Allgemeine Relativitätstheorie: Übungen & Lösungen

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Some further reading

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T.P. Cheng, *Relativity, Gravitation and Cosmology*, 2^e
Oxford Univ. Press (2010)

S. Weinberg, *Gravitation and cosmology*, Wiley (1978)

C.M. Will, *Confrontation between general relativity and experiments*,
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C.M. Will, *...und Einstein hatte doch Recht/Les enfants d'Einstein*,
Springer (1986)

Série 2

1. The invariant of Minkowski space reads, with the metric tensor $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$,

$$\mathrm{d}s^2 = \eta_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = \overline{g}_{\mu\nu}\,\mathrm{d}\overline{x}^\mu\,\mathrm{d}\overline{x}^\nu \tag{1}$$

where the coordinates $\overline{x}^\mu$ come from a coordinate transformation $x^\mu \mapsto \overline{x}^\mu$. Give the metric tensor $\overline{g}_{\mu\nu}$ in the new coordinates.

Solution:
it is enough to write out the respective derivatives (use the expansion $\mathrm{d}x^\alpha = \frac{\partial x^\alpha}{\partial \bar{x}^\mu}\mathrm{d}\bar{x}^\mu$)

$$\begin{aligned}
\mathrm{d}s^2 &= \eta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \\
&= \eta_{\alpha\beta}\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\mathrm{d}\bar{x}^\mu \frac{\partial x^\beta}{\partial \bar{x}^\nu}\mathrm{d}\bar{x}^\nu \\
&= \underbrace{\left(\eta_{\alpha\beta}\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\nu}\right)}_{=:\,\bar{g}_{\mu\nu}} \mathrm{d}\bar{x}^\mu \mathrm{d}\bar{x}^\nu \\
&= \bar{g}_{\mu\nu}\mathrm{d}\bar{x}^\mu \mathrm{d}\bar{x}^\nu
\end{aligned}$$

and one can read off

$$\boxed{\bar{g}_{\mu\nu} = \eta_{\alpha\beta}\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\nu}}$$

2. Consider the determinant of the metric tensors $g := \det g_{\mu\nu}$. Is it Lorentz-invariant ?

Solution:
The metric tensor transforms as follows

$$\bar{g}_{\mu\nu} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu}$$

and this can be viewed as a product of matrices. Taking the determinant

$$\begin{aligned} \bar{g} &:= \det\left(\bar{g}_{\mu\nu}\right) \\ &= \det\left(g_{\alpha\beta}\right) \det\left(\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\right) \det\left(\frac{\partial x^\beta}{\partial \bar{x}^\nu}\right) \\ &= g \left[\det\left(\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\right)\right]^2 \end{aligned}$$

For a general transformation $x \mapsto \bar{x}$, g is invariant if and only if $\det\left(\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\right) = 1$.
For linear transformations, $\Lambda^\alpha_\nu = \frac{\partial x^\alpha}{\partial \bar{x}^\mu}$ is a matrix with constant matrix elements. For a space rotation, $\det \Lambda = 1$ is well-known. For a Lorentz transformation (in x-direction)

$$\Lambda^\alpha_\nu = \begin{pmatrix} \gamma & \gamma v & & \\ \gamma v & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta & & \\ \sinh\theta & \cosh\theta & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \Rightarrow \det \Lambda = \begin{cases} \gamma^2 - \gamma^2 v^2 & = 1 \\ \cosh^2\theta - \sinh^2\theta & = 1 \end{cases}$$

$\Rightarrow$ g *is* Lorentz-invariant, but it is **not** invariant under general transformations.

3. Show that the invariant volume element of a four-dimensional space is given by

$$\mathrm{d}^4 V = \left(-g\right)^{1/2} \mathrm{d}^4\mathtt{x} = \left(-g\right)^{1/2} \mathrm{d}t\mathrm{d}x\mathrm{d}y\mathrm{d}z \qquad (2)$$

where $g := \det g_{\mu\nu}$ is the determinant of the metric tensor $g_{\mu\nu}$.

Solution:
Under the transformation $\mathrm{x} \to \bar{\mathrm{x}}$, the volume element is $\mathrm{d}^4\mathrm{x} = \det\left(\frac{\partial \mathrm{x}}{\partial \bar{\mathrm{x}}}\right)\mathrm{d}^4\bar{\mathrm{x}}$. One of the frames can be assumed to be Minkowski space with the metric tensor $\eta_{\mu\nu} = \mathrm{diag}\,(-1, 1, 1, 1)$. From the previous exercice

$$-\bar{g} = -\det\left(\bar{g}_{\alpha\beta}\right) = -\det\left(\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial x^\nu}{\partial \bar{x}^\beta}\eta_{\mu\nu}\right) = \left[\det\left(\frac{\partial \mathrm{x}}{\partial \bar{\mathrm{x}}}\right)\right]^2(-\det\eta)$$

Hence $\left(-g\right)^{1/2} = \det\frac{\partial \mathrm{x}}{\partial \bar{\mathrm{x}}}$. In consequence

$$\mathrm{d}^4\bar{V} := \left(-\bar{g}\right)^{1/2}\mathrm{d}^4\bar{\mathrm{x}} = \det\left(\frac{\partial \bar{\mathrm{x}}}{\partial \mathrm{x}}\frac{\partial \mathrm{x}}{\partial \bar{\mathrm{x}}}\right)\left(-g\right)^{1/2}\mathrm{d}^4\mathrm{x} = \mathrm{d}^4 V$$

4. (a) Show that the *invariant* volume element of three-dimensional space, for an observer with the four-velocity $\mathtt{u}$ is given by

$$\mathrm{d}^3 V = (-g)^{1/2}\, u^0\, \mathrm{d}^3 x \tag{3}$$

(b) Write down the *invariant* volume element of the contra-variant momentum $\mathrm{d}^4\mathtt{p}$ in four-dimensional momentum space.
(c) Write down the *invariant* three-dimensional volume element in momentum space *"on the mass shell"*, that is with the constraint $\sqrt{-\mathtt{p}\cdot\mathtt{p}} = m$.

Solution:
(a) at rest, one has clearly $\mathrm{d}^3 V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$. One wants a scalar which to reduces to this at rest.
Start from $\mathrm{d}^4 V$, and introduce the component $u^0 = \frac{\mathrm{d}t}{\mathrm{d}\tau}$ of the four-velocity u, where τ is proper time.

$$\mathrm{d}^4 V = \left(-g\right)^{1/2}\mathrm{d}t\mathrm{d}x\mathrm{d}y\mathrm{d}z = \left(-g\right)^{1/2}\mathrm{d}t\mathrm{d}x\mathrm{d}y\mathrm{d}z\frac{u^0}{u^0} = \left(-g\right)^{1/2}u^0\mathrm{d}\tau\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Since $\mathrm{d}^4 V$ and $\mathrm{d}\tau$ are scalars, $\mathrm{d}^3 V := \left(-g\right)^{1/2}u^0\mathrm{d}x\mathrm{d}y\mathrm{d}z$ must be scalar as well.
(b) The four-momentum $\mathrm{p} = (P^0, \boldsymbol{P})$ transforms as a four-vector. The invariant volume element is

$$\mathrm{d}^4 p = \left(-g\right)^{1/2}\mathrm{d}P^0\mathrm{d}P^x\mathrm{d}P^y\mathrm{d}P^z$$

(c) One has the extra constraint $(-\mathrm{p}\cdot\mathrm{p}) = m$. This gives the invariant $3D$ element

$$\mathrm{d}^3 p = \int (-g)^{1/2}\, \mathrm{d}P^0 \mathrm{d}P^x \mathrm{d}P^y \mathrm{d}P^z\, \delta\left(\left(-g_{\alpha\beta} P^\alpha P^\beta\right)^{1/2} - m \right)$$

From the theory of distributions (see e.g Gelfand & Shilov, *Generalised Functions, Vol. 1*) one recalls the identity $\int \mathrm{d}x\, \delta\big(f(x)\big) = \sum_{x_0} \frac{1}{|f'(x_0)|}$, where x_0 runs over all zeros of $f(x)$, that is $f(x_0) = 0$.

