VR08 02.10.2024

Aufgabe 3:

a.) $1+3+5+7+9+\ldots = \sum_{k=0}^{\infty}(2k+1) = \sum_{k=1}^{\infty}(2k-1)$

b.) $1+4+9+16+25+\ldots = \sum_{k=1}^{\infty}(k^2)$

c.) $3+7+11+15+\ldots = \sum_{k=0}^{\infty}(3+4k) = \sum_{k=1}^{\infty}(4k-1)$

d.) $\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\ldots = \sum_{k=0}^{\infty}\frac{1}{5^{k+1}} = \sum_{k=1}^{\infty}\frac{1}{5^k}$

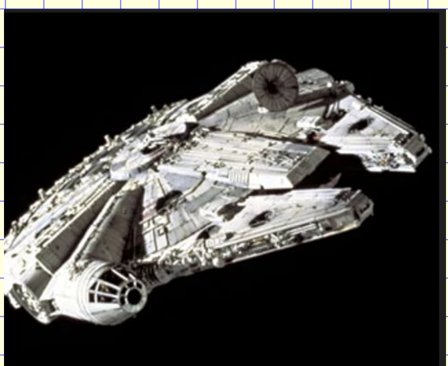
e.) $2+3+2+3+2+\ldots = \sum_{k=1}^{\infty}\frac{5}{2}+\frac{1}{2}\cdot(-1)^k$ „Alternierend"

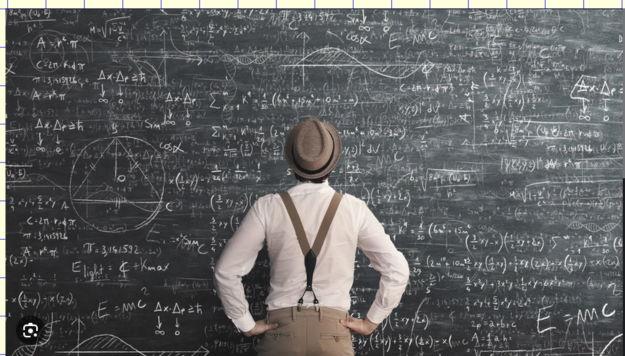
f.) $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots = \sum_{k=0}^{\infty}\frac{1}{2^k}$

g.) $1+\frac{1}{2^z}+\frac{1}{3^z}+\frac{1}{4^z}+\ldots = \zeta(z) = \sum_{k=1}^{\infty}\frac{1}{k^z}$ $\in \mathbb{C}$

„Riemannsche Zeta-Funktion"

Riemannsche Vermutung: Alle nichttriviale NS haben den Realteil $\frac{1}{2}$.





Millenium Falke $\neq$ Millenium Probleme

Die Liste enthält die folgenden sieben Probleme:

1. der Beweis der Vermutung von Birch und Swinnerton-Dyer aus der Zahlentheorie,
2. der Beweis der Vermutung von Hodge aus der algebraischen Geometrie,
3. Analyse von Existenz und Regularität von Lösungen des Anfangswertproblems der dreidimensionalen inkompressiblen Navier-Stokes-Gleichungen.
4. die Lösung des P-NP-Problems der Informatik,
5. der Beweis der Poincaré-Vermutung in der Topologie (2002 gelöst von Grigori Jakowlewitsch Perelman, die Vermutung trifft zu),
6. der Beweis der Riemannschen Vermutung der Zahlentheorie,
7. die Erforschung der Gleichungen von Yang-Mills.

b.) $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Allgemein / Ansatz: $\sum_{k=1}^{n} (a_{k+1} - a_k) = a_{n+1} - a_1$

$$\overset{n=4}{=} a_{1+1} - a_1 + a_{2+1} - a_2 + a_{3+1} - a_3 + a_{4+1} - a_4$$

$$= a_2 - a_1 + a_3 - a_2 + a_4 - a_3 + a_5 - a_4$$

$$= a_5 - a_1$$

$$(k+1)^2 - k^2 = \underbrace{k^2 + 2k + 1}_{\text{1. Binom.}} - k^2 = 2k+1$$

$$\sum_{k=1}^{n} (a_{k+1} - a_k) = \sum_{k=1}^{n} \left((k+1)^2 - k^2\right) = \sum_{k=1}^{n} (2k+1) \;*$$

$$= a_{n+1} - a_1 = (n+1)^2 - 1^2 = n^2 + 2n + 1^2 - 1^2$$

$$= n^2 + 2n$$

$$= n(n+2)$$

$*$

$$\sum_{k=1}^{n} (2k+1) = n(n+2)$$

$$\Leftrightarrow \sum_{k=1}^{n} (2k) + \sum_{k=1}^{n} (1) = n(n+2)$$

$$\Leftrightarrow 2\sum_{k=1}^{n} k + \underbrace{\sum_{k=1}^{n} 1}_{n} = n(n+2)$$

$$\Leftrightarrow 2\sum_{k=1}^{n} k + n = n(n+2) \quad | -n$$

$$\Leftrightarrow 2\sum_{k=1}^{\infty} k = n(n+2) - n \quad | \cdot \tfrac{1}{2}$$

