{\cancel{u+1}}{(u-2)\cancel{(u+1)}}\,du \qquad u \neq -1 \qquad x > \ln(2) \;|\; e^{()}$

$\Rightarrow e^x > 2 \Rightarrow u > 2$

$= \int \frac{1}{u-2}\,du = \ln(|u-2|) \overset{RS}{=} \ln(\underbrace{|e^x-2|}_{>0,\text{ da } x>\ln(2)}) = \ln(e^x-2)$

$\frac{e^{2x}+e^x}{e^{2x}-e^x-2} = \frac{e^x(e^x+1)}{(e^x+1)(e^x-2)} \rightarrow \int \frac{e^x}{e^x-2}\,dx$

$= \int \frac{f'(x)}{f(x)}\,dx = \ln(f(x)) \qquad f(x) = e^x-2$

$= \ln(e^x-2)$

iv) a) $\int \underset{f}{x} \underset{g}{\sin(x)} dx = f \cdot G - \int f' \cdot G$

$= x \cdot (-\cos(x)) + \underbrace{\int \cos(x)\, dx}_{\sin(x) + C}$

$= -x\cos(x) + \sin(x) + C$

b) $I = \int \sin^2(x)\, dx = \int \underset{f}{\sin(x)} \cdot \underset{g}{\sin(x)}\, dx$

$= -\sin(x) \cdot \cos(x) + \int \underbrace{\cos(x) \cos(x)}_{\cos^2(x) = 1 - \sin^2(x)}\, dx$

$= -\sin(x) \cdot \cos(x) + \int 1\, dx - \int \sin^2(x)\, dx \quad | + \int \sin^2(x)\, dx$

$\Rightarrow 2 \cdot \int \sin^2(x)\, dx = -\sin(x) \cdot \cos(x) + x \quad | : 2$

$\Rightarrow \int \sin^2(x)\, dx = \frac{-\sin(x)\cos(x)}{2} + \frac{x}{2} + C$

c) $\int \ln(x)\, dx$

$= \int \underset{g}{1} \cdot \underset{f}{\ln(x)}\, dx = \ln(x) \cdot x - \int \frac{1}{\cancel{x}} \cdot \cancel{x}\, dx = \ln(x) \cdot x - \int 1\, dx$

$= \ln(x) \cdot x - x + C$

Probe $\frac{d}{dx}(\ln(x) \cdot x - x) = \frac{1}{\cancel{x}} \cdot \cancel{x} + \ln(x) \cancel{-1} = \ln(x)$

Blatt 11

Aufgabe 1

i) a) $\int \frac{1}{e^{3x}+5}\, dx = I \qquad u = 3x \Rightarrow$ kein Vorteil

$u = e^{3x} \qquad \frac{du}{dx} = 3e^{3x} \quad | \cdot dx \quad | : 3e^{3x}$

$\Rightarrow dx = \frac{1}{3} \cdot \frac{1}{e^{3x}}\, du$

$$\Rightarrow \frac{1}{3}\int \frac{1}{u+5}\cdot \underbrace{\frac{1}{e^{3x}}}_{\frac{1}{u}}\,du \Rightarrow \frac{1}{3}\int \frac{1}{u^2+5u}\,du \qquad u^2+5u=0$$

$$u\cdot(u+5)=u^2+5u$$

$$\Rightarrow \frac{1}{u^2+5u}=\frac{A}{u+5}+\frac{B}{u}=-\frac{1}{5\cdot(u+5)}+\frac{1}{5\cdot u}$$

$$=\frac{A\cdot u+Bu+5B}{(u+5)\cdot u} \quad |\cdot(u+5)\cdot u$$

$$\Rightarrow 1=u\cdot(A+B)+5B$$

(1) $A+B=0 \Rightarrow A=-\frac{1}{5}$

(2) $5B=1 \Rightarrow B=\frac{1}{5}$

$$I=\frac{1}{3}\cdot\left[-\frac{1}{5}\underbrace{\int\frac{1}{u+5}\,du}_{\ln(|u+5|)}+\frac{1}{5}\underbrace{\int\frac{1}{u}\,du}_{\ln(|u|)}\right] \qquad \ln(x)-\ln(y)=\ln\left(\frac{x}{y}\right)$$

$$=\frac{1}{3}\cdot\frac{1}{5}\cdot(\ln(|u|)-\ln(|u+5|))$$

$$=\frac{1}{15}\ln\left(\frac{|u|}{|u+5|}\right)\Rightarrow\ln\left(\sqrt[15]{\frac{|u|}{|u+5|}}\right)\overset{RS}{=}\ln\left(\sqrt[15]{\frac{|e^{3x}|}{|e^{3x}+5|}}\right)$$