With the help of this, one eliminates the integration over P^0 and finds

$$\begin{aligned} \mathrm{d}^3 p &= (-g)^{1/2}\, \mathrm{d}P^x \mathrm{d}P^y \mathrm{d}P^z \left[-\frac{1}{2} \left(-g_{\alpha\beta} P^\alpha P^\beta\right)^{1/2} 2 g_{t\alpha} P^\alpha \right]^{-1} \\ &= (-g)^{1/2}\, \mathrm{d}P^x \mathrm{d}P^y \mathrm{d}P^z \left(\frac{m}{-P_0} \right) \end{aligned} \tag{4}$$

In the rest frame,this reduces indeed to $\mathrm{d}^3 p \to \mathrm{d}P^x \mathrm{d}P^y \mathrm{d}P^z$, as expected.

5. Relativistic electrodynamics is described by the field tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ where $\mathtt{A}$ is the four-vector-potential. On a test body with electric charge q then acts the Lorentz force (frz. *force de Laplace* (sic !)), with four-momentum $\mathtt{p} = m\mathtt{u}$ and proper time τ

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = qF^{\mu\nu}u_\nu \tag{5}$$

(a) Consider first the zeroth component (time component) $\mu = 0$ of the equation (5). Express it via the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ and show that

$$\frac{\mathrm{d}p^0}{\mathrm{d}t} = q\boldsymbol{v}\cdot\boldsymbol{E} \tag{6}$$

(b) Write the equation for $\mathrm{d}\boldsymbol{p}/\mathrm{d}t$, expressed via $\boldsymbol{E}$ and $\boldsymbol{B}$.
Hint: consider the space components of (5).
(c) A particle with electric charge q and mass m moves on a circle with radius R in an uniform magnetic field $\boldsymbol{B} = B\boldsymbol{e}_z$.
(i) Express B in terms of known quantities and the angular frequency ω.
(ii) In the rest system, why the magnetic field $\boldsymbol{B}$ cannot furnish work on the particle ? Was is the finding of an observer, who moves with the relative velocity $\beta\boldsymbol{e}_x$? Which velocity does he find, and in particular, which value of $u^{0'}$?
(iii) Determine $\mathrm{d}u^{0'}/\mathrm{d}\tau$ and hence also $\mathrm{d}p^{0'}/\mathrm{d}\tau$. Why can the energy of the particle change, although the magnetic field $\boldsymbol{B}$ does not furnish work ?

Solution:

(a) set $\mu = 0$ in eq. (5): $\frac{\mathrm{d}p^0}{\mathrm{d}\tau} = qF^{0\nu}u_\nu = qE^i\gamma v_i$, with $i = 1, 2, 3$. Because of $\mathrm{d}\tau = \mathrm{d}t/\gamma$, this gives indeed eq. (6).

almost identical at the non-relativistic form, but p^0 also contains the rest energy.

(b) this is worked out directly

$$\frac{\mathrm{d}p^i}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}p^i}{\mathrm{d}t} = qF^{i\nu}u_\nu = qF^{i0}u_0 + qF^{ij}u_j = q\gamma E^i + q\gamma\varepsilon^{ijk}B_k v_j$$

which is indeed the Lorentz force $\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = q\big(\boldsymbol{E} + \boldsymbol{v}\wedge\boldsymbol{B}\big)$.

(c) (i) from the Lorentz force $\omega|\boldsymbol{p}| = \left|\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}\right| = q|\boldsymbol{v}||\boldsymbol{B}|$. Then $B = |\boldsymbol{B}| = \frac{\omega}{q}\frac{|\boldsymbol{p}|}{|\boldsymbol{v}|} = \frac{m\omega}{q\sqrt{1-v^2}} = \frac{m\omega}{q\sqrt{1-\omega^2R^2}}$ (having set $c = 1$).

(ii) in (6), the change of the energy p^0 does not depend on B, hence $p^0 = \mathrm{cste.}$. No work is neither furnished, nor gained.

In the frame of the laboratory, the components of the four-velocity are

$$u^0 = \left(1-\omega^2R^2\right)^{-1/2},\ u^x = \frac{\omega y}{\sqrt{1-\omega^2R^2}},\ u^y = -\frac{\omega x}{\sqrt{1-\omega^2R^2}}$$

On the other hand, for an observer with relative velocity $\beta \boldsymbol{e}_x$, one finds from a Lorentz transformation

$$u'^0 = \gamma\big(u^0 - \beta u^x\big) = \gamma\big(1 - \beta\omega y\big)\big(1 - \omega^2 R^2\big)^{-1/2}\ ,\ \ \gamma = \big(1 - \beta^2\big)^{-1/2} \quad (*)$$

(iii) we have $\frac{\mathrm{d}p'^0}{\mathrm{d}\tau} = m\frac{\mathrm{d}u'^0}{\mathrm{d}\tau} = -\frac{m\omega\gamma\beta u^y}{\sqrt{1-\omega^2R^2}} \neq 0$.

No contradiction, since the electric/magnetic fields transform as follows

$$\begin{aligned} E'^y &= F'^{02} = \Lambda^0_\mu \Lambda^2_\nu F^{\mu\nu} \\ &= \Lambda^0_0 \Lambda^2_\nu F^{0\nu} + \Lambda^0_1 \Lambda^2_\nu F^{1\nu} \ = \ \Lambda^0_0 \Lambda^2_2 F^{02} + \Lambda^0_1 \Lambda^2_2 F^{12} \\ &= \gamma \cdot 1 \cdot E^y + \big(-\gamma\beta\big) \cdot 1 \cdot B^z \end{aligned}$$

The electric field $\boldsymbol{E}$ *does not at all transform as a vector*.

If $\boldsymbol{E} = \boldsymbol{0}$, one has $E'^y = -\gamma\beta B^z$. From (5), one expects

$$\frac{\mathrm{d}p'^0}{\mathrm{d}\tau} = qE'^y u'^y = -\frac{m\omega\gamma\beta u'^y}{\sqrt{1-\omega^2 R^2}} = -\frac{m\omega\gamma\beta u^y}{\sqrt{1-\omega^2R^2}}$$

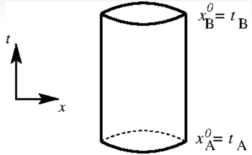
in perfect agreement with (*) above.

The electric field created by Lorentz-transforming the magnetic field $\boldsymbol{B}$ *furnishes the work.*

6. A vector field $J^\alpha(\mathtt{x})$ satisfies the continuity equation (conservation law) $\partial_\alpha J^\alpha = 0$ and for large distances $r = |\boldsymbol{r}| \to \infty$ it falls off faster than r^{-2}.
(a) Show that $Q := \int \mathrm{d}^3x\, J^0$ is constant in time.
(b) Show that Q is a Lorentz scalar, that is $\int \mathrm{d}^3x\, J^0 = \int \mathrm{d}^3x'\, J^{0'}$.

Therefore, Q is called the **conserved charge** of the conserved four-current J^α.

