Vorrechnen Blatt 9

Aufgabe 1:

$$DQ = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$DQ = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon} \leftarrow$$

i) a) $f(x) = x^2 + 1$

$$f'(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{(x+\varepsilon)^2 \cancel{+1} - x^2 \cancel{-1}}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \frac{\cancel{x^2} + 2x\varepsilon \cancel{- x^2} + \varepsilon^2}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{2x\cancel{\varepsilon}}{\cancel{\varepsilon}} + \frac{\varepsilon^{\cancel{2}}}{\cancel{\varepsilon}} = \lim_{\varepsilon \to 0} 2x + \varepsilon$$

$$= 2x$$

b) $f(x) = x^3 - 5x^2 + 6x + 2$

$$f'(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon} = \lim_{\varepsilon \to 0} \left[(x+\varepsilon)^3 - 5(x+\varepsilon)^2 + 6(x+\varepsilon) + 2 - x^3 + 5x^2 - 6x - 2 \right] \cdot \frac{1}{\varepsilon}$$

$$\left(x^3 + 3x^2\varepsilon + 3x\varepsilon^2 + \varepsilon^3 = (x+\varepsilon)^3 \right)$$

$$\begin{array}{ccccccccccc} & & & & 1 & & & & & & (a+b)^0 \\ & & & 1 & & 1 & & & & & (a+b)^1 \\ & & 1 & & 2 & & 1 & & & & (a+b)^2 \\ & 1 & & 3 & & 3 & & 1 & & & (a+b)^3 \\ 1 & & 4 & & 6 & & 4 & & 1 & & (a+b)^4 \end{array}$$

Pascal'sches Dreieck

$$= \lim_{\varepsilon \to 0} \left[\cancel{x^3} + 3x^2\varepsilon + 3x\varepsilon^2 + \varepsilon^3 \cancel{- 5x^2} - 10x\varepsilon - 5\varepsilon^2 \cancel{+ 6x} + 6\varepsilon \cancel{+ 2} \cancel{- x^3} \cancel{+ 5x^2} \cancel{- 6x} \cancel{- 2} \right] \cdot \frac{1}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \left[3x^2\varepsilon + 3x\varepsilon^2 + \varepsilon^3 - 10x\varepsilon - 5\varepsilon^2 + 6\varepsilon \right] \cdot \frac{1}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \underset{\downarrow\atop 3x^2}{3x^2} + \underset{\downarrow\atop 0}{3x\varepsilon} + \underset{\downarrow\atop 0}{\varepsilon^2} - \underset{\downarrow\atop -10x}{10x} - \underset{\downarrow\atop 0}{5\varepsilon} + \underset{\downarrow\atop 6}{6} = 3x^2 - 10x + 6$$

c) $f(x) = \frac{1}{x}$

$$f'(x) = \lim_{\varepsilon \to 0} \frac{\frac{1}{x+\varepsilon} - \frac{1}{x}}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \cdot \left[\frac{\cancel{x} - \cancel{x} - \varepsilon}{(x+\varepsilon)x} \right] = \lim_{\varepsilon \to 0} \frac{1}{\cancel{\varepsilon}} \frac{-\cancel{\varepsilon}}{x^2 + \cancel{\varepsilon x}} = -\frac{1}{x^2}$$

d) $f(x) = \frac{x}{x-1}$

$$f'(x) = \lim_{\varepsilon \to 0} \left(\frac{x+\varepsilon}{x+\varepsilon-1} - \frac{x}{x-1} \right) \cdot \frac{1}{\varepsilon} = \lim_{\varepsilon \to 0} \left(\frac{(x+\varepsilon)(x-1) - x \cdot (x+\varepsilon-1)}{(x+\varepsilon-1)(x-1)} \right) \cdot \frac{1}{\varepsilon}$$

$$= \lim_{\varepsilon \to 0} \left(\frac{\cancel{x^2} \cancel{- x} \cancel{+ \varepsilon x} - \cancel{\varepsilon} \cancel{- x^2} \cancel{- \varepsilon x} \cancel{+ x}}{x^2 + \varepsilon x - x - x - \varepsilon + 1} \right) \cdot \frac{1}{\cancel{\varepsilon}} = \lim_{\varepsilon \to 0} \left(-\frac{1}{x^2 - 2x + 1 + \varepsilon \cdot \cancel{(x-1)}^{\to 0}} \right)$$