$$=\ln\left(\sqrt[15]{\frac{e^{3x}}{e^{3x}+5}}\right)+C$$

$$u=e^{3x}+5$$

$$\frac{du}{dx}=3e^{3x} \quad = \text{Alternative}$$

b) $\int \underset{f(x)}{e^x}\,\underset{g(x)}{\cos(x)}\,dx=e^x\sin(x)-\int \underset{\cancel{g(x)}}{e^x}\,\underset{\cancel{f(x)}}{\sin(x)}\,dx$

Terme vertauschen? NEIN! :(

$$\int e^x\sin(x)\,dx=\sin(x)e^x-\int\cos(x)e^x\,dx$$

einsetzen:

$$\int e^x\cos(x)\,dx=\cancel{e^x\sin(x)}-\cancel{\sin(x)e^x}+\int\cos(x)e^x\,dx$$

$$\int e^x \cos(x)\,dx = \int e^x \cos(x)\,dx$$

Statt dessen: Terme unvertauscht lassen

$$\int e^x \cos(x)\,dx = e^x \sin(x) - \int \underset{\substack{\uparrow \\ f(x)}}{e^x} \underset{\substack{\uparrow \\ g(x)}}{\sin(x)}\,dx$$

NR:

$$\int e^x \sin(x)\,dx = -e^x \cos(x) + \int e^x \cos(x)\,dx$$

einsetzen:

$$\int e^x \cos(x)\,dx = e^x \sin(x) - \left(-e^x \cos(x) + \int e^x \cos(x)\,dx\right)$$

$$= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x)\,dx \quad \Big| + \int e^x \cos(x)\,dx \quad \Big| : 2$$

$$\Rightarrow \int e^x \cos(x)\,dx = \frac{1}{2} \cdot e^x \left(\sin(x) + \cos(x)\right)$$

c) $$\int \sqrt{\frac{1-x}{1+x}} \cdot \frac{1}{(1-x)^2}\,dx = \int \frac{1}{\sqrt{\frac{1+x}{1-x}}} \cdot \frac{1}{(1-x)^2}\,dx$$

$$u = \frac{1+x}{1-x} \qquad \frac{du}{dx} \overset{QR}{=} \frac{\left(\frac{d}{dx}(1+x)\right)(1-x) - \left(\frac{d}{dx}(1-x)\right)(1+x)}{(1-x)^2}$$

$$= \frac{2}{(1-x)^2}$$

$$\Rightarrow \int \frac{1}{\sqrt{u}} \cdot \frac{1}{2} \cdot \frac{du}{\cancel{dx}}\,\cancel{dx} = \underbrace{\int \frac{1}{2\sqrt{u}}\,du}_{\sqrt{u}} = \sqrt{u} \overset{RS}{=} \sqrt{\frac{1+x}{1-x}}$$

d) $$\int \underset{\substack{\uparrow \\ g}}{x^2} \underset{\substack{\uparrow \\ f}}{\ln(x)}\,dx$$

$$= \frac{1}{3}x^3 \cdot \ln(x) - \int \frac{1}{3} x^{\cancel{3}\,2} \cdot \frac{1}{\cancel{x}}\,dx$$

$$= \frac{1}{3}x^3 \cdot \ln(x) - \frac{1}{3}\int x^2\,dx = \frac{1}{3}x^3 \ln(x) - \frac{1}{3} \cdot \frac{1}{3}x^3 = \frac{1}{3}x^3 \ln(x) - \frac{1}{9}x^3$$

$$= \frac{1}{3}x^3 \cdot \left(\ln(x) - \frac{1}{3}\right)$$

e) $I = \int \underbrace{\sin(4x)}_{g} \underbrace{\sin(6x)}_{f} dx = -\frac{1}{4}\cos(4x)\sin(6x) + \frac{6}{4}\int \underbrace{\cos(4x)}_{g} \underbrace{\cos(6x)}_{f} dx$

$= -\frac{1}{4}\cos(4x)\sin(6x) + \frac{6}{4}\left(\frac{1}{4}\sin(4x)\cos(6x) + \frac{6}{4}\int \sin(4x)\sin(6x)dx\right)$

$= -\frac{1}{4}\cos(4x)\sin(6x) + \underbrace{\frac{6}{16}}_{\frac{3}{8}}\sin(4x)\cos(6x) + \underbrace{\frac{36}{16}}_{\frac{9}{4}}\int \sin(4x)\sin(6x)dx$

