Vorlesung 24.09.24 – Algebra II

1.5. Gleichungen mit einer Variablen

1.5.1 Lineare Gleichungen

$$ax=b \Rightarrow x=\frac{b}{a} \quad ; \quad a\neq 0$$

1.5.2. quadratische Gleichungen

$$ax^2+bx+c=0 \quad ; \quad (x-x_1)(x-x_2)=0$$

Lösungen durch x_1, x_2

Mitternachtsformel $$x_{1/2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \quad ; \quad a\neq 0$$

p,q-Formel $$x_{1/2}=-\frac{p}{2}\pm\sqrt{\left(\frac{p}{2}\right)^2-q}$$

1.5.3 Exponentialgleichungen

$$e^x=a \Rightarrow x=\ln a$$

Bsp. $$3^{x^2-4}=6^{-x}$$

$$3^{x^2-4}=3^{-x}\cdot 2^{-x} \qquad | :3^{-x}$$

$$3^{x^2-4}\,3^{x}=2^{-x}$$

$$3^{x^2+x-4}=2^{-x} \qquad | \log_3(..)$$

$$x^2+x-4=\log_3\left(2^{-x}\right)=-x\log_3 2$$

$$x^2+x(1+\log_3 2)-4=0$$

$$x_{1/2}=\frac{-(1+\log_3 2)\pm\left((1-\log_3 2)^2+16\right)^{1/2}}{2}$$

$$x_1=-2{,}98 \quad ; \quad x_2=1{,}34$$

1.5.4. Faktorisieren

Bsp. $x^4 - x^2 = 0 \iff x^2(x^2-1) = 0$

$\Rightarrow x^2 = 0 \lor x^2 - 1 = 0$ „$\lor$" $\hat{=}$ oder

„$\land$" $\hat{=}$ und

$x \in \{-1, 0, 1\}$

1.5. 5. Wurzelgleichungen

Lösung durch Quadrieren ! keine Äquivalenzumformung

$\Rightarrow$ immer Probe machen

Bsp. $\sqrt{4x^2 + \sqrt{x^2} + 1} = x + 1 \quad | (\,)^2$

$4x^2 + \sqrt{x^2} + 1 = x^2 + 2x + 1$

$\sqrt{x^2} = -3x^2 + 2x \quad | (\,)^2$

$x^2 = (-3x^2 + 2x)^2 = 9x^4 - 12x^3 + 4x^2$

$9x^4 - 12x^3 + 3x^2 = 0 \Rightarrow x^2(9x^2 - 12x + 3)$

$x^2 = 0 \lor 3x^2 - 4x + 1 = 0$

$x = 0 \qquad x_{1/2} = \dfrac{4 \pm \sqrt{16-12}}{6}$; $x_1 = \frac{1}{3}$

$x_2 = 1$

Probe: $x_0 = 0$ und $x_1 = \frac{1}{3}$

$\sqrt{4x^2 + \sqrt{x^2} + 1} = x + 1$

$x_0 = 0 \Rightarrow 1 = 1$ ✓ okay

$x_1 = \frac{1}{3}$ $\sqrt{\frac{4}{9} + \sqrt{\frac{1}{9}} + 1} = \frac{1}{3} + 1 = \frac{4}{3}$ ✓ okay

$x_2 = 1$ $\sqrt{4 + 1 + 1} = \sqrt{6} = 1 + 1 = 2$ ↯ Widerspruch $\Rightarrow x_2$ entfällt.

