Vorrechnen Blatt 12

1) i) a) $z_1 = 1 + i$

$Re = 1 \quad , \quad Im = 1$

$|z| = \sqrt{Re^2 + Im^2} = \sqrt{2}$

b) $z_2 = 2 - 3i$

$Re = 2 \quad , \quad Im = -3$

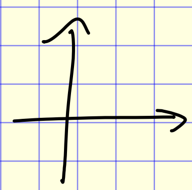
$|z| = \sqrt{4 + 9} = \sqrt{13}$

c) $z_3 = \sqrt{3} + i$

$Re = \sqrt{3} \quad , \quad Im = 1$

$|z| = \sqrt{3 + 1} = 2$

ii) a) $\underbrace{|Re(z)|}_{x} + \underbrace{|Im(z)|}_{y} \leq 4$



$|x| + |y| \leq 4$

Fall 1: $x \geq 0 \Rightarrow |x| = x \Rightarrow x + |y| \leq 4$

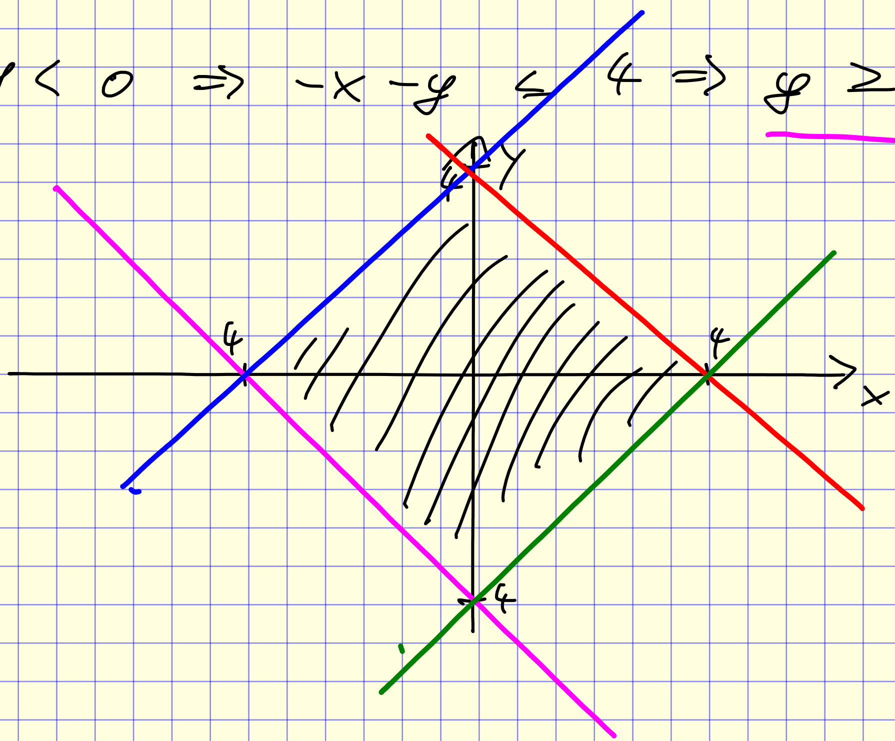
a) $y \geq 0 \Rightarrow x + y \leq 4 \Rightarrow \underline{y \leq 4 - x}$

b) $y < 0 \Rightarrow |y| = -y \Rightarrow x - y \leq 4 \Rightarrow \underline{y \geq x - 4}$

Fall 2: $x < 0 \Rightarrow |x| = -x \Rightarrow -x + |y| \leq 4$

a) $y \geq 0 \Rightarrow -x + y \leq 4 \Rightarrow \underline{y \leq 4 + x}$

b) $y < 0 \Rightarrow -x - y \leq 4 \Rightarrow \underline{y \geq -x - 4}$



b) $|z| \leq 2 \cdot \underbrace{Re(z)}_{\Downarrow\ x}$ $\qquad |z| = \sqrt{x^2 + y^2}$

