Vorrechnen Blatt 11

Aufgabe 1

i) a) $\int \frac{1}{e^{3x}+5}\,dx \qquad u = e^{3x} \Leftrightarrow \frac{du}{dx} = 3e^{3x} \quad | \cdot dx \,|: 3e^{3x}$

$\Leftrightarrow dx = \frac{1}{3}\frac{1}{e^{3x}}\,du$

$= \int \frac{1}{u+5}\,\frac{1}{3}\,\frac{1}{e^{3x}}\,du = \int \frac{1}{u+5}\cdot\frac{1}{3}\,\frac{1}{u}\,du$

$= \frac{1}{3}\cdot\int\frac{1}{(u+5)u}\,du$

$\frac{1}{\cancel{(u+5)u}} = \frac{A}{u+5} + \frac{B}{u} = \frac{1}{5}\cdot\frac{1}{u} - \frac{1}{5}\cdot\frac{1}{u+5}$

$= \frac{A\cdot u + Bu + 5B}{\cancel{(u+5)u}}$

$= \frac{1}{3}\cdot\left(\frac{1}{5}\cdot\int\frac{1}{u}\,du - \frac{1}{5}\int\frac{1}{u+5}\,du\right) \qquad 0\cdot u^1 + 1\cdot u^0 = u^1\cdot(A+B) + 5B\cdot u^0$

$= \frac{1}{3}\cdot\left(\frac{1}{5}\ln(|u|) - \frac{1}{5}\ln(|u+5|)\right) \Leftrightarrow A+B=0 \qquad 5B = 1 \quad |:5$

$\Leftrightarrow A + \frac{1}{5} = 0 \qquad \Leftrightarrow B = \frac{1}{5}$

$\overset{RS}{=} \frac{1}{15}\ln(|e^{3x}|) - \frac{1}{15}\ln(|e^{3x}+5|) \qquad \Leftrightarrow A = -\frac{1}{5}$

$= \ln\left(\sqrt[15]{\frac{e^{3x}}{e^{3x}+5}}\right) + C$

b) $\int \underset{f}{e^x}\,\underset{g}{\cos(x)}\,dx = \underset{f}{e^x}\cdot\underset{G}{\sin(x)} - \int \overset{f'}{e^x}\,\overset{G}{\sin(x)}\,dx$

$\int$ nochmal PI: $f(x) = e^x$, $g(x) = \sin(x)$

$= e^x\sin(x) - \left(e^x(-\cos(x)) - \int e^x(-\cos(x))\,dx\right)$

$= e^x\sin(x) + e^x\cos(x) - \int e^x\cos(x)\,dx \quad | + \int e^x\cos(x)\,dx$

$\Leftrightarrow 2\cdot\int e^x\cos(x)\,dx = e^x\sin(x) + e^x\cos(x) \quad |:2$

$\Leftrightarrow \int e^x\cos(x)\,dx = \frac{1}{2}(e^x\sin(x) + e^x\cos(x)) + C$

c) $\int\sqrt{\frac{1-x}{1+x}}\cdot\frac{1}{(1-x)^2}\,dx \qquad u = \frac{1+x}{1-x} \quad \frac{du}{dx} = \frac{2}{(1-x)^2} \quad |\cdot dx\,|: \frac{2}{(1-x)^2}$

$\Leftrightarrow dx = \frac{1}{2}\cdot(1-x)^2\cdot du$

$= \int\frac{1}{\sqrt{u}}\cdot\frac{1}{\cancel{(1-x)^2}}\cdot\frac{1}{2}\cdot\cancel{(1-x)^2}\cdot du = \frac{1}{2}\cdot\int\frac{1}{\sqrt{u}}\,du = \sqrt{u} \overset{}{\underset{RS}{=}} \sqrt{\frac{1+x}{1-x}} + C$

d) $\int \underset{g}{x^2}\,\underset{f}{\ln(x)}\,dx \underset{PI}{=} \frac{1}{3}x^3\ln(x) - \int\frac{1}{3}x^{\cancel{3}^2}\frac{1}{\cancel{x}}\,dx = \frac{1}{3}x^3\ln(x) - \frac{1}{9}x^3$

$= \frac{1}{3}x^3\left(\ln(x) - \frac{1}{3}\right) + C$

e) $\int \underset{g}{\sin(4x)}\,\underset{f}{\sin(6x)}\,dx = -\frac{1}{4}\cos(4x)\sin(6x) + \frac{6}{4}\int \underset{g}{\cos(4x)}\,\underset{f}{\cos(6x)}\,dx$