Solution:



(a) take a domain Ω bounded in time by x^0_A below and x^0_B above and with spatial sides far from the origin

using Gauss's theorem

$$
\begin{aligned}
0 &= \int_\Omega \mathrm{d}^4 V \partial_\alpha J^\alpha = \int_\Omega \mathrm{d}t\, \partial_\alpha J^\alpha \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \int_{t_A} J^\alpha \mathrm{d}^3\Sigma^\alpha + \int_{t_B} J^\alpha \mathrm{d}^3\Sigma^\alpha \\
&= \int_{t_B} J^0 \mathrm{d}x\mathrm{d}y\mathrm{d}z - \int_{t_A} J^0 \mathrm{d}x\mathrm{d}y\mathrm{d}z \;=\; Q(t_B) - Q(t_A)
\end{aligned}
$$

$\mathrm{d}\Sigma^\alpha$ is the surface element, oriented *normal* to the surface

- for the only surface here in finite distances, in time-direction

(b) write the charge as $Q = \int \mathrm{d}^4 x\, J^\alpha \partial_\alpha \Theta(n_\beta x^\beta)$, where $n_0 = 1$, $n_1 = n_2 = n_3 = 0$ and $\Theta(u) = \begin{cases} 1 & \text{if } u > 0 \\ 0 & \text{if } u \leq 0 \end{cases}$. To see this, note that Q only contains Lorentz-invariant quantities. It is enough to check it at rest.

$$Q = \int \mathrm{d}^4x\, J^0 \partial_{x^0} \Theta(n_\beta x^\beta) = \int \mathrm{d}^4x\, J^0(x)\delta(x^0) = \int \mathrm{d}^3x J^0(0, \boldsymbol{x}) = Q(0)$$

that is $Q = Q(t) = Q(0)$ is time-independent.
Under a Lorentz transform $Q \mapsto Q' = \int \mathrm{d}^4x\, J^\alpha \partial_\alpha \Theta(n'_\beta x'^\beta)$, with $n'_\beta = \Lambda^\gamma_\beta n_\gamma$.
Hence

$$Q' - Q = \int \mathrm{d}^4x\, \partial_\alpha \Big(J^\alpha(x) \big(\Theta(n'_\beta x'^\beta) - \Theta(n_\beta x^\beta)\big) \Big)$$

Since one knows that (i) $J^\alpha(x) \to 0$ if $|\boldsymbol{x}| \to \infty$ fast enough and (ii) $\Theta(n'_\beta x'^\beta) - \Theta(n_\beta x^\beta) \to 0$ if $|t| \to \infty$, one can again apply Gauss's theorem in $4D$ and express $Q' - Q$ as surface integrals.
This implies $Q' - Q = 0$, hence Q is scalar.

7. Show that the two-dimensional space with the metric

$$\mathrm{d}s^2 = \mathrm{d}v^2 - v^2\mathrm{d}u^2 \tag{7}$$

is identical to the flat two-dimensional Minkowski-space with the metric $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2$.
Hint: find a coordinate transformation $t = t(v, u)$ and $x = x(v, u)$ which sends the Minkowski metric into the metric (7).

Also show that for a non-accelerated particle the contra-variant component p_u of the 'four-momentum' p is constant. Is this also true for the component p_v ?

Solution:
one might use the analogy with polar coordinates as inspiration

make the ansatz $t = v \sinh u$, $x = v \cosh u$, hence $x^2 - t^2 = v^2$ and $x/t = \coth u$.

$$\begin{aligned} \mathrm{d}t &= \mathrm{d}v \sinh u + \mathrm{d}u \, v \cosh u \\ \mathrm{d}x &= \mathrm{d}v \cosh u + \mathrm{d}u \, v \sinh u \end{aligned}$$

and furthermore $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 = \mathrm{d}v^2 - v^2 \mathrm{d}u^2$. Inverting the above infinitesimal transformation gives
$\mathrm{d}v = \mathrm{d}x \cosh u - \mathrm{d}t \sinh u$ and $\mathrm{d}u = v^{-1}\big(\mathrm{d}t \cosh u - \mathrm{d}x \sinh u\big)$. Next,

$$p_u = g_{uu} p^u = -mv^2 \frac{\mathrm{d}u}{\mathrm{d}\tau} = -mv \cosh u \frac{\mathrm{d}t}{\mathrm{d}\tau} + mv \sinh u \frac{\mathrm{d}x}{\mathrm{d}\tau} = -mx \frac{\mathrm{d}t}{\mathrm{d}\tau} + mt \frac{\mathrm{d}x}{\mathrm{d}\tau}$$

Non-accelerated particle: $x(t) = x_0 + \frac{\mathrm{d}x}{\mathrm{d}t} t$, $\frac{\mathrm{d}t}{\mathrm{d}\tau} = \mathrm{cste.}$, $\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{cste.}$. Hence
$p_u = -m \frac{\mathrm{d}t}{\mathrm{d}\tau} x_0 = \mathrm{cste.}$, as claimed.
Since $-m^2 = \mathtt{p} \cdot \mathtt{p} = g^{vv} \big(p_v\big)^2 + g^{uu} \big(p_u\big)^2 = \big(p_v\big)^2 - \frac{1}{v^2} \big(p_u\big)^2 \Rightarrow p_v \neq \mathrm{cste.}$.

8. Show that the metric of the surface of the three-dimensional sphere S^3 embedded into $4D$ euclidean space reads:

$$\mathrm{d}s^2 \quad = \quad R^2 \left[\mathrm{d}\alpha^2 + \sin^2\alpha \, \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2\right)\right] \qquad (8)$$

$$(R \text{ is the constant radius of the sphere})$$

<u>**Hint:**</u> how would you formulate $4D$ spherical coordinates ?

Solution:
a sphere S^3 with radius R is given by $x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2$. Then introduce the coordinates

$$\begin{aligned}
x_4 &= R\cos\alpha \\
x_3 &= R\sin\alpha\cos\theta \\
x_2 &= R\sin\alpha\sin\theta\cos\phi \\
x_1 &= R\sin\alpha\sin\theta\sin\phi
\end{aligned}$$

In cartesian coordinates, the metric is $\mathrm{d}s^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2$ and reproducing (8) is straightforward.

start with $\mathrm{d}x_4 = -R\sin\alpha\mathrm{d}\alpha$ etc.

9. Hyperboloide haben die folgende Parameterdarstellung im dreidimensionalen Raum $\mathbb{R}^3$

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a\sqrt{s^2+d}\ \cos\varphi \\ b\sqrt{s^2+d}\ \sin\varphi \\ c\,s \end{pmatrix} \quad \text{so daß} \quad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = d$$

wobei a, b, c Konstante sind und $d = \pm 1$. Für $d = +1$ hat man ein **einschaliges Hyperbolid** (**hyperboloïde à une nappe**) H_1 und für $d = -1$ ein **zweischaliges Hyperbolid** (**hyperboloïde à deux nappes**) H_2.
Für ein einschaliges Hyperbolid kann man wählen $s = \sinh\xi$ und für ein zweischaliges Hyperbolid $s = \cosh\xi$.

Geben Sie die Parameterdarstellung in beiden Fällen an und ebenfalls, welche geometrische Bedingung diese beiden Flächen erfüllen. Wie kann man diese geometrisch veranschaulichen ? Wie lautet die Metrik $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2$ (für $a = b = c$) und insbesondere der metrische Tensor in beiden Fällen ?

Solution:

(a) einschaliges Hyperbolid $d = +1$: setzt man $s = \sinh\xi$, so findet man

$\boldsymbol{r} = \begin{pmatrix} a\ \cosh\xi\cos\varphi \\ b\ \cosh\xi\sin\varphi \\ c\ \sinh\xi \end{pmatrix}$, was die Oberfläche $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = 1$ parametrisiert.