$\Rightarrow \underline{I} = \int \sin(4x)\sin(6x)dx = -\frac{1}{4}\cos(4x)\sin(6x) + \frac{3}{8}\sin(4x)\cos(6x) + \frac{9}{4}\int \sin(4x)\sin(6x)dx \quad \Big| -\frac{9}{4}I$

$\Rightarrow -\frac{5}{4}I = -\frac{1}{4}\cos(4x)\sin(6x) + \frac{3}{8}\sin(4x)\cos(6x) \quad \Big| \cdot \left(-\frac{4}{5}\right)$

$\Rightarrow \underline{I} = \frac{1}{5}\cos(4x)\sin(6x) - \frac{3}{10}\sin(4x)\cos(6x)$

f) $\int \sin(x)\, e^{\cos(x)} dx = -e^{\cos(x)}$

$u = \cos(x)$

$\frac{du}{dx} = -\sin(x) \quad |\cdot dx \,|:(-\sin(x))$

$\Rightarrow dx = -\frac{1}{\sin(x)}du \quad \Rightarrow \quad \int \cancel{\sin(x)}\, e^u \cdot \left(-\frac{1}{\cancel{\sin(x)}}\right)du = -\int e^u du$

$= -e^u$

$\overset{RS}{=} -e^{\cos(x)}$

g) $\int \frac{\sqrt{1+x}}{x} dx \qquad u = 1+x \Rightarrow$ bringt nichts

$u = \sqrt{1+x} \qquad \frac{du}{dx} = \frac{1}{2\sqrt{1+x}} \quad |\cdot dx \,|\cdot 2\sqrt{1+x} \Rightarrow dx = 2\sqrt{1+x}\,du$

$\int \frac{\sqrt{1+x}}{x}dx = \int \frac{\sqrt{1+x}}{x}\cdot 2\sqrt{1+x}\,du = 2\int \frac{1+x}{x}du \qquad u = \sqrt{1+x} \quad |()^2 \,| -1$

$\Rightarrow x = u^2 - 1$

$= 2\cdot \int \frac{\cancel{1}+u^2\cancel{-1}}{u^2-1}du$

Grad Zähler = Grad Nenner

$\Rightarrow$ deshalb Polynomdivision

$$u^2 : (u^2-1) = 1 + \frac{1}{u^2-1}$$
$$\underline{-(u^2-1)}$$
$$1$$

$$\underline{I} = 2\int 1\,du + 2\cdot\underbrace{\int \frac{1}{u^2-1}\,du}_{PBZ}$$

$$\frac{1}{u^2-1} = \frac{A}{u+1} + \frac{B}{u-1} = -\frac{1}{2\cdot(u+1)} + \frac{1}{2\cdot(u-1)}$$

$\Rightarrow 1 = (A+B)u + B - A$

① $B - A = 1 \Rightarrow 2B = 1 \Rightarrow B = \frac{1}{2}$

② $A + B = 0 \Rightarrow A = -B$

$\Rightarrow A = -\frac{1}{2}$

$$I = 2\cdot\underbrace{\int 1\,du}_{u} + \cancel{2}\cdot\int -\frac{1}{\cancel{2}\cdot(u+1)}\,du + \cancel{2}\cdot\int\frac{1}{\cancel{2}\cdot(u-1)}\,du$$

$$= 2u - \ln(|u+1|) + \ln(|u-1|) \overset{RS}{=} 2\cdot(\sqrt{1+x}) - \ln(|\sqrt{1+x}+1|) + \ln(|\sqrt{1+x}-1|)$$

$$= 2\cdot\sqrt{1+x} - \ln\left(\frac{|\sqrt{1+x}+1|}{|\sqrt{1+x}-1|}\right) + C$$

h) $\int\frac{1}{\sin(x)}\,dx$ $\quad u = \sin(x)$ $\quad \cos(x) = \sqrt{1-\sin^2(x)}$

$$\frac{du}{dx} = \cos(x) \mid \cdot dx \mid : \cos(x) \Rightarrow dx = \frac{1}{\cos(x)}\cdot du$$

$$\int\frac{1}{\sin(x)}\frac{1}{\cos(x)}\,du = \int\frac{1}{u}\cdot\frac{1}{\sqrt{1-u^2}}\,du \qquad u = \cos(x) \quad \frac{du}{dx} = -\sin(x).$$