$\sqrt{x^2 + y^2} \leq 2x \qquad | ( )^2$

$x^2 + y^2 \leq 4x^2$

$y^2 \leq 4x^2 - x^2$

$y^2 \leq 3x^2 \qquad | \sqrt{\ }$

$\sqrt{y^2} \leq \sqrt{3} \sqrt{x^2}$

$|y| \leq \sqrt{3} \; |x|$

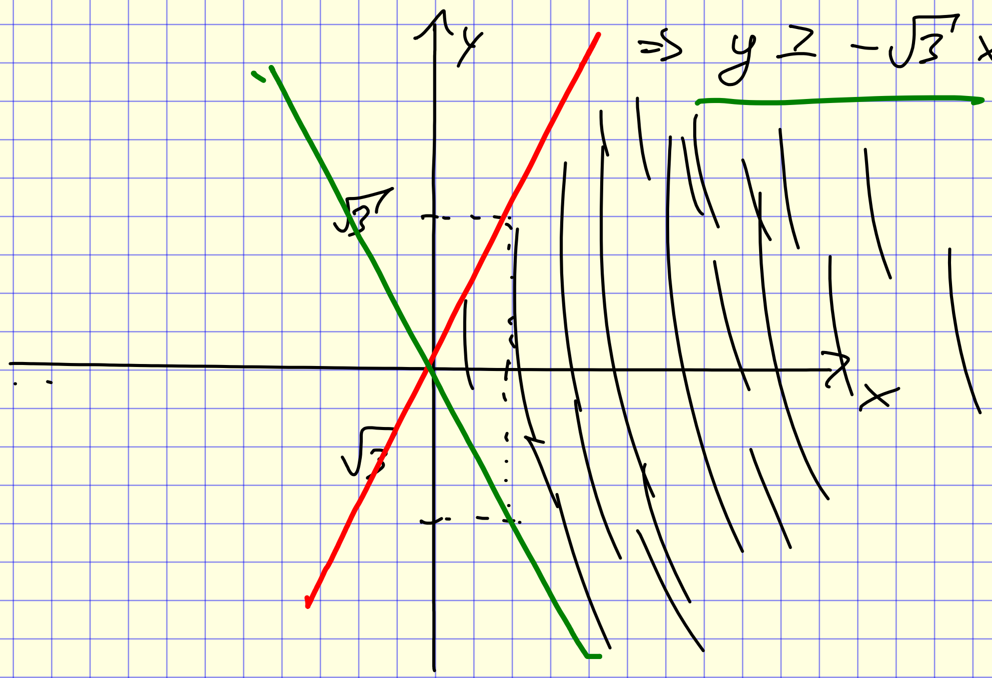
Fall 1: $y \geq 0 \Rightarrow y \leq \sqrt{3}\,|x|$

a) $x \geq 0 \Rightarrow \underline{y \leq \sqrt{3}\,x}$

b) $x < 0 \Rightarrow$ ausschließlich dann: $x \geq \frac{1}{2}\sqrt{x^2+y^2} \geq 0$

Fall 2: $y < 0 \Rightarrow |y| = -y \Rightarrow -y \leq \sqrt{3}\,|x|$

$\Rightarrow \underline{y \geq -\sqrt{3}\,x}$



iii) $x^2 - 10x + 34 = 0$, $x_{1,2} = 5 \pm \sqrt{25-34}$

$x_{1,2} = 5 \pm \sqrt{-9}$

$x_{1,2} = 5 \pm \sqrt{-1} \cdot \sqrt{9}$

$x_{1,2} = 5 \pm i\,3$

$x_1 = 5 - 3i$

$x_2 = 5 + 3i$

iv) a) $(1+i)^5 = \underbrace{(1+i)^2}_{\left(\begin{array}{l}1+2i+i^2\\ = 2i\end{array}\right)} \cdot (1+i)^2 \cdot (1+i)$

$$(1+i)^5 = 2i \cdot 2i \cdot (1+i)$$
$$= 4i^2 (1+i)$$
$$= -4(1+i)$$
$$= -4-4i$$

b) $\frac{1}{2+3i} = \frac{1}{(2+3i)} \cdot \frac{(2-3i)}{(2-3i)} = \frac{(2-3i)}{4-6i+6i-9i^2} = \frac{(2-3i)}{4+9}$