Es ist äquivalent, $a = b = c$ zu setzen und als Oberfläche zu nehmen $x^2 + y^2 - z^2 = a^2$. Die Parametrisierung verifiziert diese Oberfläche, weil

$$x^2 + y^2 - z^2 = a^2\left(\cosh^2\xi\cos^2\varphi + \cosh^2\xi\sin^2\varphi - \sinh^2\xi\right) = a^2\left(\cosh^2\xi - \sinh^2\xi\right) = a^2$$

Damit wird die Metrik

$$\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2 \\
&= \left(a\sinh\xi\cos\varphi\mathrm{d}\xi - a\cosh\xi\sin\varphi\mathrm{d}\varphi\right)^2 + \left(a\sinh\xi\cos\varphi\mathrm{d}\xi + a\cosh\xi\cos\varphi\mathrm{d}\varphi\right)^2 - \left(a\cosh\xi\mathrm{d}\xi\right)^2 \\
&= \left(a^2\sinh^2\xi\cos^2\varphi\mathrm{d}\xi^2 - 2a^2\cosh\xi\sinh\xi\cos\varphi\sin\varphi\,\mathrm{d}\xi\mathrm{d}\varphi + a^2\cosh^2\xi\sin^2\varphi\mathrm{d}\varphi^2\right) \\
&\quad + \left(a^2\sinh^2\xi\sin^2\varphi\mathrm{d}\xi^2 + 2a^2\cosh\xi\sinh\xi\cos\varphi\sin\varphi\mathrm{d}\xi\mathrm{d}\varphi + a^2\cosh^2\xi\cos^2\varphi\mathrm{d}\varphi^2\right) - a^2\cosh^2\xi\mathrm{d}\xi^2 \\
&= a^2\sinh^2\xi\,\mathrm{d}\xi^2 + a^2\cosh^2\xi\,\mathrm{d}\varphi^2 - a^2\cosh^2\xi\,\mathrm{d}\xi^2 \\
&= a^2\left(-\mathrm{d}\xi^2 + \cosh^2\xi\,\mathrm{d}\varphi^2\right)
\end{aligned}$$

Mit der Notation $(x^1, x^2) = (\xi, \varphi)$ hat man $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ mit dem metrischen Tensor $g_{\mu\nu} = \begin{pmatrix} -a^2 & 0 \\ 0 & a^2\cosh^2\xi \end{pmatrix}$.

(b) zweischaliges Hyperbolid $d = -1$: setzt man $s = \cosh\xi$, so findet man

$\boldsymbol{r} = \begin{pmatrix} a\ \sinh\xi\cos\varphi \\ b\ \sinh\xi\sin\varphi \\ \pm c\ \cosh\xi \end{pmatrix}$, was die Oberfläche $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = -1$ parametrisiert. Es ist äquivalent, $a = b = c$ zu setzen und als Oberfläche zu nehmen $x^2 + y^2 - z^2 = -a^2$. Die Parametrisierung verifiziert diese Oberfläche, weil

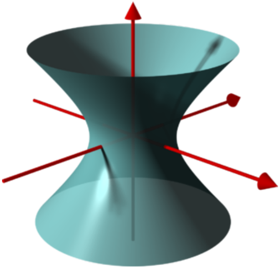
$$x^2 + y^2 - z^2 = a^2\left(\sinh^2\xi\cos^2\varphi + \sinh^2\xi\sin^2\varphi - \cosh^2\xi\right) = a^2\left(-\cosh^2\xi + \sinh^2\xi\right) = -a^2$$

Damit wird die Metrik

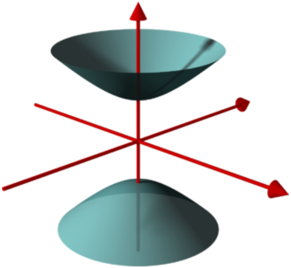
$$\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2 \\
&= \left(a\cosh\xi\cos\varphi\mathrm{d}\xi - a\sinh\xi\sin\varphi\mathrm{d}\varphi\right)^2 + \left(a\cosh\xi\cos\varphi\mathrm{d}\xi + a\sinh\xi\cos\varphi\mathrm{d}\varphi\right)^2 - \left(a\sinh\xi\mathrm{d}\xi\right)^2 \\
&= \left(a^2\cosh^2\xi\cos^2\varphi\mathrm{d}\xi^2 - 2a^2\cosh\xi\sinh\xi\cos\varphi\sin\varphi\,\mathrm{d}\xi\mathrm{d}\varphi + a^2\sinh^2\xi\sin^2\varphi\mathrm{d}\varphi^2\right) \\
&\quad + \left(a^2\cosh^2\xi\sin^2\varphi\mathrm{d}\xi^2 + 2a^2\cosh\xi\sinh\xi\cos\varphi\sin\varphi\,\mathrm{d}\xi\mathrm{d}\varphi + a^2\sinh^2\xi\cos^2\varphi\mathrm{d}\varphi^2\right) - a^2\sinh^2\xi\mathrm{d}\xi^2 \\
&= a^2\cosh^2\xi\,\mathrm{d}\xi^2 + a^2\sinh^2\xi\,\mathrm{d}\varphi^2 - a^2\sinh^2\xi\,\mathrm{d}\xi^2 \\
&= a^2\left(\mathrm{d}\xi^2 + \sinh^2\xi\,\mathrm{d}\varphi^2\right)
\end{aligned}$$

Mit der Notation $(x^1, x^2) = (\xi, \varphi)$ hat man $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ mit dem metrischen Tensor $g_{\mu\nu} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2\sinh^2\xi \end{pmatrix}$.

eine geometrische Vorstellung ergibt sich aus den Abbildungen:



einschalig/ une nappe



zweischalig/ deux nappes

Physikalische Deutung: falls man die (ausgezeichnete) z-Richtung als Zeitachse in einem Zeit-Raum-Diagramm interpretiert, so ist das zweischalige Hyperboloid eine Illustration des Lichtkegels der Viererimpulses eines massiven Teilchens.

Bildquelle: `https://de.wikipedia.org/wiki/Hyperboloid`

10. (a) In euclidean spaces the angle θ between two vectors $\boldsymbol{U}$ and $\boldsymbol{V}$ can be found from the scalar product, since $\cos\theta = \frac{\boldsymbol{U}\cdot\boldsymbol{V}}{|\boldsymbol{U}|\,|\boldsymbol{V}|}$. Consider more general spaces, with a metric tensor $g_{\mu\nu}$. How to define the angle between two vectors in such a case ?
(b) Consider **conformal transformations** $x^\mu \mapsto \overline{x}^\mu$, for which the metric tensor transforms as follows, by definition

$$g_{\alpha\beta} \mapsto f(\mathrm{x})g_{\alpha\beta} \tag{9}$$

where $f = f(\mathrm{x}) = f(x^\mu) \neq 0$ is an arbitrary (differentiable) function. Show that conformal transformations keep all angles invariant. How do light-like curves transform ?