$$\Rightarrow \int\frac{1}{\sin(x)}\cdot\frac{1}{-\sin(x)}\,du = -\int\frac{1}{\underbrace{\sin^2(x)}_{1-\cos^2(x) = 1-u^2}}\,du$$

$$= -\int \frac{1}{1-u^2}\,du = \int \frac{1}{u^2-1}\,du \Rightarrow \ln\left(\sqrt{\frac{|u-1|}{|u+1|}}\right) = \ln\left(\sqrt{\frac{|\cos(x)-1|}{|\cos(x)+1|}}\right)$$

ii) $\int \frac{1}{(x^2+1)^3}\,dx$

a) $$\int \frac{1}{(x^2+1)^n}\,dx = \int \frac{1}{(x^2+1)^{n-1}} \cdot \frac{1}{x^2+1}\,dx$$

zum Int. / Ableit. schlecht

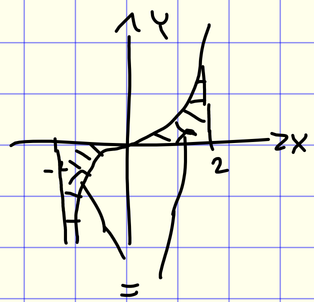
$$\int \underset{f(x)}{\frac{1}{(x^2+1)^n}} \cdot \underset{g(x)}{1}\,dx = \frac{x}{(x^2+1)^n} + 2n \int \underbrace{\frac{x^2}{(x^2+1)^{n+1}}}\,dx$$

$$\frac{x^2}{(x^2+1)^{n+1}} = \frac{x^2+1-1}{(x^2+1)^{n+1}} = \frac{\cancel{x^2+1}}{(x^2+1)^{n+1}} - \frac{1}{(x^2+1)^{n+1}}$$

damit

$$\int \frac{1}{(x^2+1)^n}\,dx = \frac{x}{(x^2+1)^n} + \underbrace{2n \int \frac{1}{(x^2+1)^n}\,dx}_{2n\cdot I} - 2n \int \frac{1}{(x^2+1)^{n+1}}\,dx \quad \Big|\, -2n\cdot I$$

$$(1-2n)\cdot I = \frac{x}{(x^2+1)^n} - 2n \underbrace{\int \frac{1}{(x^2+1)^{n+1}}\,dx}_{I_{n+1}}$$



$$\Rightarrow I_{n+1} = \frac{x}{(x^2+1)^n} \cdot \frac{1}{2n} + \frac{2n-1}{2n} \cdot I_n$$

b) $$I_n = \frac{x}{(x^2+1)^{n-1}} \cdot \frac{1}{2\cdot(n-1)} + \frac{2n-3}{2n-2} \cdot I_{n-1}$$

Aufgabe 2

i) a) $$\int_{-2}^{2} (x^3-x)\,dx = \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_{-2}^{2}$$

$= 0$ Punktsymmetrie / ungerade $f(-x) = -f(x)$

b) $\int_0^1 \frac{x^2}{1+x^2}\,dx$

$$x^2 : (x^2+1) = 1 - \frac{1}{x^2+1}$$
$$\underline{-(x^2+1)}$$
$$-1$$

$$1+\tan^2(u) = \sec^2(u)$$

$$= \int_0^1 \left(1-\frac{1}{x^2+1}\right)dx = \underbrace{\int_0^1 1\,dx}_{1} - \int_0^1 \frac{1}{1+x^2}\,dx$$

$$x = \tan(u) \qquad \frac{dx}{du} = \frac{1}{\sec^2(u)}$$

$$\Rightarrow dx = \sec^2(u)\,du$$

$$= 1 - \int_0^1 \frac{1}{\cancel{\sec^2(u)}}\cancel{\sec^2(u)}\,du = 1 - \int_0^1 1\,du \qquad x = \tan(u) \mid \arctan()$$

$$\Rightarrow \arctan(x) = \underline{\arctan}(\cancel{\tan}(u))$$

$$= 1 - [u]_0^1$$

$$\Rightarrow u = \arctan(x)$$

$$= 1 - (\underbrace{\arctan(1)}_{\frac{\pi}{4}} - \underbrace{\arctan(0)}_{0})$$

$$= 1 - \frac{\pi}{4}$$

c) $\int_2^4 \frac{|x-3|}{x^2}\,dx$

$$|x-3| = \begin{cases} x-3, & x \geq 3 \\ -x+3, & x < 3 \end{cases}$$

$$= \int_2^3 \frac{|x-3|}{x^2}\,dx + \int_3^4 \frac{|x-3|}{x^2}\,dx = \int_2^3 \frac{-x+3}{x^2}\,dx + \int_3^4 \frac{x-3}{x^2}\,dx$$

