Integralrechnung II

Substitution

wdhl: Differentiation – Kettenregel

$$[f(g(x))]' = f'(g(x))\, g'(x) \quad ; \quad \boxed{\frac{d}{dx} f(g(x)) = \frac{df}{\cancel{dg}} \cdot \frac{\cancel{dg}}{dx}}$$

Bsp. $y(x) = \sqrt{x^2+x+5}$; $f(g) = \sqrt{g}$; $g = x^2+x+5$

$$\left.\begin{aligned} \frac{df}{dg} &= \frac{d}{dg} g^{1/2} = \frac{1}{2} g^{-1/2} = \frac{1}{2\sqrt{g}} \\ \frac{dg}{dx} &= 2x+1 \end{aligned}\right\} \; y'(x) = \frac{1}{2\sqrt{x^2+x+5}} \cdot (2x+1)$$

Beim Integrieren: (zuerst ohne Grenzen)

Formal $$\int f(g(x)) \cdot g'(x)\,dx = \int f(g)\,dg \quad ; \quad \int f(g) \cdot \frac{dg}{dx}\,dx = \int f(g)\,dg$$

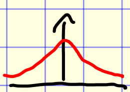
Bsp. i) $\int x \cos(x^2)\,dx$; $g(x) = x^2$; $\frac{dg}{dx} = 2x$; $dg = 2x\,dx$; $\frac{dg}{2x} = dx$

$$= \int x \cos(x^2) \frac{dg}{2x} = \int \frac{1}{2} \cos g \, dg = \frac{1}{2} \sin g + C' = \underline{\underline{\frac{1}{2} \sin x^2 + C'}}$$

passt gerade

ii) $\int e^{-x^2}\,dx$; $g(x) = x^2$; $dg = 2x\,dx$; $\frac{dg}{2x} = dx$; $\frac{dg}{2\sqrt{g}} = dx$

$$= \int e^{-x^2}\,dx = \int e^{-g} \cdot \frac{1}{2\sqrt{g}}\,dg = \frac{1}{2} \int \frac{e^{-g}}{\sqrt{g}}\,dg$$; hilft nicht.



$\int e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}\, \mathrm{erf}(x)$; erf(x) – Error-Funktion, Geogebra

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mit Grenzen:

$$\int_0^2 x \cos(x^2+1)\,dx \quad ; \quad g(x) = x^2+1 \; ; \; dg = 2x\,dx \; ; \; \frac{dg}{2x} = dx$$

$5 = 2^2+1$

$$\int_0^2 x \cos(x^2+1)\,dx = \frac{1}{2} \int_1^5 \cos g \, dg = \left[\frac{1}{2} \sin g\right]_1^5 = \frac{\sin 5}{2} - \frac{\sin 1}{2}$$

$g = x^2+1 = 0^2+1 = 1$

$$= \left[\frac{1}{2} \sin(x^2+1)\right]_0^2$$

$$\int x\cos(x^2+1)\,dx = \int \frac{1}{2}\cos g\,dg = \frac{1}{2}\sin g + C \; ; \; g = x^2+1$$
$$= \left[\frac{1}{2}\sin(x^2+1)\right]_0^2$$

partielle Integration: aus Produktregel beim Diff.

$$[f(x)\cdot g(x)]' = f'\cdot g + g'\cdot f \quad \Big| \int\text{- Operator}$$

$$\int [f(x)\cdot g(x)]' = \int f'g + \int g'f$$

$$\int f'g = [f\cdot g] - \int g'f$$; g - ableiten, f aber integrieren

Bsp.