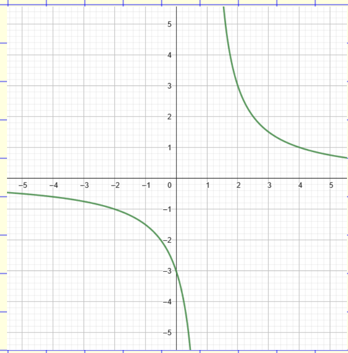
$$\Leftrightarrow \sum_{k=1}^{\infty} k = \frac{n(n+2) - n}{2}$$

$$\Leftrightarrow \sum_{k=1}^{\infty} k = \frac{n^2 + 2n - n}{2} = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}$$

$$\Leftrightarrow \boxed{\sum_{k=1}^{\infty} k = \frac{n(n+1)}{2}}$$ Gaußsche Summenformel

Aufgabe 1:

i.) a.) $f(x) = \frac{3}{x-1}$, $x \to \infty$



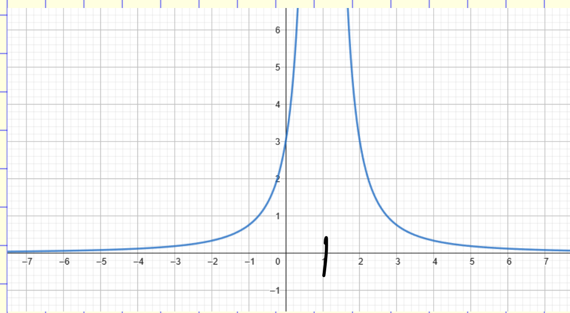
$$\lim_{x \to \infty} f(x) = \lim_{x\to\infty} \frac{3}{x-1} = 0^+$$

$$= \lim_{x\to\infty} \frac{\overbrace{\frac{3}{x}}^{\to 0}}{1-\underbrace{\frac{1}{x}}_{\to 0}} = \frac{0}{1} = 0$$

b.) $f(x) = \frac{3}{(x-1)^2}$, $x \to 1$

Parabel um 1 nach rechts

$$\lim_{x \to 1} \frac{3}{(x-1)^2} = \infty$$

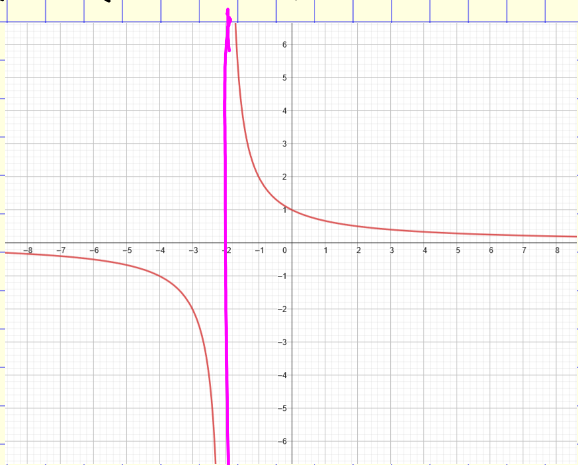


c.) $\frac{2(x-2)}{(x^2-4)}$, $x \to 2$

3. Binom.

$$\lim_{x\to 2} 2(x-2) = 0 = \lim_{x\to 2} (x^2-4)$$

$$\Leftrightarrow \lim_{x\to 2} \frac{2(x-2)}{(x^2-4)} = \lim_{x\to 2} \frac{2(x-2)}{(x+2)(x-2)} = \frac{2}{x+2} = \frac{2}{2+2} = \frac{2}{4} = \frac{1}{2}$$



d.) $\frac{2(x-2)}{x^2-4}$, $x \to -2$

(wegen oben)