Solution:
(a) the cosine θ between two vectors is *defined* as

$$\cos\theta := \frac{\boldsymbol{U}\cdot\boldsymbol{V}}{|\boldsymbol{U}||\boldsymbol{V}|} = \frac{g_{\mu\nu}U^\mu V^\nu}{\left(g_{\mu\nu}U^\mu U^\nu\, g_{\alpha\beta}V^\alpha V^\beta\right)^{1/2}}$$

(b) under a conformal transformation $\mathtt{x}\mapsto\bar{\mathtt{x}}$ one has

$$\cos\theta \mapsto \cos\bar{\theta} = \frac{f(\mathtt{x})g_{\mu\nu}U^\mu V^\nu}{\left(f(\mathtt{x})g_{\mu\nu}U^\mu U^\nu\, f(\mathtt{x})g_{\alpha\beta}V^\alpha V^\beta\right)^{1/2}} = \cos\theta$$

invariance of angles under conformal transformations
* light-like curves maintain this property, since

$$0 = \mathtt{x}\cdot\mathtt{x} = g_{\mu\nu}x^\mu x^\nu \mapsto f(\mathtt{x})g_{\mu\nu}x^\mu x^\nu = 0 = \bar{\mathtt{x}}\cdot\bar{\mathtt{x}}$$

11. Consider the metric

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \left(\frac{3}{13}\mathrm{d}x + \frac{4}{13}\mathrm{d}y + \frac{12}{13}\mathrm{d}z\right)^2 \qquad (10)$$

Is this really a three-dimensional space ? Try to find new coordinates ζ, η such that $\mathrm{d}s^2 = \mathrm{d}\zeta^2 + \mathrm{d}\eta^2$.

Solution:
Criterion: $3D$ space iff $\mathrm{d}^3V = g^{1/2}\mathrm{d}x\mathrm{d}y\mathrm{d}z \neq 0$.
Hence work out the determinant

$$\mathrm{d}^3V = \begin{vmatrix} 1-\left(\frac{3}{13}\right)^2 & -\frac{3}{13}\frac{4}{13} & -\frac{12}{13}\frac{3}{13} \\ -\frac{3}{13}\frac{4}{13} & 1-\left(\frac{4}{13}\right)^2 & -\frac{4}{13}\frac{12}{13} \\ -\frac{12}{13}\frac{3}{13} & -\frac{4}{13}\frac{12}{13} & 1-\left(\frac{12}{13}\right)^2 \end{vmatrix}^{1/2} \mathrm{d}x\mathrm{d}y\mathrm{d}z = 0$$

$\Rightarrow$ the space must be either $1D$ or $2D$.
Since the metric does not depend explicitly on z, one can consider the projection into the xy-plane where

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - \left(\frac{3}{13}\mathrm{d}x + \frac{4}{13}\mathrm{d}y\right)^2$$

$$g = \det\begin{pmatrix} 1-\left(\frac{3}{13}\right)^2 & -\frac{3}{13}\frac{4}{13} \\ -\frac{3}{13}\frac{4}{13} & 1-\left(\frac{4}{13}\right)^2 \end{pmatrix} = \frac{14336}{169^2} \neq 0$$

shows that this projection is indeed $2D$. One can diagonalise $g_{\mu\nu}$ and find $\mathrm{d}s^2 = \mathrm{d}\zeta^2 + \mathrm{d}\eta^2$, where

$$\zeta = \frac{12}{5}\left(\frac{3}{13}x + \frac{4}{13}y\right) \ , \ \ \eta = \frac{12}{5}\left(-\frac{4}{13}x + \frac{3}{13}y\right)$$

Série 3

1. The covariant derivatives of the metric tensor are defined as follows

$$\begin{aligned} g_{\mu\nu;\lambda} &:= g_{\mu\nu,\lambda} - g_{\sigma\nu}\Gamma^{\sigma}_{\mu\lambda} - g_{\mu\sigma}\Gamma^{\sigma}_{\nu\lambda} \qquad (1) \\ \text{with } \Gamma^{\mu}_{\nu\lambda} &= g^{\mu\rho}\Gamma_{\rho\nu\lambda} = \frac{1}{2}g^{\mu\rho}\left(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho}\right) \end{aligned}$$

where $\Gamma^{\mu}_{\nu\lambda}$ denote the Christoffel symbols. Show that the metric tensor $g_{\mu\nu}$ always has a vanishing covariant derivative, that is $\boxed{g_{\mu\nu;\lambda} = 0}$.

N.B.: *this* **compatibility property** *of the metric is characteristic for Einstein's theory of gravitation. In particular, such metrics are also compatible with flat spaces with a Minkowski metric tensor.*

Solution:
Begin with recalling from (1) the definition of $g_{\mu\nu;\lambda}$. For comparing the Christoffel symbols, it is useful to put all indices down-stairs

$$\Gamma_{\rho\nu\lambda} = \frac{1}{2}\left(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho}\right)$$

We then have

$$\begin{aligned} g_{\mu\nu;\lambda} &= g_{\mu\nu,\lambda} - \Gamma_{\nu\mu\lambda} - \Gamma_{\mu\nu\lambda} \\ &= g_{\mu\nu,\lambda} - \frac{1}{2}\left(g_{\nu\mu,\lambda} + g_{\nu\lambda,\mu} - g_{\mu\lambda,\nu}\right) - \frac{1}{2}\left(g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu} - g_{\nu\lambda,\mu}\right) \\ &= g_{\mu\nu,\lambda} - g_{\mu\nu,\lambda} \;=\; 0 \end{aligned}$$

the symmetry $g_{\mu\nu} = g_{\nu\mu}$ has been frequently used

2. Show that for a *diagonal* metric with metric tensor

$$g_{\mu\nu} = \mathrm{diag}\,(g_{00}, g_{11}, g_{22}, g_{33}) = \begin{pmatrix} g_{00} & & & \\ & g_{11} & & \\ & & g_{22} & \\ & & & g_{33} \end{pmatrix} \qquad (2)$$

the Christoffel symbols have the following values:

$$\Gamma^{\mu}{}_{\nu\lambda} = 0 \quad ; \quad \Gamma^{\mu}{}_{\lambda\lambda} = -\frac{1}{2g_{\mu\mu}}\frac{\partial g_{\lambda\lambda}}{\partial x^{\mu}}$$

$$\Gamma^{\mu}{}_{\mu\lambda} = \frac{\partial}{\partial x^{\lambda}} \ln\sqrt{|g_{\mu\mu}|} \quad ; \quad \Gamma^{\mu}{}_{\mu\mu} = \frac{\partial}{\partial x^{\mu}} \ln\sqrt{|g_{\mu\mu}|} \qquad (3)$$

Herein is always $\mu \neq \nu \neq \lambda \neq \mu$ and there is **no summation** over repeated indices !

Solution:
The Christoffel symbols are given by

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} \left(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho} \right) \qquad (*)$$

(a) since g is diagonal, one must have $\rho = \mu$ in (*). But since $\mu \neq \nu \neq \lambda \neq \mu$, none of the components $\Gamma^{\mu}_{\nu\lambda}$ is non-zero.
(b) if we set $\nu = \lambda$ in (*), we find

$$\Gamma^{\mu}_{\lambda\lambda} = \frac{1}{2} g^{\mu\rho} \left(g_{\rho\lambda,\lambda} + g_{\rho\lambda,\lambda} - g_{\lambda\lambda,\rho} \right) \; = \; -\frac{1}{2} g^{\mu\rho} g_{\lambda\lambda,\rho} \; = \; -\frac{1}{2} \left(g_{\mu\mu} \right)^{-1} g_{\lambda\lambda,\mu}$$

(c) if we set $\nu = \mu$ in (*), we find

$$\Gamma^{\mu}_{\mu\lambda} = \frac{1}{2} g^{\mu\rho} \left(g_{\rho\mu,\lambda} + g_{\rho\lambda,\mu} - g_{\mu\lambda,\rho} \right) \; = \; \frac{1}{2} \left(g_{\mu\mu} \right)^{-1} g_{\mu\mu,\lambda} = \frac{\partial}{\partial x^{\lambda}} \ln\left(|g_{\mu\mu}|^{1/2} \right)$$

(d) simply set $\mu = \lambda$ in (c) and obtain

$$\Gamma^{\mu}_{\mu\mu} = \frac{\partial}{\partial x^{\mu}} \ln\left(|g_{\mu\mu}|^{1/2} \right)$$

3. Die **Pseudosphäre** P^2 hat die Metrik $\mathrm{d}s^2 = a^2\big(\mathrm{d}\xi^2 + \sinh^2\xi\,\mathrm{d}\varphi^2\big)$ eines Hyperboloides. Was ist die Form der geodätischen Kurven ?