$= -\frac{1}{4}\cos(4x)\sin(6x) + \frac{6}{4}\left(\frac{1}{4}\sin(4x)\cos(6x) + \frac{6}{4}\int \sin(4x)\sin(6x)\,dx\right)$

$= -\frac{1}{4}\cos(4x)\sin(6x) + \frac{3}{8}\sin(4x)\cos(6x) + \frac{9}{4}\int \sin(4x)\sin(6x)\,dx \quad \Big| - \frac{9}{4}\int \sin(4x)\sin(6x)\,dx$

$\Leftrightarrow -\frac{5}{4}\cdot\int \sin(4x)\sin(6x)\,dx = -\frac{1}{4}\cos(4x)\sin(6x) + \frac{3}{8}\sin(4x)\cos(6x) \quad \Big| : \left(-\frac{5}{4}\right)$

$\Leftrightarrow \int \sin(4x)\sin(6x)\,dx = \frac{1}{5}\cos(4x)\sin(6x) - \frac{3}{10}\sin(4x)\cos(6x) + C$

f) $\int \sin(x)e^{\cos(x)}\,dx = \int \cancel{\sin(x)}\cdot e^{u}\cdot\left(-\frac{1}{\cancel{\sin(x)}}\right)du = -\int e^{u}\,du = -e^{u} \overset{RS}{=} -e^{\cos(x)} + C$

$u = \cos(x)$

$\frac{du}{dx} = -\sin(x) \quad | \cdot dx \,|: (-\sin(x))$

$\Leftrightarrow dx = -\frac{1}{\sin(x)}du$

g) $\int \frac{\sqrt{1+x}}{x}\,dx \qquad u = \sqrt{1+x} \qquad \frac{du}{dx} = \frac{1}{2\sqrt{x+1}} \quad | \cdot dx \,| \cdot 2\sqrt{x+1}$

$\Leftrightarrow dx = 2\sqrt{x+1}\,du$

$= \int \frac{\sqrt{x+1}}{x}\cdot 2\sqrt{x+1}\,du = 2\cdot\int \frac{1+x}{x}\,du \qquad x = u^2 - 1$

$= 2\int \frac{\cancel{1} + u^2 \cancel{-1}}{u^2-1}\,du = 2\cdot\left(\int 1\,du + \int \frac{1}{u^2-1}\,du\right)$

PD:

$u^2 : (u^2-1) = 1 + \frac{1}{u^2-1}$

$-(u^2-1)$

1

NR: $\frac{1}{u^2-1} = \frac{A}{u+1} + \frac{B}{u-1} = \frac{1}{2}\cdot\frac{1}{u-1} - \frac{1}{2}\cdot\frac{1}{u+1}$

$= Au - A + Bu + B$

$1 = u\cdot\underbrace{(A+B)}_{A=-B} + \underbrace{B - A}_{2B=1 \Rightarrow B=\frac{1}{2},\ A=-\frac{1}{2}}$

$= 2\cdot\left(u + \frac{1}{2}\int \frac{1}{u-1}\,du - \frac{1}{2}\cdot\int \frac{1}{u+1}\,du\right)$

$= 2\cdot\left(u + \frac{1}{2}\cdot\ln(|u-1|) - \frac{1}{2}\cdot\ln(|u+1|)\right)$

$= 2u + \ln(|u-1|) - \ln(|u+1|) \overset{RS}{=} 2\cdot\sqrt{1+x} + \ln\left(\frac{|\sqrt{1+x}-1|}{|\sqrt{1+x}+1|}\right) + C$

h) $\int \frac{1}{\sin(x)}\,dx \qquad u = \sin(x)$

Weg Nr. 1 $\qquad \frac{du}{dx} = \cos(x) \quad |\cdot dx \quad |:\cos(x) \iff dx = \frac{1}{\cos(x)}\,du$

$$= \int \frac{1}{u} \cdot \frac{1}{\cos(x)}\,du \qquad \cos(x) = \sqrt{1-\sin^2(x)}$$

$$\sin^2(x) + \cos^2(x) = 1$$

$$= \int \frac{1}{u} \cdot \frac{1}{\sqrt{1-\sin^2(x)}}\,du = \int \frac{1}{u} \cdot \frac{1}{\sqrt{1-u^2}}\,du = \dots$$