$\mathbb{L} = \{0; \frac{1}{3}\}$

Achtung $\sqrt{x^2} = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

1.5.6. Betragsgleichungen

Bsp. $|x-1| + |x-2| = 5$

$$|x-1| = \begin{cases} x-1, & x \geq 1 \\ -(x-1), & x < 1 \end{cases}$$

$$|x-2| = \begin{cases} x-2, & x \geq 2 \\ -(x-2), & x < 2 \end{cases}$$

① $x \in (-\infty, 1)$ also $x < 1$

$|x-1| + |x-2| = 5 \iff (1-x) + (2-x) = 5 \Rightarrow -2x = 2$

$x = -1$

okay

② $x \in [1, 2)$

$|x-1| + |x-2| = 5 \iff (x-1) + (2-x) = 5 \Rightarrow 1 = 5$ ↯

entfällt

③ $x \in [2, \infty)$

$|x-1| + |x-2| = 5 \iff (x-1) + (x-2) = 5$

$2x = 8 \Rightarrow x = 4$

$$\Rightarrow \mathbb{L} = \{-1, 4\}$$

1.5.7. Substitution

Ersetzen eines Teilterms durch neue Variable

Bsp $(\ln x)^2 - 4 \ln x + 4 = 0$; $\ln x = y$

$(bla)^2 - 4\,bla + 4 = 0$

$y^2 - 4y + 4 = 0$

$(y-2)^2 = 0 \quad y_0 = y_{1/2} = 2$

Rücksubstitution $\ln x = z$

$$x_0 = e^{y_0} = e^2 = \exp(2) = 7{,}38\ldots$$

1.5.8. Ungleichungen mit Brüchen

Bsp. $\frac{x-1}{x+3} \geq 1$; $D = \mathbb{R} \setminus \{-3\}$ $\qquad -6 < 2 \quad | \cdot (-2)$

$\qquad 12 > -4$

Fallunterscheidung ① $x+3 > 0$; $x > -3$

$x - 1 \geq x + 3 \Rightarrow -1 \geq 3$ ↯

② $x + 3 < 0 \Rightarrow x < -3$

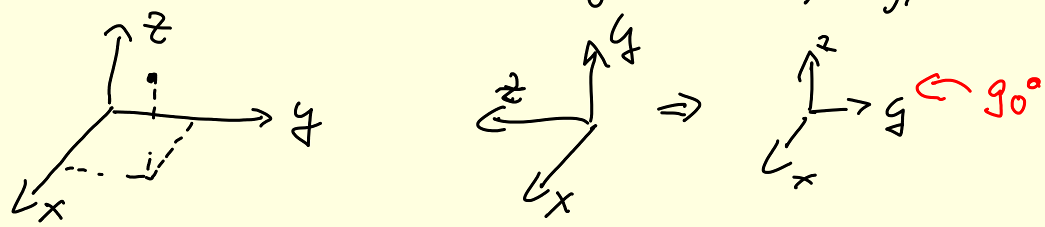
$x - 1 \leq x + 3 \Rightarrow -1 \leq 3$ okay

$\Rightarrow \mathcal{L} = \{(-\infty, -3)\}$

2. Elementare Geometrie

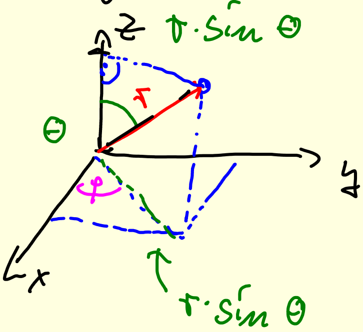
2.1. Koordinatensystem

- kartesisches Koordinatensystem KS, x, y, z



≙ 3 Zahlen

- Kugelkoordinaten: von Abstand r zum Ursprung und zwei Winkel Θ, φ



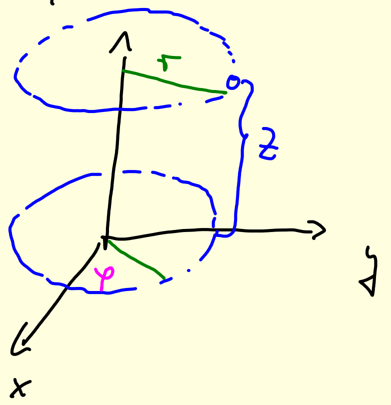
$$x = r \cdot \sin\Theta \cdot \cos\varphi$$
$$y = r \cdot \sin\Theta \cdot \sin\varphi \; ; \quad r = \sqrt{x^2 + y^2 + z^2}$$
$$z = r \cdot \cos\Theta$$

$\Theta = \arccos \frac{z}{r}$

$\varphi = \arcsin \frac{y}{r \sin\Theta} \ldots$

• Zylinderkoor.

r, φ ebene Polarkoor.