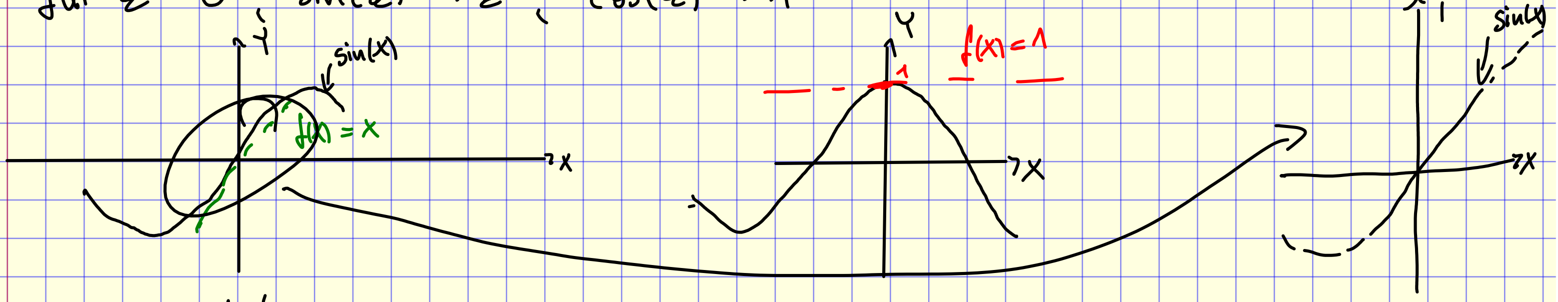
$$= -\frac{1}{x^2 - 2x + 1} = -\frac{1}{(x-1)^2}$$

QR: $f'(x) = \frac{\cancel{x} - 1 \cancel{- x}}{(x-1)^2} = \frac{-1}{(x-1)^2}$

e) $f(x) = \sin(x)$

$$f'(x) = \lim_{\varepsilon \to 0} \frac{\sin(x+\varepsilon) - \sin(x)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \cdot \left(\sin(x) \underbrace{\cos(\varepsilon)}_{\to 1} + \underbrace{\sin(\varepsilon)}_{\to \varepsilon} \cos(x) - \sin(x) \right)$$

für $\varepsilon \to 0$, $\sin(\varepsilon) \to \varepsilon$, $\cos(\varepsilon) \to 1$



$$f'(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\cancel{\sin(x)} + \varepsilon \cos(x) \cancel{- \sin(x)} \right) = \lim_{\varepsilon \to 0} \frac{\cancel{\varepsilon}}{\cancel{\varepsilon}} \cos(x) = \cos(x) \,//$$

d) $f(x) = \cos(x)$

$$f'(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (\cos(x+\varepsilon) - \cos(x)) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(\cancel{\cos(x)} \underbrace{\cos(\varepsilon)}_{\to 1} - \sin(x) \underbrace{\sin(\varepsilon)}_{\to \varepsilon} \cancel{- \cos(x)} \right)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\cancel{\varepsilon}} \cdot \cancel{\varepsilon} \cdot (-\sin(x)) = -\sin(x) \,//$$

$x \cdot \cancel{6}^{x-1}$

ii) i) $6^x x^6 \sin(x) = f(x) = g(x) \cdot h(x) \cdot u(x)$ $\qquad g'(x) = 6^x \ln(6)$