Weg Nr. 2 $\quad u = \cos(x) \qquad \frac{du}{dx} = -\sin(x) \quad |\cdot dx \quad |:(-\sin(x)) \iff dx = \left(-\frac{1}{\sin(x)}\right)du$

$$\int \frac{1}{\sin(x)} \cdot \left(-\frac{1}{\sin(x)}\right)du = -\int \frac{1}{\sin^2(x)}\,dx$$

$$\sin^2(x) = 1-\cos^2(x) = 1-u^2$$

$$= -\int \frac{1}{1-u^2}\,du = \int \frac{1}{u^2-1}\,du = \ln\left(\sqrt{\frac{|u-1|}{|u+1|}}\right) = \ln\left(\sqrt{\frac{|\cos(x)-1|}{|\cos(x)+1|}}\right) + C$$

i) $\int \frac{1}{(x^2+1)^3}\,dx$

a) $\int \frac{1}{(x^2+1)^n}\,dx \left(= \int \frac{1}{(x^2+1)^{n-1}} \cdot \frac{1}{x^2+1}\,dx \right) \qquad \int \frac{1}{(x^2+1)^n}\,dx = I_n \qquad \int \frac{1}{(x^2+1)^{n+1}}\,dx = I_{n+1}$

$$= \int \underbrace{\frac{1}{(x^2+1)^n}}_{f} \cdot \underbrace{1}_{g}\,dx = \frac{x}{(x^2+1)^n} + 2n \cdot \int \frac{x^2}{(x^2+1)^{n+1}}\,dx$$

NR:

$$\frac{x^2}{(x^2+1)^{n+1}} = \underbrace{\frac{x^2}{x^2+1}}_{PD} \cdot \frac{1}{(x^2+1)^n} \qquad x^2 : (x^2+1) = 1 - \frac{1}{x^2+1}$$

$$\begin{array}{l} -(x^2+1) \\ \hline -1 \end{array}$$

$$\int \frac{1}{(x^2+1)^n}\,dx = \frac{x}{(x^2+1)^n} + 2n \cdot \left(\int \frac{1}{(x^2+1)^n}\,dx - \int \frac{1}{(x^2+1)^{n+1}}\,dx \right) \quad \left|-2n\left(\int \frac{1}{(x^2+1)^n}\,dx\right)\right.$$

$$(1-2n)\underbrace{\int \frac{1}{(x^2+1)^n}\,dx}_{I_n} = \frac{x}{(x^2+1)^n} - 2n \cdot \underbrace{\int \frac{1}{(x^2+1)^{n+1}}\,dx}_{I_{n+1}}$$

$$I_{n+1} = \frac{x}{(x^2+1)^n} \cdot \frac{1}{2n} + \frac{2n-1}{2n} \cdot I_n \quad , \quad I_n = \frac{x}{(x^2+1)^{n-1}} \cdot \frac{1}{2(n-1)} + \frac{2n-3}{2n-2} \cdot I_{n-1}$$

$n=1 \quad \int \frac{1}{x^2+1}\,dx \qquad x = \tan(u)$

$\qquad u = \tan^{-1}(x)$

$$\frac{dx}{du} = \frac{1}{\cos^2(u)} \Big| \cdot du \iff dx = \frac{1}{\cos^2(u)} du$$

$$I = \int \frac{1}{x^2+1} \frac{dx}{dy} \cdot dy = \int \frac{1}{\tan^2(u)+1} \cdot \frac{1}{\cos^2(u)} du = \int \cancel{\cos^2(u)} \cdot \frac{1}{\cancel{\cos^2(u)}} du = u$$

$$\overset{RS}{=} \arctan(x)$$

$$\frac{1}{\tan^2(u)+1} = \frac{1}{\frac{\sin^2(u)}{\cos^2(u)}+1} = \frac{\cos^2(u)}{\underbrace{\sin^2(u)+\cos^2(u)}_{=1}} = \cos^2(u)$$

$\underline{n=2}$ $\quad I_2 = \frac{x}{x^2+1} \cdot \frac{1}{2} + \frac{1}{4-2} I_1 = \frac{1}{2} \cdot \left(\frac{x}{x^2+1} + \arctan(x)\right)$

$\underline{n=3}$ $\quad I_3 = \dots = \frac{1}{8}\left(\frac{3x^2+5x}{(x^2+1)^2} + 3\arctan(x)\right)$