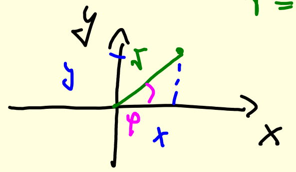
$x = r\cos\varphi$; φ - Azimut

$y = r\sin\varphi$

$z = z$

ebene Polarkoor.

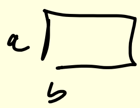
$r = \sqrt{x^2+y^2}$



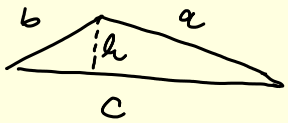
2.2. Elementare geom. Objekte

2D - Flächeninhalt / Umfang

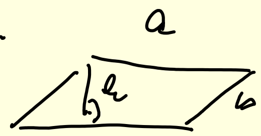
• Kreis  $A = \pi r^2$; $U = 2\pi r$

• Rechteck 

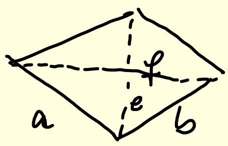
$A = a\cdot b$; $U = 2(a+b)$

• Dreieck 

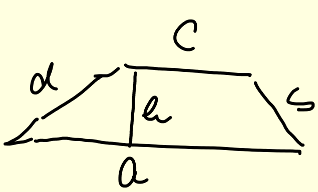
$A = \frac{1}{2} c\cdot h$; $U = a+b+c$

• Parallelogramm 

$A = a\cdot h$; $U = 2(a+b)$

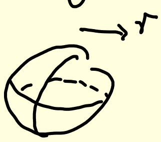
• Drachen 

$A = \frac{1}{2} e\cdot f$; $U = 2(a+b)$

• Trapez 

$A = \frac{a+c}{2}\cdot h$; $U = a+b+c+d$

3D - Volumen Oberfläche

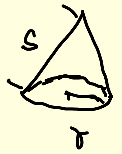
• Kugel 

$V = \frac{4}{3}\pi r^3$; $O = 4\pi r^2$

• Zylinder 

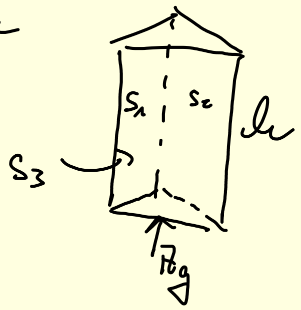
$V = \pi r^2 \cdot h$; $O = 2\pi r h + 2\pi r^2$

• Kegel



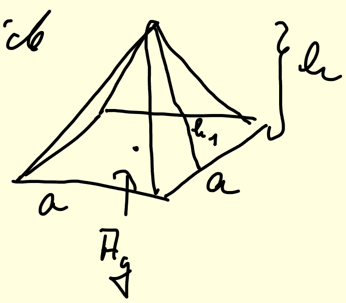
$V=\frac{1}{3}A_g\cdot h=\frac{1}{3}\pi r^2\cdot h$; $O=\pi r^2+\pi r s$

• Senkrechtes Prisma



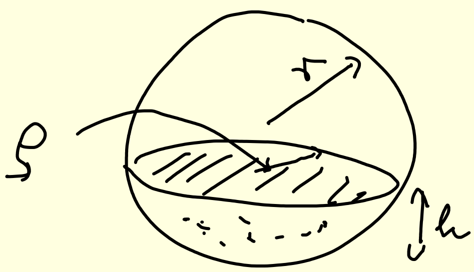
$V=A_g\cdot h$; $O=A_g\cdot 2+S_1+S_2+S_3\ \dots$

• Senkrechte Pyramide



$V=\frac{1}{3}A_g\cdot h$; $O=A_g+\frac{1}{2}h_1\cdot a\cdot 4$

• Kugelsegment



$V=\frac{\pi}{6}h\left(3\rho^2+h^2\right)=\frac{\pi}{3}h^2\cdot(3r-h)$

$O=2\pi r h+\pi\rho^2=\pi\cdot(2rh+\rho^2)$? ?

$\rho=\sqrt{h(2r-h)}$

• Kugelschicht