$f'(x) \overset{PP}{=} g'(x) \cdot h(x) \cdot u(x) + g(x) \cdot h'(x) \cdot u(x) + g(x) \cdot h(x) \cdot u'(x)$ $\qquad h'(x) = 6x^5$

$= 6^x \ln(6) \cdot x^6 \cdot \sin(x) + 6^x 6x^5 \sin(x) + 6^x x^6 \cos(x) \,//$ $\qquad u'(x) = \cos(x)$

j) $\underbrace{f(x) = \ln(\overbrace{\sqrt{e^x + x^4}}^{g(x)})}_{f(x) = \ln(g(x))}$

$g(x) = (e^x + x^4)^{\frac{1}{2}}$

$g'(x) = \frac{1}{2}(e^x + x^4)^{-\frac{1}{2}}(e^x + 4x^3)$

$= \dfrac{(e^x + 4x^3)}{2\sqrt{e^x + x^4}}$

$f'(x) = f'(g(x)) \cdot g'(x)$

$= \dfrac{1}{\sqrt{e^x + x^4}} \cdot \dfrac{(e^x + 4x^3)}{2\sqrt{e^x + x^4}}$

$= \dfrac{e^x + 4x^3}{2e^x + 2x^4} \,//$

$f(x) = \ln(x)$, $f'(x) = \frac{1}{x}$

$\sin^2(x) + \cos^2(x) = 1$

p) $f(x) = \sin((3-x^2)^2) + \cos((3-x^2)^2) + \overbrace{\sin^2(3-x^2) + \cos^2(3-x^2)}^{=1}$

$f'(x) = \cos((3-x^2)^2) \cdot (2(3-x^2) \cdot (-2x))$

$\quad - \sin((3-x^2)^2)(2 \cdot (3-x^2) \cdot (-2x))$

$= -4x(3-x^2)(\cos((3-x^2)^2) - \sin((3-x^2)^2)) = -4x(3-x^2)\sqrt{2} \cdot \sin\left(\frac{\pi}{4} - (3-x^2)^2\right) \,//$

r) $f(x) = \left(\frac{x^2+1}{x^2+3}\right)^{\sin(2x)}$ Erinnerung: b^x

$$= e^{\ln\left(\left(\frac{x^2+1}{x^2+3}\right)^{\sin(2x)}\right)} = e^{\sin(2x)\ln\left(\frac{x^2+1}{x^2+3}\right)}$$

$\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$

$$= e^{\sin(2x)\cdot\ln(x^2+1) - \sin(2x)\ln(x^2+3)}$$

$$f'(x) = \left(e^{\sin(2x)\cdot\ln(x^2+1) - \sin(2x)\ln(x^2+3)}\right)\cdot\left(2\cos(2x)\ln(x^2+1) + \sin(2x)\frac{2x}{x^2+1} - 2\cos(2x)\ln(x^2+3) - \sin(2x)\frac{2x}{x^2+3}\right]$$

$$= \left(\frac{x^2+1}{x^2+3}\right)^{\sin(2x)}\left(2\cos(2x)\ln\left(\frac{x^2+1}{x^2+3}\right) + \sin(2x)\cdot\underbrace{\left(\frac{1}{x^2+1} - \frac{1}{x^2+3}\right)\cdot 2x}_{\frac{2x}{(x^2+1)(x^2+3)}}\right)$$

$$= \left(\frac{x^2+1}{x^2+3}\right)^{\sin(2x)}\left(2\cos(2x)\ln\left(\frac{x^2+1}{x^2+3}\right) + \sin(2x)\frac{2x}{(x^2+1)(x^2+3)}\right) \quad //$$

a) $2x$ d) $\frac{1}{2}\cdot\frac{1}{\sqrt{x+1}}\cdot(x^2+1) + (x+1)^{\frac{1}{2}}\cdot 2x$ g) $\frac{-x^2+5}{(x^2+5)^2}$ QR

b) $8x^3 - 9x^2 + 7$ e) $3x^2 + 4x - 4$ h) $(4 - 6\sin(x)\cos(x))\cdot 5\cdot(4x + 3\cos^2(x))^4$