Aufgabe 2

i) a) $\int_{-2}^{2} (x^3-x)dx = \left[\frac{1}{4}x^4 - \frac{1}{2}x^2\right]_{-2}^{2} = 0$

$f(-x) = -f(x)$ + sym. Grenzen $\Rightarrow$ I wird 0

b) $\int_0^1 \frac{x^2}{1+x^2} dx \qquad \frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}$

$$= \int_0^1 1\, dx - \int_0^1 \frac{1}{1+x^2} dx = [x]_0^1 - [\arctan(x)]_0^1$$

$$= 1 - (\underbrace{\arctan(1)}_{\frac{\pi}{4}} - \underbrace{\arctan(0)}_{0})$$

$$= 1 - \frac{\pi}{4}$$

c) $\int_2^4 \frac{|x-3|}{x^2} dx \qquad |x-3| = \begin{cases} x-3, & x \geq 3 \\ -x+3, & x < 3 \end{cases}$

$\int \frac{|x-3|}{x^2} dx$

Fall 1: $x \geq 3 \Rightarrow \int \frac{x-3}{x^2} dx$

Fall 2: $x < 3 \Rightarrow \int \frac{-x+3}{x^2} dx$

$$\int_2^3 \frac{-x+3}{x^2} dx = \int_2^3 \left(-\frac{x}{x^2} + \frac{3}{x^2}\right) dx = -\int_2^3 \frac{1}{x} dx + 3 \cdot \int_2^3 \frac{1}{x^2} dx$$

$$= \int_2^3 \frac{-x+3}{x^2} dx + \int_3^4 \frac{x-3}{x^2} dx$$

$$= -\underbrace{\int_2^3 \frac{1}{x} dx}_{\ln(3)-\ln(2)} - 3 \cdot \underbrace{\int_2^3 \frac{1}{x^2} dx}_{\left[-\frac{1}{x}\right]_2^3} + \underbrace{\int_3^4 \frac{1}{x} dx}_{\ln(4)-\ln(3)} - 3 \cdot \underbrace{\int_3^4 \frac{1}{x^2} dx}_{\left[-\frac{1}{x}\right]_3^4} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$-\left(\ln\left(\frac{3}{2}\right)\right) \qquad = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

$= (-1) \cdot \ln\left(\frac{3}{2}\right) = \ln\left(\left(\frac{3}{2}\right)^{-1}\right) = \ln\left(\frac{2}{3}\right)$

$$= \ln\left(\frac{2}{3}\right) - \frac{3}{6} + \ln\left(\frac{4}{3}\right) - \frac{3}{12} = \ln\left(\frac{2}{3}\cdot\frac{4}{3}\right) - \frac{3}{4} = \ln\left(\frac{8}{9}\right) - \frac{3}{4}$$

d) $$\int_0^3 \underset{f}{x}\,\underset{g}{e^{3x}}\,dx \overset{PI}{=} \left[\frac{1}{3}xe^{3x}\right]_0^3 - \frac{1}{3}\cdot\int_0^3 e^{3x}\,dx = \left[\frac{1}{3}xe^{3x}\right]_0^3 - \left[\frac{1}{9}e^{3x}\right]_0^3$$

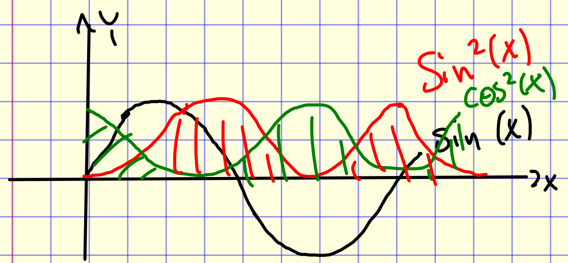
$$= e^9 - \frac{1}{9}e^9 - \Big[0 - \frac{1}{9}\underbrace{e^0}_{1}\Big]$$

$$= \underline{\underline{\frac{8}{9}e^9 + \frac{1}{9}}} \quad (\approx 7202{,}9)$$

e) $$\int_1^{\sqrt{3}} \frac{1}{x^2+1}\,dx = [\arctan(x)]_1^{\sqrt{3}} = \arctan(\sqrt{3}) - \arctan(1) = \frac{\pi}{12}$$