NR: $\left[-\frac{1}{x}\right]_2^3$

$$= \left(\left(-\frac{1}{x}\right) - \left(-\frac{1}{x}\right)\right)$$
$$= -\frac{1}{3} - \left(-\frac{1}{2}\right)$$
$$= -\frac{1}{3} + \frac{1}{2}$$
$$= \frac{1}{2} - \frac{1}{3}$$

$$= -\underbrace{\int_2^3 \frac{1}{x}\,dx}_{\ln(3)-\ln(2)} - 3\cdot\underbrace{\int_2^3 \frac{1}{x^2}\,dx}_{-\frac{1}{x}\Big|_2^3} + \underbrace{\int_3^4 \frac{1}{x}\,dx}_{\ln(4)-\ln(3)} - 3\cdot\underbrace{\int_3^4 \frac{1}{x^2}\,dx}_{-\frac{1}{x}\Big|_3^4}$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{3}{6} - \frac{2}{6} = \frac{1}{6}$$

$$= \underbrace{\frac{1}{3} - \frac{1}{4}}_{\frac{4}{12} - \frac{3}{12}} = \frac{1}{12}$$

$$= \ln\left(\frac{3}{2}\right) - \frac{1}{2} + \ln\left(\frac{4}{3}\right) - \frac{1}{4} = \ln\left(\frac{3}{2}\cdot\frac{4}{3}\right) \underbrace{-\frac{1}{2} - \frac{1}{4}}_{-\frac{3}{4}} = \ln(2) - \frac{3}{4}$$

d) $\int_0^3 \underset{\uparrow f}{x}\,\underset{\uparrow g}{e^{3x}}\,dx \overset{P.I.}{=} \frac{1}{3}xe^{3x}\Big|_0^3 - \frac{1}{3}\int_0^3 e^{3x}\,dx$

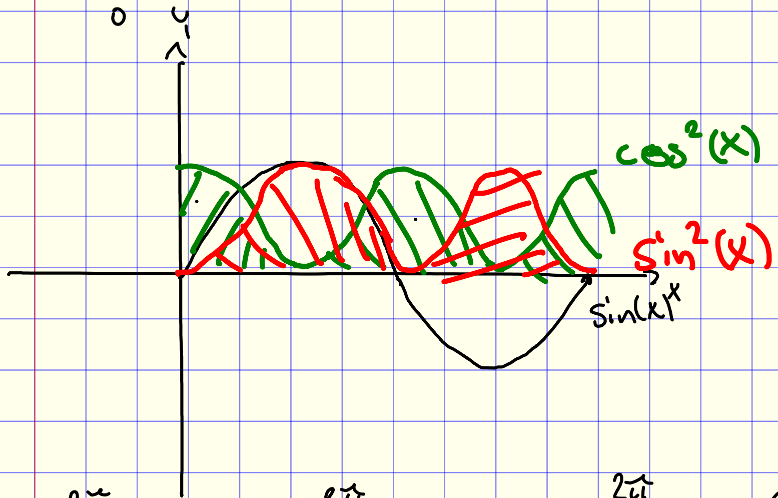
$$= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\Big|_0^3$$

$$= \underbrace{e^9 - \frac{1}{9}e^9}_{\frac{8}{9}e^9} - \Big[0 - \frac{1}{9}\underbrace{e^0}_{1}\Big]$$

$$\frac{8}{9}e^9 + \frac{1}{9} \approx 7202{,}9$$

e) $\int_1^{\sqrt{3}} \frac{1}{x^2+1}\,dx = \big[\arctan(x)\big]_1^{\sqrt{3}} = \underbrace{\arctan(\sqrt{3})}_{60^\circ} - \underbrace{\arctan(1)}_{\frac{\pi}{4} = 45^\circ} = 15^\circ = \frac{\pi}{12}$

f) $\int_0^{2\pi} \sin^2(x)\,dx$ P.I. ...



$$I = \int_0^{2\pi} \sin^2(x)\,dx = \int_0^{2\pi} \underbrace{\cos^2(x)}_{1-\sin^2(x)}\,dx = \int_0^{2\pi} 1\,dx - \underbrace{\int_0^{2\pi} \sin^2(x)\,dx}_{I} \quad \Big| + I$$

$$2I = \int_0^{2\pi} 1\,dx = 2\pi \quad \Big| :2 \Rightarrow I = \pi$$