c) $\frac{1}{3\sqrt[3]{(x+4)^2}}$ f) $5x^4 + \frac{4}{x^3}$ k) $\frac{(4x^2+5)^{-\frac{3}{2}}}{2\cdot\sqrt{2x-3}}(-8x^2 + 24x + 10)$

l) $6\cdot(e^{2x+3} + 4x + 8)^5\cdot(e^{2x+3}\cdot 2 + 4)$ m) $2\sin(x)\cos(x)\cdot(\ln(x) + 2) + \frac{\sin^2(x)+1}{x}$

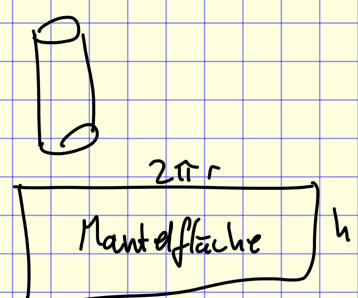
n) $e^{\frac{x^2+3}{x^2+1}}\cdot\left(-\frac{4x}{(x^2+1)^2}\right)$ o) $\frac{1}{2}\cdot\frac{1}{x^3\cdot e^{2x}\cdot\ln(x)}\cdot\left(3x^2\cdot e^{2x}\cdot\ln(x) + x^3\cdot 2e^{2x}\cdot\ln(x) + x^{\frac{2}{3}}e^{2x}\cdot\frac{1}{x}\right)$

q) $\frac{\sin(ax+b)(3-x^2) + \cos(ax+b)(ax^3+3ax)}{(x^2+3)^2}$

Aufgabe 2: $V = X$ $O(r,h)$ muss minimal werden

$\pi r^2 h = X$ Nebenbedingung

$O(r,h) = 2\pi r^2 + 2\pi r h$ Hauptbedingung

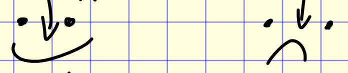


$h = \frac{X}{\pi r^2}$

$\Leftrightarrow$ in $O(r,h)$ einsetzen $O(r) = 2\pi r^2 + 2\pi r\cdot\frac{X}{\pi r^2} = 2\pi r^2 + \frac{2X}{r}$

$O'(r) = 4\pi r - \frac{2X}{r^2} \overset{!}{=} 0$ muss gleich sein

Tiefpunkt   Hochpunkt



positiv   negativ

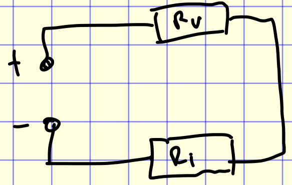
$\Leftrightarrow 4\pi r = \frac{2X}{r^2} \quad |\cdot r^2 \quad |:4\pi$ $O''(r) = 4\pi + \frac{4X}{r^3}$

$\Leftrightarrow r^3 = \frac{1}{2}\frac{X}{\pi} \quad |\sqrt[3]{\ }$ $O''\left(\sqrt[3]{\frac{1}{2}\frac{X}{\pi}}\right) = 4\pi + \frac{4X}{\frac{1}{2}\frac{X}{\pi}} = 4\pi + 8\pi = 12\pi > 0 \Rightarrow$ Min.

$\Leftrightarrow r = \sqrt[3]{\frac{1}{2}\frac{X}{\pi}}$ $h = \frac{X}{\pi\cdot\left(\sqrt[3]{\frac{1}{2}\frac{X}{\pi}}\right)^2}$