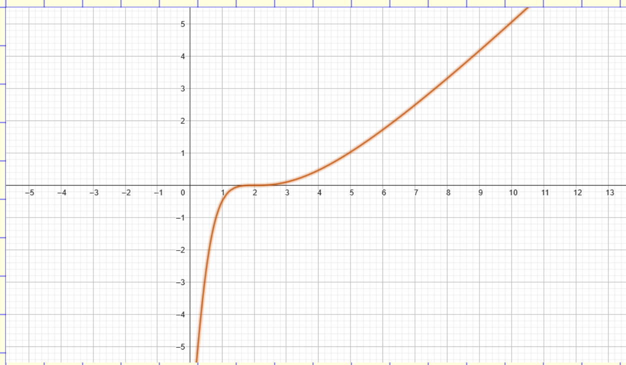
$$\lim_{x \to -2} \frac{2}{x+2} = \pm\infty$$

weil Definitionslücke (Nullstelle im Nenner) e.) $\lim\limits_{x\to\infty} \dfrac{(x-2)^3 \to \infty}{x^2+1 \to \infty}$, „aber schneller"

(Pascalsches Dreieck)

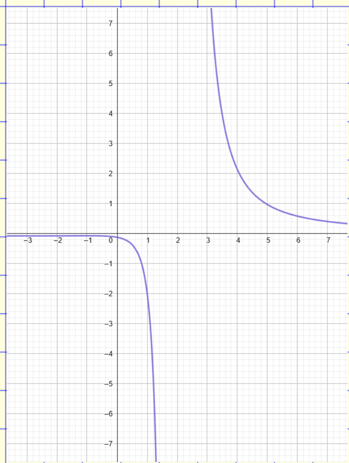
$$\Leftrightarrow \quad \frac{(x-2)^3}{x^2+1} = \frac{x^3-6x^2+12x-8}{x^2+1}$$

$$\lim_{x\to\infty} \frac{1-\overset{\to 0}{\frac{6}{x}}+\overset{\to 0}{\frac{12}{x^2}}-\overset{\to 0}{\frac{8}{x^3}}}{\underset{\to 0}{\frac{1}{x}}+\underset{\to 0}{\frac{1}{x^3}}} = \infty$$



f.) $\dfrac{x^2+1}{(x-2)^3}$, $x \to \infty$ „Kehrwert" von $f(x)$ bei e

$$\lim_{x\to\infty} \frac{x^2+1}{(x-2)^3} = 0$$



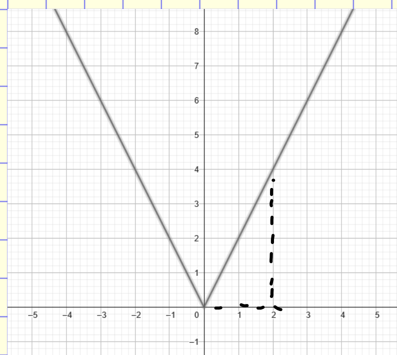
$\sqrt{x^2} \neq (\sqrt{x})^2$

ii) a.) $f(x) = 2|x| = \begin{cases} -2\cdot x, & x<0 \\ 0, & x=0 \\ 2x, & x>0 \end{cases}$ $\Rightarrow$ Krit. Stelle bei 0

$$\lim_{x\to 0^+} (2x) = 0^+$$

$$\lim_{\to 0^-} (-2x) = 0^+$$

$$f(0)=0$$

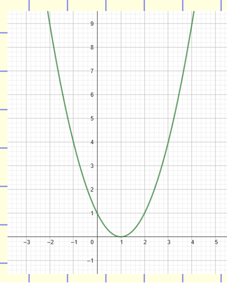


$\Rightarrow$ Stetig (rechtsseitiger/linksseitiger Grenzwert stimmen überein)

b.) $f(x) = |(x-1)^2| = (x-1)^2$, da $x \in \mathbb{R}$ $\Rightarrow$ $(x-1)^2 \geq 0$

$\to$ Parabel um 1 nach rechts

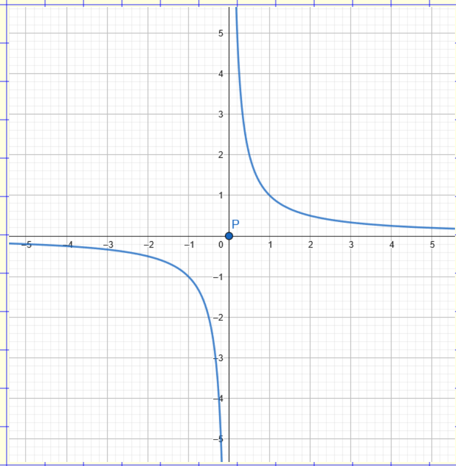
$\Rightarrow$ Stetig auf $\mathbb{R}$



c.) $f(x) = \begin{cases} 1/x & , x \neq 0 \quad ① \\ 0 & , x = 0 \quad ② \end{cases}$

① $\begin{cases} \lim\limits_{x \to 0^-} \frac{1}{x} = -\infty \\ \lim\limits_{x \to 0^+} \frac{1}{x} = \infty \end{cases}$

x	-2	-1	$-0,5$	
$f(x)$	$-\frac{1}{2}$	-1	-2	....