$$= \frac{2}{13} - \frac{3}{13} i$$

c) $\frac{(-i+1)^3}{(i+2)}$

$$(-i+1)^3 = (-i+1)^2 \cdot (-i+1)$$
$$= ((-i)^2 - 2i + 1) \cdot (-i+1)$$
$$= (-1 - 2i + 1)(-i+1)$$
$$= -2i(-i+1)$$
$$= 2i^2 + -2i$$
$$= -2-2i$$

$$= \frac{-2-2i}{(i+2)} = \frac{(-2-2i)(i-2)}{(i+2)(i-2)} = \frac{-2i+4-2i^2+4i}{i^2-4}$$

$$= \frac{2i+4+2}{-5} = \frac{6+2i}{-5} = -\frac{6}{5} - \frac{2}{5} i$$

$$= -\frac{2}{5}(3+i)$$

vi) $z = a + ib = r \cdot e^{i\phi}$ $\qquad r = \sqrt{a^2 + b^2}$

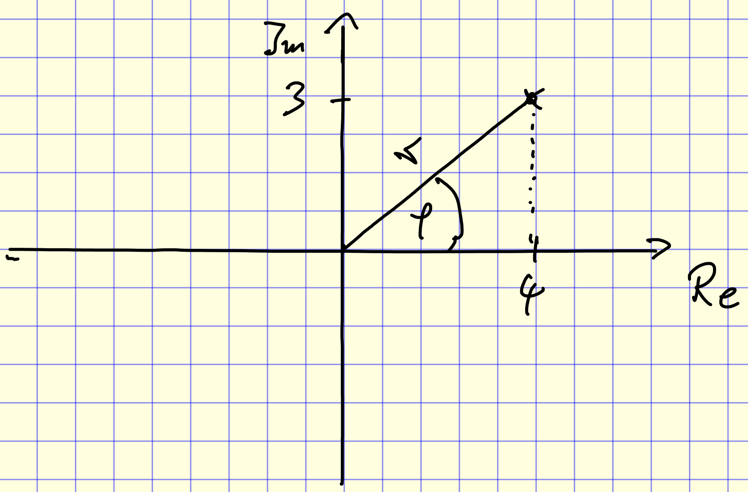
$$\ln(z) = \ln(3i + 4)$$

$$\ln(r \cdot e^{i\varphi}) = \ln(r) + \ln(e^{i\varphi}) = \ln(r) + i\varphi$$

$$r = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$\varphi = \arctan\left(\frac{b}{a}\right) = \arctan\left(\frac{3}{4}\right) \approx 36{,}87^\circ \approx \underline{\underline{0{,}644}}$$

$$\ln(3i + 4) = \ln(5) + i \cdot 0{,}644$$



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$$\tan(\varphi) = \frac{G}{A}$$

vii) $z = 1 + i$ $\qquad |z| = \sqrt{2}$

$$\varphi = \underline{\arctan\left(\frac{1}{1}\right)} = \frac{\pi}{4}$$

$$z = \sqrt{2} \cdot e^{i\frac{\pi}{4}}$$

$$z^5 = (1+i)^5 = \underbrace{(1+i)^2}_{\underbrace{1 + 2i - 1}_{2i}} (1+i)^2 (1+i) = 2i \cdot 2i \cdot (1+i)$$

$$= 4i^2 \cdot (1+i)$$

$$= -4 - 4i$$

vgl: iv, a)

viii) a) $i^3 = i^2 \cdot i = -1i = -i$

b) $i^4 = i^2 \cdot i^2 = -1 \cdot (-1) = 1$

c) $i^{13} = i^4 \cdot i^4 \cdot i^4 \cdot i = 1 \cdot 1 \cdot 1 \cdot i = i$

d) $(-i)^3 = (-i)^2 \cdot (-i) = -1 \cdot (-i) = i$

2)

a) $$\int \frac{\frac{\sin^2(x)}{\cos^4(x)} - \frac{4}{\cos^2(x)}}{\tan^3(x) - \tan^2(x) - 4\tan(x) + 4} \, dx$$