The **pseudo-sphere** P^2 has the metric $\mathrm{d}s^2 = a^2\big(\mathrm{d}\xi^2 + \sinh^2\xi\,\mathrm{d}\varphi^2\big)$ of a hyperboloïde. What is the form of the geodetic curves ?

Solution:
This is the metric of a two-sheeted hyperboloïde H_2, as seen in an earlier exercice. Label the coordinates as $(x^1, x^2) = (\xi, \varphi)$. The geodesics are obtained as solutions of the *geodesic equations*

$$\ddot{x}^\mu + \Gamma^\mu_{\kappa\lambda}\dot{x}^\kappa\dot{x}^\lambda = 0 \ \ , \ \ \Gamma^\mu_{\kappa\lambda} = \frac{1}{2}g^{\mu\rho}\big(g_{\rho\kappa,\lambda} + g_{\rho\lambda,\kappa} - g_{\kappa\lambda,\rho}\big) = \Gamma^\mu_{\lambda\kappa}$$

First one must find the non-vanishing Christoffel symbols. Since the metric is diagonal, one can use the technique explained in the previous exercice. Also, the non-zero elements of the inverse metric tensor are found, e.g. via $g^{11} = \frac{1}{g_{11}}$. The only non-vanishing Christoffel symbols are

$$\begin{aligned}
\Gamma^1_{22} &= -\frac{1}{2}g^{11}\frac{\partial}{\partial\xi}g_{22} = -\frac{1}{2}\cdot 1 \cdot \big(2\sinh\xi\cosh\xi\big) = -\sinh\xi\cosh\xi \\
\Gamma^2_{21} &= \Gamma^2_{12} = \frac{1}{2}g^{22}\frac{\partial}{\partial\xi}g_{22} = \frac{1}{2}\frac{2\sinh\xi\cosh\xi}{\sinh^2\xi} = \coth\xi
\end{aligned}$$

Then the two geodesic equations read

$$\begin{aligned}
\ddot{x}^1 + \Gamma^1_{22}\,\dot{x}^2\dot{x}^2 = 0 \quad &\Rightarrow \quad \ddot{\xi} - \sinh\xi\cosh\xi\,\dot{\varphi}^2 = 0 \\
\ddot{x}^2 + 2\Gamma^2_{12}\,\dot{x}^1\dot{x}^2 = 0 \quad &\Rightarrow \quad \ddot{\varphi} + 2\coth\xi\,\dot{\xi}\dot{\varphi} = 0
\end{aligned}$$

If $\dot\varphi \neq 0$, the second of these gives

$$\frac{1}{\dot\varphi}\frac{\mathrm{d}\dot\varphi}{\mathrm{d}\sigma} + 2\coth\xi\,\frac{\mathrm{d}\xi}{\mathrm{d}\sigma} = 0 \;\Rightarrow\; \ln\dot\varphi + 2\ln\bigl(\sinh\xi\bigr) = \text{cste.} \;\Rightarrow\; \boxed{\dot\varphi\sinh^2\xi = h = \text{cste}}$$

Rather than solving the last remaining geodesic equation, it is more simple to go back to the metric, if one chooses the parameter $\sigma \stackrel{!}{=} s$ as the arc length. From the metric

$$1 = a^2\left(\frac{\mathrm{d}\xi}{\mathrm{d}s}\right)^2 + a^2\sinh^2\xi\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2 = a^2\left(\frac{\mathrm{d}\xi}{\mathrm{d}s}\right)^2 + \frac{a^2h^2}{\sinh^2\xi}$$

where the conservation law derived above (implicitly taking $\sigma = s$) was inserted. From this, the geodesic equations can be written as

$$\frac{\mathrm{d}\xi}{\mathrm{d}s} = \pm\frac{\sqrt{\sinh^2\xi - a^2h^2}}{a\sinh\xi} \quad , \quad \frac{\mathrm{d}\varphi}{\mathrm{d}s} = \frac{h}{\sinh^2\xi}$$

Since we *require the geometric form of the geodesic*, we are really looking for the **orbit**, which we seek in the form $\varphi = \varphi(\xi)$. Hence

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = \frac{\mathrm{d}\varphi}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}\xi} = \pm\frac{h}{\sinh^2\xi}\frac{a\sinh\xi}{\sqrt{\sinh^2\xi - a^2h^2}} = \pm\frac{\mathrm{d}}{\mathrm{d}\xi}\arccos\left(\frac{h}{\sqrt{1/a^2+h^2}}\coth\xi\right)$$

the details of this integration will be spelled out below

This form of the orbit can be re-expressed as $\cos\left(\varphi - \varphi_0\right) - \frac{h}{\sqrt{1/a^2+h^2}} \coth\xi = 0$, which can be re-stated as $A\cos\varphi + B\sin\varphi + C\coth\xi = 0$ with known constantes A, B, C. This can also be rephrased as

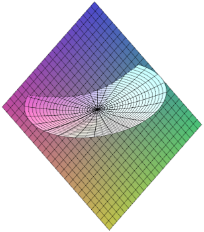
$$\boxed{A\sinh\xi\cos\varphi + B\sinh\xi\sin\varphi + C\cosh\xi = 0} \qquad (*)$$

Equations of this kind arise form the intersection of a hyperboloïde given by $x^2 + y^2 - z^2 = -a^2$, and a plane going through the origin which is described by $\alpha x + \beta y + \gamma z = 0$.

recall HESSE's normal form $\boldsymbol{n} \cdot \boldsymbol{r} = d$ for the equation of a plane, with distance d to the origin

Recall that a two-sheeted hyperboloïde has the parametrisation $x = a\sinh\xi\cos\varphi$, $y = a\sinh\xi\cos\varphi$ and $z = \pm a\cosh\xi$. Inserting this into the equation for the plane produces an equation of the form (*).

The figure shows an example of such an intersection.



[Mathematical remarks on the details of the integration:

want to integrate $\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = \pm \frac{ah}{\sinh^2 \xi} \frac{\sinh \xi}{\sqrt{\sinh^2 \xi - a^2 h^2}}$ (it is better **not** to cancel $\sinh \xi$)

$$\varphi - \varphi_0 = \pm ah \int \frac{\mathrm{d}\xi}{\sinh^2 \xi} \frac{1}{\sqrt{1 - \frac{a^2 h^2}{\sinh^2 \xi}}}$$

notice $\frac{1}{\sinh^2 \xi} = \coth^2 \xi - 1$

$$= \pm ah \int \frac{\mathrm{d}\xi}{\sinh^2 \xi} \left[1 - a^2 h^2 \left(\coth^2 \xi - 1\right)\right]^{-1/2}$$

notice $\frac{\mathrm{d} \coth \xi}{\mathrm{d}\xi} = -\frac{1}{\sinh^2 \xi}$

$$= \pm ah \int \frac{\mathrm{d}\xi}{\sinh^2 \xi} \left[1 + a^2 h^2 - a^2 h^2 \coth^2 \xi\right]^{-1/2}$$

set $u = \coth \xi \Rightarrow \mathrm{d}u = -\frac{\mathrm{d}\xi}{\sinh^2 \xi}$

$$= \mp ah \int \mathrm{d}u \, \frac{1}{ah} \left[\frac{1 + a^2 h^2}{a^2 h^2} - u^2\right]^{-1/2}$$

set $u = \sqrt{\frac{1}{a^2 h^2} + 1} \cos \alpha \Rightarrow \mathrm{d}u = -\sqrt{\frac{1}{a^2 h^2} + 1} \sin \alpha \mathrm{d}\alpha$

$$= \pm \frac{\sqrt{\frac{1}{a^2 h^2} + 1}}{\sqrt{\frac{1}{a^2 h^2} + 1}} \int \mathrm{d}\alpha \, \frac{\sin \alpha}{\sin \alpha} = \pm \alpha$$

$$= \pm \arccos \left(\frac{1}{\sqrt{\frac{1}{a^2 h^2} + 1}} \coth \xi \right) = \pm \arccos \left(\frac{h}{\sqrt{\frac{1}{a^2} + h^2}} \coth \xi \right)$$

as claimed.]