Aufgabe 3:

ges: $\frac{dP}{dR_V} \overset{!}{=} 0 \;(\Rightarrow R_{V_1} \quad \frac{d^2P(R_{V_1})}{dR_V^2} < 0)$



$$P = U_V \cdot I = I \cdot R_V \cdot I \qquad I = \frac{U}{R_i + R_V}$$
$$= I^2 \cdot R_V$$
$$= \frac{U^2}{(R_i + R_V)^2} \cdot R_V$$

$$\frac{dP}{dR_V} = \frac{U^2}{(R_V+R_i)^2} - \frac{2U^2 R_V}{(R_V+R_i)^3}$$
$$= -\frac{U^2(R_V - R_i)}{(R_V+R_i)^3} \qquad \frac{dP}{dR_V} \overset{!}{=} 0$$

$$\Leftrightarrow -\frac{U^2(R_V - R_i)}{(R_V+R_i)^3} = 0 \quad | \cdot \left(\frac{1}{U^2}\right) \quad | \cdot (-1) \quad | \cdot (R_V + R_i)^3$$

$$\Leftrightarrow R_V - R_i = 0 \Leftrightarrow R_V = R_i \,//$$

Aufgabe 4:

i) a) $f(x) = 8 - 7x \qquad f'(x) = -7 < 0 \quad \forall x \in \mathbb{R}$ ← „für alle"

($\exists$ ← „für min. ein" $\quad \exists_1$ ← „für genau ein")

↳ streng monoton fallend

$f'(x) \neq 0 \;\; \forall x \in \mathbb{R} \Rightarrow$ keine lokalen Extrema

$W = \mathbb{R}$

b) $f(x) = x^2 + 3x - 28 \qquad f'(x) = 2x + 3$ Monotonie: $f'(x) < 0 \Rightarrow 2x + 3 < 0$

$\Leftrightarrow x < -\frac{3}{2}$

$x < -\frac{3}{2}$: streng monoton fallend

$x \leq -\frac{3}{2}$: monoton fallend

$x > -\frac{3}{2}$: streng monoton steigend

$x \geq -\frac{3}{2}$: monoton steigend

$f'(x) > 0 \Rightarrow x > -\frac{3}{2}$

$f'(x) = 0 \Rightarrow x = -\frac{3}{2}$

Extrema: $x = -\frac{3}{2} \qquad f''(x) = 2 > 0 \;\forall x \in \mathbb{R} \Rightarrow f''(-\frac{3}{2}) > 0 \rightarrow$ Minimum

$f(-\frac{3}{2}) = -\frac{121}{4}, \quad f(x) \geq -\frac{121}{4} \qquad W = [-\frac{121}{4}, \infty)$

c) $f(x) = x^3 + 27$

$f'(x) = 3x^2 \quad f'(x) = 0 \Leftrightarrow x = 0 \qquad f'(x) > 0 \quad \forall x \neq 0$

↳ $f(x)$ streng monoton steigend auf $\mathbb{R} \setminus \{0\}$

$f''(x) = 6x, \; f'''(x) = 6, \; f^{(n)}(x) = 0$ f. $n > 3$

$$f''(x) = \begin{cases} < 0, & x < 0 \quad \text{linksgekr.} \\ > 0, & x > 0 \quad \text{rechtsgekr.} \\ 0, & x = 0 \end{cases}$$

$\Rightarrow f(x)$ hat bei $x=0$ einen Sattelpunkt $(0, 27)$ $W = \mathbb{R}$

d) $f(x) = x^3 - 27x$, $f'(x) = 3x^2 - 27$

Monotonie: $3x^2 - 27 = 0 \Leftrightarrow x^2 = 9 \Leftrightarrow x = \pm 3$

$\lim\limits_{x \to \pm\infty} f'(x) = \lim\limits_{x \to \pm\infty} 3x^2 - 27 = +\infty$ $f'(x) < f.\ x \in (-3; 3)$

$f(x)$ streng mon. fal. f. $x \in (-3; 3)$

$f(x)$ str. mon. steigend f. $x \in (-\infty; -3)$, $x \in (3, \infty)$

Extrema: $f''(x) = 6x$

$\hookrightarrow f''(-3) < 0 \rightarrow$ Hochpunkt

$f''(+3) > 0 \rightarrow$ Tiefpunkt