② $f(0) = 0$ => Polstelle

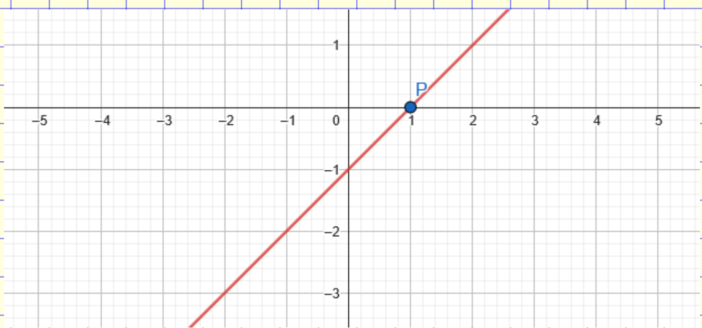
=> nicht stetig

d) $f(x) = \begin{cases} \frac{x^2 - 2x + 1}{x - 1} & , x \neq 1 \quad ① \\ 0 & , x = 1 \quad ② \end{cases}$

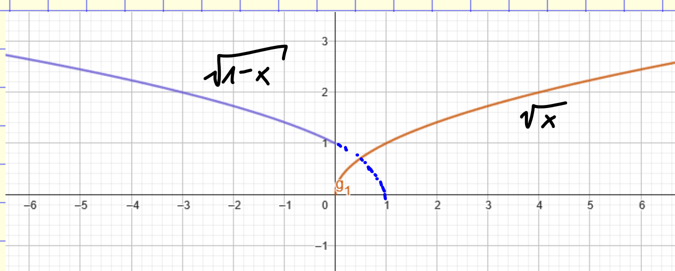
① 2. Binom. Formel (Zähler): $f(x) = \frac{x^2 - 2x + 1}{x - 1} = \frac{(x-1)^{2}}{x - 1} = x - 1$

$\lim\limits_{x \to 1^\pm} f(x) = \lim\limits_{x \to 1^\pm} (x - 1) = 0^\pm$ → Gerade mit NS bei $x = 1$

② $f(1) = 0$ // passt => stetig



e.) $f(x) = \begin{cases} \sqrt{x} & , x \geq 0 \quad ① \\ \sqrt{1-x} & , x < 0 \quad ② \end{cases}$



① $\lim\limits_{x \to 0^+} (\sqrt{x}) = 0^+$

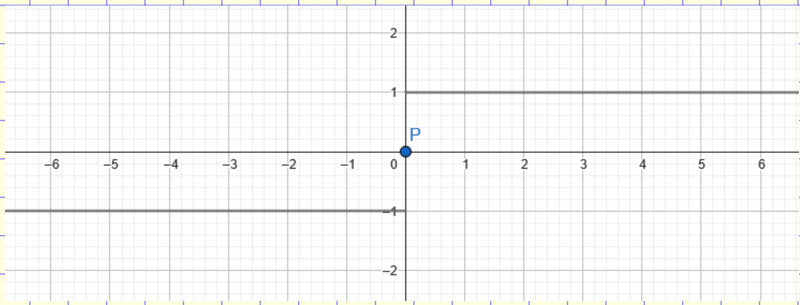
② $\lim\limits_{x \to 0^-} (\sqrt{1-x}) = 1^+ = \lim\limits_{x \to 0^-} (\sqrt{-x+1})$

↳ Spiegelung an y-Achse

Verschiebung um 1 nach links

③ $f(0) = \sqrt{0} = 0$ $\Rightarrow$ unstetigkeit, da $0 \neq 1$

f.) $f(x) = \text{Sign}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$



$$\left.\begin{array}{l} \lim\limits_{x \to 0^+} (f(x)) = 1 \\ \lim\limits_{x \to 0^-} (f(x)) = -1 \\ f(0) = 0 \end{array}\right\} \Rightarrow \text{unstetig}$$

g.) $f(x) = \begin{cases} \frac{x+2}{x^2+x-2}, & x \neq -2 \quad ① \\ 1/3, & x = -2 \quad ② \end{cases}$ $① \to \frac{-2+2}{4-2-2} = \frac{0}{0}$ ↯