4. In less than 4 dimensions, the Riemann tensor admits simple forms.

Write down simple explicit expressions for the Riemann tensor in $d = 1, 2, 3$ dimensions. How many independent components of the Riemann tensor do you find in each case ?
Hint: for $d < 4$ the Riemann tensor can be expressed uniquely through the Ricci scalar R and the Ricci tensor. Use the known symmetries of the Riemann tensor

$$R_{\mu\nu\lambda\sigma} = R_{\lambda\sigma\mu\nu} \;\; ; \;\; R_{\mu\nu\lambda\sigma} = -R_{\nu\mu\lambda\sigma} = -R_{\mu\nu\sigma\lambda} \;\; ; \;\; R_{\mu[\nu\lambda\sigma]} = 0 \tag{4}$$

☞ In empty space, the field equations of gravitation are $R_{\mu\nu} = 0$. What follows about gravitation in empty space in $d = 2$ or $d = 3$ dimensions ?

Solution:

(a) $d=1$: the Riemann tensor has a single component $R_{1111}=0$ because of the symmetries. ☞ *all* $1D$ *spaces are flat*.

(b) $d=2$: the Riemann tensor has a single independent component. One can take into account the symmetries and write the Riemann tensor in the form

$$R_{\alpha\beta\gamma\delta}=\left(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma}\right)r$$

with a scalar r.

Verifying the first three symmetries in (4) is obvious. For the last one, the *Bianchi identity*, consider

$$\begin{aligned}R_{\alpha[\beta\gamma\delta]} &:= \frac{1}{3}\left(R_{\alpha\beta\gamma\delta}+R_{\alpha\gamma\beta\delta}+R_{\alpha\delta\beta\gamma}\right)\\ &= \frac{r}{3}\left(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma}+g_{\alpha\delta}g_{\gamma\beta}-g_{\alpha\beta}g_{\gamma\delta}+g_{\alpha\beta}g_{\gamma\delta}-g_{\alpha\gamma}g_{\delta\beta}\right) = 0\end{aligned}$$

We now compute the Ricci scalar

$$R=R^{\alpha\beta}{}_{\alpha\beta}=\left(g^{\alpha}{}_{\alpha}g^{\beta}{}_{\beta}-g^{\alpha}{}_{\beta}g^{\beta}{}_{\alpha}\right)r=\left(2\cdot 2-2\right)r=2r$$

so that we finally have

$$\boxed{R_{\alpha\beta\gamma\delta}=\frac{1}{2}\left(g_{\alpha\gamma}g_{\beta\delta}-g_{\alpha\delta}g_{\beta\gamma}\right)R}$$

(c) $d=3$: The Riemann tensor has 6 independent components. Since in $3D$, the Ricci tensor $R_{\mu\nu}=R_{\nu\mu}$ has 6 independent components as well, one may try to express the Riemann tensor through the $R_{\mu\nu}$. First, the following ansatz takes the symmetries into account

$$\begin{aligned} R_{\mu\nu\lambda\sigma} &= A\big(g_{\mu\lambda}R_{\nu\sigma}-g_{\nu\lambda}R_{\mu\sigma}-g_{\mu\sigma}R_{\nu\lambda}+g_{\nu\sigma}R_{\mu\lambda}\big) \\ &\quad +B\big(g_{\mu\lambda}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\lambda}\big)R \end{aligned}$$

where the constants A,B are to be found. (the second line is the same as in $2D$)
By contraction, one obtains

$$\begin{aligned} R^{\mu}{}_{\nu\mu\sigma} &= R_{\nu\sigma} \\ &= A\big(3R_{\nu\sigma}-R_{\nu\sigma}-R_{\nu\sigma}+g_{\nu\sigma}R\big)+B\big(3g_{\nu\sigma}-g_{\nu\sigma}\big)R \\ &= AR_{\nu\sigma}+g_{\nu\sigma}R\big(A+2B\big) \end{aligned}$$

This gives $A=1$ and $B=-\frac{1}{2}$.
for a check, contract once more: $R=AR(1+3)+BR2\cdot 3=\big(4\cdot 1-\frac{1}{2}\cdot 6\big)R=\big(4-3\big)R$.

* In empty space, the Ricci tensor vanishes $R_{\mu\nu}=0$ (hence $R=0$ as well).
Therefore, the $2D/3D$ full Riemann tensor vanishes in empty space
$\Rightarrow$! *no gravitational force in empty space for $d=2$ or $d=3$* !

☞ long-range gravitational forces across empty space need $d=4$ time-space dimensions

5. In the rest frame of a perfect fluid with (proper) mass density ρ and pressure p the energy-momentum tensor has the form

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} . \tag{5}$$

Find the energy-momentum tensor for an element of the liquid with proper mass density ρ and proper pressure p, which moves with the four-velocity $\mathfrak{u}$.

Solution:
In the rest frame, one has

explicitly assumed here: cartesian coordinates

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

In addition, one has $u^0 = 1$, $u^i = 0$, with $i = 1, 2, 3$. This suggests to propose the form

$$T^{\mu\nu} = p g^{\mu\nu} + (\rho + p) u^\mu u^\nu$$

* this reduces to the known expression in the rest frame, where $g^{\mu\nu} = \eta^{\mu\nu}$.
* the proposed form is generally co-variant.

Hence, by the principle of general co-variance, it will hold in general, for all coordinate systems.

N.B.: ermöglicht auf sehr billige Art, $T^{\mu\nu}$ in nichtkartesischen Koordinaten explizit hinzuschreiben, (nehme das Ruhesystem !) sobald man nur den metrischen Tensor $g^{\mu\nu}$ in diesen Koordinaten kennt

Série 4

1. Show that the gravitational force on a test body inside a gravitating hollow sphere vanishes.
Hint: Birkhoff's theorem states, that the Schwarzschild metric is a solution of the field equations $R_{\mu\nu} = 0$ in the case of spherical symmetry. Can such a solution have singularities in the inside of a sphere ?
You may remember an analogous result in Newton's theory of gravity or else in electrostatics.

Solution:
Because of the spherical symmetry, Birkhoff's theorem states that $g_{\mu\nu}$ must be Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{\mathscr{R}}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{\mathscr{R}}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

where $\mathrm{d}\Omega$ is the element of the solid angle.
If $\mathscr{R} \neq 0$, the solution has a singularity at $r = \mathscr{R}$ ($\mathscr{R}$ plays here the rôle of an integration constant). However, singularities are physically inadmissible, since one is in the interior of a sphere,without any masses. Hence $\mathscr{R} = 0$. Then the metric must be

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

which is the flat Minkowski metric. Since the corresponding Riemann tensor vanishes, there is *no gravitational force*.