f) $$\int_0^{2\pi} \sin^2(x)\,dx = \int_0^{2\pi} \underset{f}{\sin(x)}\,\underset{g}{\sin(x)}\,dx \overset{PI}{=} \dots$$

alternativ:



$$\int_0^{2\pi} \sin^2(x)\,dx = \int_0^{2\pi} \underbrace{\cos^2(x)}_{1-\sin^2(x)}\,dx = \int_0^{2\pi} 1\,dx - \underbrace{\int_0^{2\pi} \sin^2(x)\,dx}_{I} \quad |+I$$

$$\Leftrightarrow 2\cdot\int_0^{2\pi} \sin^2(x)\,dx = 2\pi \quad |:2$$

$$\Leftrightarrow \int_0^{2\pi} \sin^2(x)\,dx = \pi$$

ii) $$\int_0^{\frac{1}{3}} \frac{e^{3x}}{e^{3x}+5}\,dx \qquad u = e^{3x} \qquad \frac{du}{dx} = 3e^{3x} \quad |\cdot dx \quad |:3e^{3x}$$

$$\Leftrightarrow dx = \frac{1}{3}\cdot\frac{1}{e^{3x}}\,du$$

$$= \int_0^{\frac{1}{3}} \frac{\cancel{u}}{u+5}\cdot\frac{1}{3}\cdot\frac{1}{\cancel{e^{3x}}_{u}}\,du = \frac{1}{3}\cdot\int_0^{\frac{1}{3}} \frac{1}{u+5}\,du = \frac{1}{3}\cdot[\ln(|u+5|)]_0^{\frac{1}{3}} \overset{RS}{=} \frac{1}{3}[\ln(|e^{3x}+5|)]_0^{\frac{1}{3}}$$

$$= \frac{1}{3}\cdot(\ln(e^1+5) - \ln(e^0+5))$$

$$= \ln\left(\sqrt[3]{\frac{e+5}{6}}\right) \;//$$

iii) a) $$\int_2^{11} \frac{1}{\sqrt{x-2}}\,dx \qquad u(x) = x-2 \qquad \frac{du}{dx} = 1 \quad |\cdot dx \Leftrightarrow du = dx$$

$u(x) = x-2$
$u(11) = 9$
$u(2) = 0$

$$= \int_0^9 \frac{1}{\sqrt{u}}\,du = [2\sqrt{u}]_0^9 = 2\cdot\sqrt{9} - 2\cdot\sqrt{0} = 2\cdot 3 = 6$$

b) $$\int_0^\infty 2xe^{-2x}\,dx = 2\cdot\int_0^\infty \underset{f}{x}\,\underset{g}{e^{-2x}}\,dx = 2\cdot\left(\left[-\frac{1}{2}xe^{-2x}\right]_0^\infty + \frac{1}{2}\cdot\int_0^\infty e^{-2x}\,dx\right)$$

$$= 2\cdot\left(\left[-\frac{1}{2}xe^{-2x}\right]_0^\infty + \left[-\frac{1}{4}e^{-2x}\right]_0^\infty\right)$$

$$\lim_{a\to\infty}\left[-\frac{1}{2}xe^{-2x}\right]_0^a + \lim_{a\to\infty}\left[-\frac{1}{4}e^{-2x}\right]_0^a$$

$$= \lim_{a\to\infty}\left(\cancel{-\frac{1}{2}ae^{-2a}}^{\to 0} + \cancel{\frac{1}{2}\cdot 0\, e^{-2\cdot 0}}^{0}\right) + \lim_{a\to\infty}\left(\cancel{-\frac{1}{4}e^{-2a}}^{\to 0} + \frac{1}{4}\cdot e^{0}\right)$$

$$= \frac{1}{4} \quad //$$

c) $\displaystyle\int_0^\infty \frac{1}{(x-1)^3}\,dx \qquad u = x-1 \iff du = dx$

$$= \int_{-1}^{\infty} \frac{1}{u^3}\,du = \left[-\frac{1}{2}\cdot\frac{1}{u^2}\right]_{-1}^{\infty} \implies \lim_{a\to\infty}\left[-\frac{1}{2}\cdot\frac{1}{u^2}\right]_{-1}^{a} = \lim_{a\to\infty} \cancel{-\frac{1}{2}\cdot\frac{1}{a^2}}^{\to 0} + \frac{1}{2}\cdot\frac{1}{(-1)^2}$$

$$= \frac{1}{2} \quad //$$