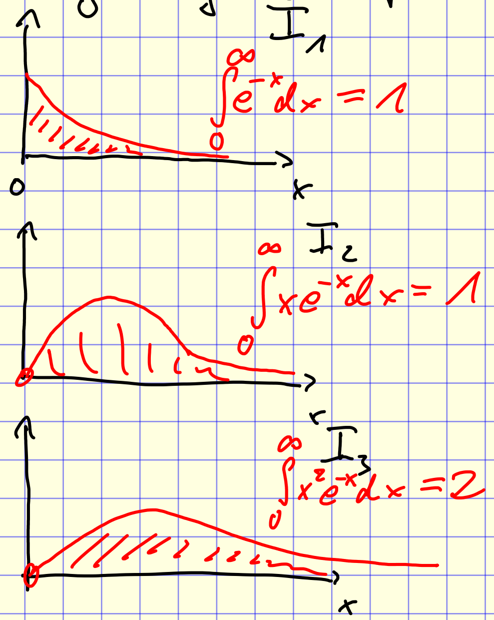
$$\int_0^1 \underset{g}{x}\,\underset{f'}{e^{-x}}\,dx = \left[-\underset{f}{e^{-x}}\cdot \underset{g}{x}\right]_0^1 - \int_0^1 1\cdot(-e^{-x})\,dx$$
$$= \left[-e^{-x}\cdot x\right]_0^1 + \int_0^1 e^{-x}\,dx = \left[-e^{-x}x\right]_0^1 + \left[-e^{-x}\right]_0^1$$
$$= -e^{-1} - 0 + (-e^{-1}) - (-1) = 1 - \frac{2}{e}$$

$$\int \underset{g}{x}\,\underset{f'}{e^{-x}}\,dx = -\underset{f}{e^{-x}}\cdot \underset{g}{x} - \int -\underset{f}{e^{-x}}\cdot \underset{g'}{1}\,dx = \underline{\underline{-e^{-x}\cdot x - e^{-x}}}$$

$$\int_0^1 x e^{-x}\,dx = \left[e^{-x}(-x-1)\right]_0^1 = -\frac{2}{e} - (-1) = 1 - \frac{2}{e}$$

Beispiel für partielle Integration ⟹ verallgemeinerte Fakultät

Betrachte
$$I_n = \int_0^\infty \underbrace{e^{-x}}_{f'} \cdot \underbrace{x^{n-1}}_{g}\, dx \; ; \; n \geq 1$$



$$I_n = [fg]_0^\infty - \int_0^\infty f g'$$

$$I_n = \underbrace{\left[-e^{-x} \cdot x^{n-1}\right]_0^\infty}_{0-0=0} + \int_0^\infty e^{-x} (n-1) x^{n-2}\, dx$$

$$I_n = (n-1) \cdot \int_0^\infty e^{-x} x^{n-2}\, dx = (n-1) \cdot I_{n-1}$$

z.B. $I_3 = (3-1) \cdot I_2 = 2 \cdot I_2 = 2 \cdot 1 \cdot I_1 = 2 \cdot 1 \cdot 1$

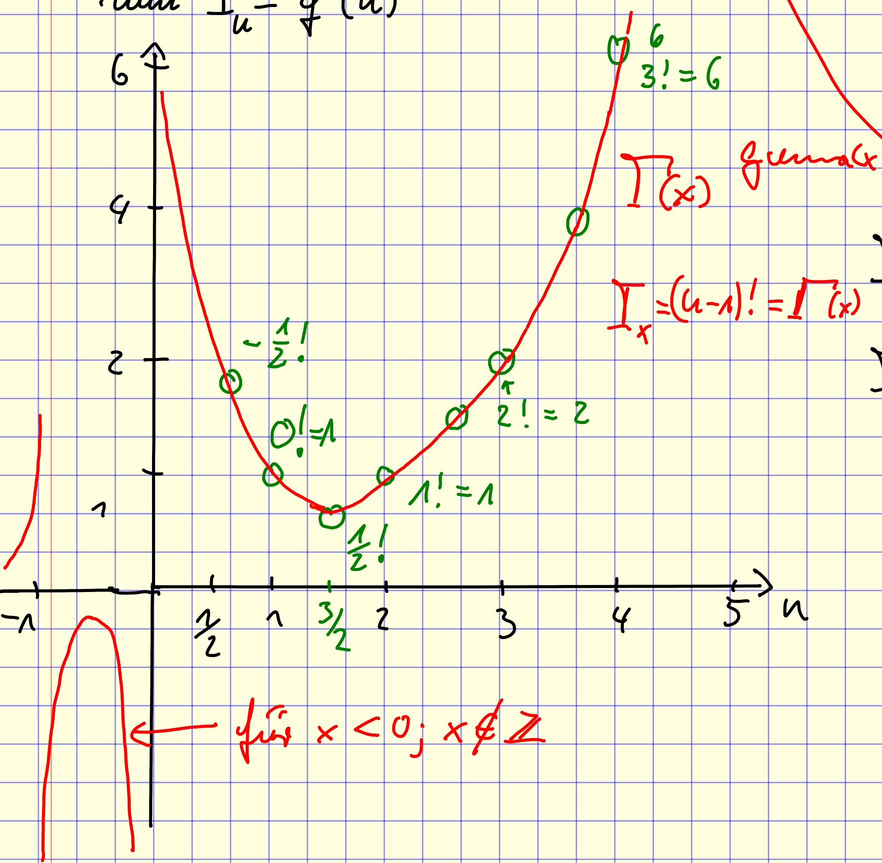
$I_4 = (4-1) \cdot I_3 = 3 \cdot I_3 = 3 \cdot 2 \cdot 1 \cdot 1 \; ; \; I_5 = 4 \cdot 3 \cdot 2 \cdot 1 \; ; \; I_6 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