① Nullstellen Nenner: $x^2 + x - 2 \stackrel{!}{=} 0 \Rightarrow x = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 2}$

$\Leftrightarrow x = -\frac{1}{2} \pm \sqrt{\frac{9}{4}}$

$\Leftrightarrow x = -\frac{1}{2} \pm \frac{3}{2}$

$\Leftrightarrow x = 1 \vee x = -2$

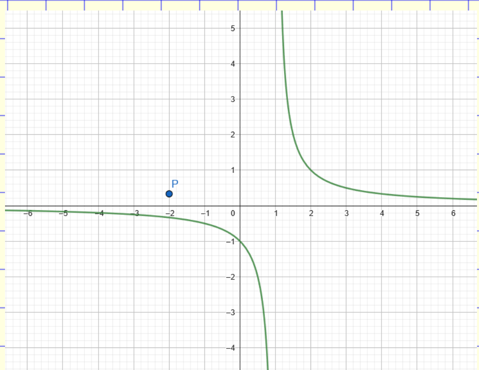
Linearfaktoren:

$x^2 + x - 2 = (x-1)(x+2)$

$f(x)$ ① $\Rightarrow \frac{x+2}{x^2+x-2} = \frac{x+2}{(x-1)(x+2)} = \frac{1}{x-1}$

$\Rightarrow$ hebbare Def. Lücke bei $x = -2$

① $\lim\limits_{x \to -2^{\pm}} \left(\frac{1}{x-1}\right) = -\frac{1}{3}^{\mp}$ ② $f(-2) = \frac{1}{3}$



$-\frac{1}{3} \neq \frac{1}{3} \Rightarrow$ unstetigkeit h.) $f(x) = \dfrac{2x^2+4x-6}{x^2+x-2} = \dfrac{2(x^2+2x-3)}{x^2+x-2}$ ← ① ← ②

① $2x^2+4x-6 \overset{!}{=} 0 \quad | \cdot \frac{1}{2}$

$x^2+2x-3=0$

pq-Formel: $x = -1 \pm \sqrt{4} \Rightarrow x = -1 \lor x = 2$

$\Rightarrow (x+1)(x-2)$

② $x^2+x-2 = 0$

$x = -\frac{1}{2} \pm \sqrt{\frac{1}{4}+2} \Rightarrow x = 1 \lor x = -2$

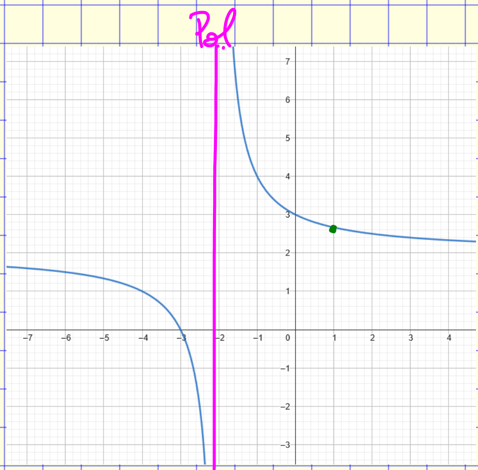
$\Rightarrow (x-1)(x+2)$

$\Rightarrow f(x) = \dfrac{2\cdot(x-1)(x+3)}{(x-1)(x+2)} = \dfrac{2(x+3)}{(x+2)}$

- hebbare Def. Lücke bei $x = 1$
- Polstelle bei $x = -2$

$\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} \dfrac{2(x+3)}{(x+2)} = \dfrac{2(1+3)}{1+2} = \dfrac{2+6}{3} = \dfrac{8}{3}$

stetig auf $\mathbb{R} \setminus \{-2\}$



iii)

a.) $\dfrac{2-x^2}{1-x} \Rightarrow$ Kritische Stelle bei 1

$\lim\limits_{x \to \pm\infty} \dfrac{2-x^2}{1-x} \overset{\cdot \frac{1}{x^2}}{=} \lim\limits_{x \to \pm\infty} \left(\dfrac{\underbrace{\frac{2}{x^2}}_{\to 0} - 1}{\underbrace{1/x^2}_{\to 0} - \underbrace{\frac{1}{x}}_{\to 0}} \right) = \lim\limits_{x \to \pm\infty} \left(\dfrac{-1}{\underbrace{\frac{1}{x^2}}_{\to 0} - \underbrace{\frac{1}{x}}_{\to 0}} \right) = \pm\infty$