$y = \tan(x)$ $$\frac{dy}{dx} = \frac{1}{\cos^2(x)}$$

$$dx = \cos^2(x) \, dy$$

$$\frac{\sin^2(x)}{\cos^4(x)} - \frac{4}{\cos^2(x)} = (\underbrace{\tan^2(x)}_{y^2} - 4) \cdot \frac{1}{\cos^2(x)}$$

$$\int \frac{\frac{(y^2-4)}{\cos^2(x)} \cdot \cos^2(x)}{y^3 - y^2 - 4y + 4} \, dy$$

$$\Rightarrow \int \frac{y^2 - 4}{y^3 - y^2 - 4y + 4} \, dy$$

→ Polynom division :

$$(y^3 - y^2 - 4y + 4) : (y - 1) = y^2 - 4$$

$$y^3 - y^2 - 4y + 4 = (y-1) \cdot (y^2 - 4)$$

$$\int \frac{\cancel{(y^2-4)}}{(y-1)\cancel{(y^2-4)}} \, dy = \int \frac{1}{y-1} \, dy = \ln(|y-1|) + C$$

$$\Rightarrow \ln(|\tan(x) - 1|) + C$$

b) $\int_{-1}^{1} \frac{x^2}{\sqrt{1-x}} \, dx$

$$\int f(x) \cdot g'(x) = f(x) \cdot g(x) - \int f'(x) \cdot g(x)$$

$f(x) = x^2 \qquad g'(x) = \frac{1}{\sqrt{1-x}}$

P.I. : $-2\sqrt{1-x}\, x^2 + 4 \underline{\int \sqrt{1-x}\, x \, dx}$

$g'(x) = \sqrt{1-x}$

$f(x) = x$

$$= -2\sqrt{1-x}\, x^2 + 4 \left[-\frac{2}{3} (1-x)^{3/2} x + \frac{2}{3} \underbrace{\int (1-x)^{3/2} dx}_{-\frac{2}{5}(1-x)^{5/2}} \right]$$

$$\int_{-1}^{1} \frac{x^2}{\sqrt{1-x}} \, dx = \left[-2\sqrt{1-x}\, x^2 - \frac{8}{3} (1-x)^{3/2} x - \frac{16}{15} (1-x)^{5/2} \right]_{-1}^{1}$$

$$= -\left(-2\sqrt{2} - \frac{8}{3} 2^{3/2} (-1) - \frac{16}{5} 2^{5/2}\right)$$

$$= \sqrt{2} \left(2 - \frac{8}{3} 2 + \frac{16}{15} 2^2 = \frac{\sqrt{2}}{3} \left(6 - 16 + \frac{16}{5} \cdot 4\right)\right.$$

$$= \frac{\sqrt{2}}{3} \cdot \left(-\frac{50}{5} + \frac{64}{5}\right) = \frac{\sqrt{2}}{3} \cdot \frac{14}{5} = \frac{\sqrt{2} \cdot 14}{15} \approx 1{,}319\ldots$$

1/ v) $\underline{x^3+27} \; \underline{=0}$ $\qquad x_n = a_n + i b_n$

$$(a+ib)^3 = -27 \quad \Big| \sqrt[3]{\ }$$

$$a+ib = -3$$

$$(x^3+27):(x+3) = \underbrace{x^2-3x+9}_{0}$$
$$\frac{-(x^3+3x^2)}{-3x^2+27}$$

$$x_{1,2} = \frac{3}{2} \pm \sqrt{\frac{9}{4}-9}$$

$$x_{1,2} = \frac{3}{2} \pm \sqrt{\frac{9}{4}-\frac{36}{4}}$$

$$= \frac{3}{2} \pm \sqrt{-\frac{27}{4}}$$

$$= \frac{3}{2} \pm i\sqrt{\frac{27}{4}}$$

$$x_1 = \frac{3}{2} + i\sqrt{\frac{27}{4}}$$

$$x_2 = \frac{3}{2} - i\sqrt{\frac{27}{4}}$$