Rekursionsvorschrift

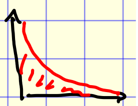
$$\boxed{I_n = (n-1) I_{n-1}}$$

$I_n = (n-1)!$ Fakultät, wenn I_n auch für $n \notin \mathbb{N}$

wenn $I_n = f(n)$



z.B. $I_{1/2 = n} = \int_0^\infty e^{-x} \cdot x^{\frac{1}{2}-1} dx = \int_0^\infty \frac{e^{-x}}{\sqrt{x}} dx$



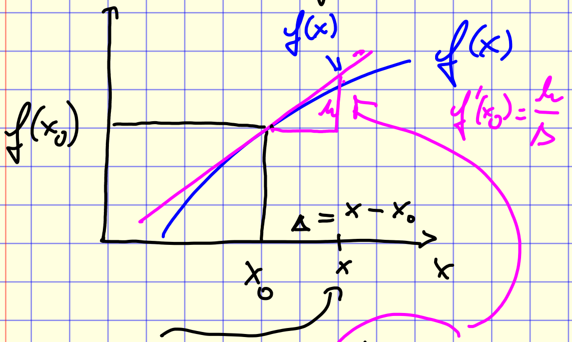
$= \sqrt{\pi}$, siehe erf(x)

$= 1.77\ldots$

$I_{3/2 = n} = \left(\frac{3}{2} - 1\right) \cdot I_{1/2} = \frac{1}{2}\sqrt{\pi} = 0{,}886 = \frac{1}{2}!$

$I_{5/2 = n} = \left(\frac{5}{2} - 1\right) I_{3/2} = \ldots$

Ausblick: Differentialrechnung / Taylorentwicklung



$$f(x) = f(x_0) + f'(x_0)\cdot\Delta + ???$$

Ziel: $f(x)$ als Polynom in x; $p(x)$

Idee: wir bauen Polynome so, dass gilt

$$p(x_0) = f(x_0)$$
$$p'(x_0) = f'(x_0)$$
$$p''(x_0) = f''(x_0)$$

$$f(x) = \underbrace{\frac{f(x_0)}{0!}(x-x_0)^0 + \frac{f'(x_0)}{1!}(x-x_0)^1 + \overbrace{\frac{f''(x_0)}{2!}}^{\text{Zahl}}\underbrace{(x-x_0)^2}_{\text{Polynom}} + \dots}_{p(x)}$$

checken

$$p(x = x_0) = f(x_0)$$
$$p'(x)\Big|_{x=x_0} = f'(x_0)$$
$$p''(x)\Big|_{x=x_0} = \frac{f''(x_0)}{2!}\cdot 2\cdot 1$$

da $\frac{d^2}{dx^2}(x-x_0)^2 = \frac{d}{dx}(2(x-x_0)) = 2$

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{d^n f(x)}{dx^n}\Big|_{x=x_0} \cdot \frac{(x-x_0)^n}{n!}}$$

oft wird $x_0 = 0$ $\qquad x_0 = 0 \Rightarrow x - x_0 = x$

z.B. $\sin(x) = 0 + \frac{d}{dx}\sin(x)\Big|_{x=0} \cdot \frac{(x-x_0)^1}{1!} + \frac{1}{2!}\frac{d^2}{dx^2}\sin x\Big|_{x=0} \cdot x^2 + \dots$

$$\sin(x) = 1\cdot x - \frac{1}{3!}\cdot x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

analog: $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots = \frac{d}{dx}\sin(x)$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

z.B. $e^1 = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots = 2{,}718281828\dots$

so rechnet der Taschenrechner $\sin x$, $\cos x$, e^x, ... aus.