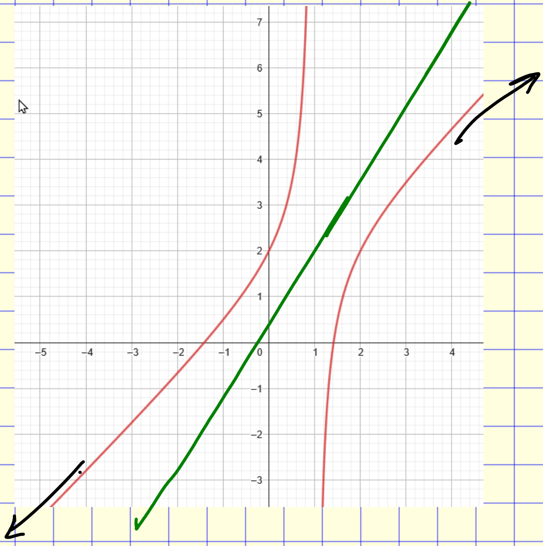
< 0 für $+\infty$

> 0 für $-\infty$

Zähler $\geq$ Nenner:

$$\frac{2-x^2}{1-x} \overset{\cdot(-1)}{=} \frac{x^2-2}{x-1}$$

$$(x^2-2) : (x-1) = \cancel{x} + 1$$
$$-(x^2-x)$$
$$x-2$$
$$-(x-1)$$
$$-1$$



$$\Rightarrow \frac{x^2-2}{x-1} = (x+1) - \underbrace{\frac{1}{x-1}}_{\text{vernachlässigen}}$$

Asympt.

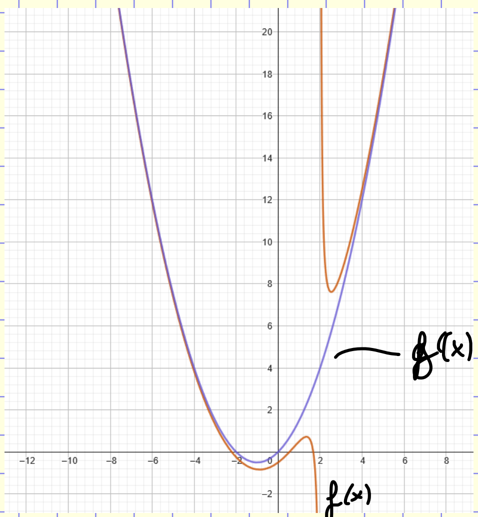
$$\lim_{x\to\pm\infty} \frac{2-x^2}{1-x} = \lim_{x\to\pm\infty}\left(x+1-\underbrace{\frac{1}{x-1}}_{\to 0}\right) = \lim_{x\to\pm\infty}(x+1)$$

$$= \lim_{x\to\pm\infty}(x)$$

- NS: $x = \pm\sqrt{2}$
- Polstelle $x = 1$

b.) $f(x) = \dfrac{\frac{x^3}{2} - 2x + 1}{x-2}$ → Asympt. $x = 2$

$$\lim_{x\to\pm\infty} f(x) = \frac{\frac{x^2}{x} - 2 + \frac{1}{x} \nearrow 0}{1 - \frac{2}{x} \searrow 0} = \lim_{x\to\pm\infty} \frac{\frac{x^2}{2}-2}{1} = \lim\left(\frac{x^2}{2}-2\right) = \infty$$



$f(x) = \frac{x^2}{2} - 2$

$f(x)$

Polynomdivision: $\left(\frac{x^3}{2} - 2x + 1\right) : (x-2) = \frac{x^2}{2} + x + \frac{1}{x-2}$
$$-\left(\frac{x^3}{2} - x^2\right)$$
$$x^2 - 2x + 1$$
$$-(x^2 - 2x)$$
$$1$$

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^2}{2} + x + \underbrace{\frac{1}{x-2}}_{\downarrow\, 0} = \frac{x^2}{2} + x$$

$$\frac{x^2}{2} + x = \frac{1}{2}(x+1)^2 - \frac{1}{2}$$

$y = -\frac{1}{2}$

$x = -1$

- Polstelle bei $x = 2$
- $\lim\limits_{x \to \pm\infty} f(x) = \infty$