2. Mathematically a curved ('Riemann') space in d dimensions has a *constant curvature*, if the Riemann tensor $R_{\mu\nu\lambda\kappa}$ can be expressed as follows through the metric tensor $g_{\mu\nu}$ (with $\det g \neq 0$):

$$R_{\mu\nu\lambda\kappa} = K \left(g_{\mu\lambda} g_{\nu\kappa} - g_{\mu\kappa} g_{\nu\lambda} \right) \tag{1}$$

where the constant K describes the constant curvature. Show that the Ricci tensor has the form

$$R_{\mu\nu} := g^{\sigma\tau} R_{\sigma\mu\tau\nu} = K(d-1) g_{\mu\nu} \tag{2}$$

Can one make a statement about two-dimensional spaces ($d = 2$) ?
Hints: $g^{\sigma\tau} g_{\sigma\tau} = d$. Recall the general form of $R_{\mu\nu\lambda\kappa}$ for $d = 2$.

Solution:
A direct calculation gives

$$
\begin{aligned}
R_{\mu\nu} &= g^{\sigma\tau} R_{\sigma\mu\tau\nu} \\
&= K g^{\sigma\tau}\left(g_{\sigma\tau} g_{\mu\nu} - g_{\sigma\nu} g_{\mu\tau}\right) \\
&= K d g_{\mu\nu} - K g^{\tau}_{\nu} g_{\mu\tau} \;=\; K(d-1) g_{\mu\nu}
\end{aligned}
$$

If $d = 2$, we had from a previous exercice that $R_{\alpha\beta\gamma\delta} = \frac{R}{2}\left(g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma}\right)$. Comparison shows that $K = \frac{1}{2}R$. A conformal transformation can be used to make $R = \mathrm{cste.}$.
Hence, for $d = 2$ any space is conformally related to a Riemann space of constant curvature.

3. A detailed study of the mouvement of galaxies, far from earth, has raised interest in the following variant of Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu} \quad , \quad \kappa := -\frac{8\pi G}{c^2} \tag{3}$$

where Λ is called *cosmological constant*. G is Newton's gravitational constant.

a) What is the dimension of Λ ?

b) Show, via a convenient contraction, that the Ricci scalar $R = \kappa T + 4\Lambda$ (here $T := T^{\mu}_{\mu}$) and derive from (3) the following alternative form of the field equations

$$R_{\mu\nu} = \Lambda g_{\mu\nu} - \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \tag{4}$$

c) Without external sources, that is for $T_{\mu\nu} = 0$, the flat Minkowski metric a solution of the field equations (4) ? Compare the field equations (4) without sources with the form of a Riemann space with constant curvature, as derived in a previous exercice. Can you interpret geometrically the cosmological constant Λ ?

d) In the non-relativistic limit the component $\mu = \nu = 0$ from eq. (4) reproduces Newton's equations. Show that for $\Lambda \neq 0$ one obtains a generalised Poisson's equation

$$\Delta\phi + \Lambda c^2 = 4\pi G\rho \tag{5}$$

where $\phi = -\frac{c^2}{2}h_{00}$ is Newton's gravitational potential, $g_{\mu\nu} = \eta_{\mu\nu} - h_{\mu\nu} + \ldots$ and Δ denotes the usual Laplace operator.

e) Show that for $\Lambda \neq 0$ one has phenomenologically an additional force $F_\Lambda = \frac{1}{3}\Lambda c^2 r$ in the distance r from the centre of the force. For which class of objects would you expect measurable effects of the constant Λ ?

<u>**Hint:**</u> in $3D$ spherical coordinates the Laplace operator reads $\Delta f(r) = \frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(rf(r)\right)$, for the case of a spherical symmetry.

Solution:
(a) Since $R_{\mu\nu}$ contains two derivatives $\Rightarrow$ Λ has dimension $[\mathrm{length}^{-2}]$.
(b) From (3) have $R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} - \kappa T_{\mu\nu}$. Contracting, one finds $R = R^{\mu}{}_{\mu} = \frac{1}{2} \cdot 4R - 4\Lambda - \kappa T$, or $R = 4\Lambda + \kappa T$. Insert this into (3) and find

$$R_{\mu\nu} = \frac{1}{2} g_{\mu\nu}\big(4\Lambda + \kappa T\big) - \Lambda g_{\mu\nu} - \kappa T_{\mu\nu} = \Lambda g_{\mu\nu} - \kappa\big(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\big)$$

(c) The Minkowski metric is solution of $R_{\mu\nu} = 0$, which is different from (4) with $\Lambda \neq 0$ and $T_{\mu\nu} = 0$. In a previous exercice we have seen that for spaces with a constant curvature $R_{\mu\nu} = K(d-1) g_{\mu\nu}$. Comparison gives $\Lambda = 3K$. In the absence of sources, *the resulting space has the constant curvature* $K = \frac{\Lambda}{3}$.
(d) In principle, it is enough to repeat the calculations for obtaining the non-relativistic limit from the lecture. For clarity, the main steps are repeated here.

The newtonian limit is a *weak-field limit* where one sets $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with h 'small'. In addition, as $c \to \infty$, one expects $\tau \simeq t$, $\frac{\mathrm{d}x^0}{\mathrm{d}\tau} \simeq c$, $\frac{\mathrm{d}x^i}{\mathrm{d}\tau} \simeq \frac{\mathrm{d}x^i}{\mathrm{d}t} = v^i \ll c$. Furthermore, this is a *static* approximation where the potentials are time-independent. The three spatial geodesic equations become

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + c^2 \Gamma^i_{00}\big(1 + \mathrm{O}(1/c)\big) = 0 \quad \Rightarrow \quad \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -c^2 \Gamma^i_{00} = a^i \quad \text{acceleration}$$

which begins to look like a newtonian equation of motion.

One must now work out, in the static approximation and to linear order in h:

$$\Gamma^i_{00} = \frac{1}{2} g^{i\nu} \big(\underbrace{2 g_{\nu 0,0}}_{=0} - g_{00,\nu} \big) = -\frac{1}{2} g^{ik} g_{00,k} \simeq -\frac{1}{2} \eta^{ik} h_{00,k} + \mathrm{O}(h^2) = -\frac{1}{2} \nabla^i h_{00}$$

This gives the equation of motion $\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -c^2 \Gamma^i_{00} = \frac{c^2}{2} \nabla^i h_{00}$ and should be compared with the newtonian equation $\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\nabla^i \phi$. One identifies the newtonian gravitational potential $\boxed{h_{00} = -\frac{2}{c^2} \phi}$, or $g_{00} = -\big(1 + \frac{2}{c^2} \phi\big)$.

In order to find the newtonian limit of the field equation, consider again

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \Rightarrow \quad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathrm{O}(h^2)$$

[to see this: $g_{\mu\nu} g^{\nu\kappa} = \big(\eta_{\mu\nu} + h_{\mu\nu}\big)\big(\eta^{\nu\kappa} - h_{\nu\kappa}\big) \simeq \eta_{\mu\nu}\eta^{\nu\kappa} - \eta_{\mu\nu} h_{\nu\kappa} + h_{\mu\nu}\eta^{\nu\kappa} + \mathrm{O}(h^2)$
$= \delta^\kappa_\mu - h^\kappa_\mu + h^\kappa_\mu = \delta^\kappa_\mu$]
Then

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} \big(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho} \big) \simeq \frac{1}{2} \eta^{\mu\rho} \big(h_{\rho\nu,\lambda} + h_{\rho\lambda,\nu} - h_{\nu\lambda,\rho} \big) + \mathrm{O}(h^2)$$

which itself is of first order in h throughout.

Recall the computation of the Ricci tensor

$$
\begin{aligned}
R_{\mu\nu} &= \Gamma^{\kappa}_{\mu\nu,\kappa} - \Gamma^{\kappa}_{\mu\kappa,\nu} + \underbrace{\Gamma^{\kappa}_{\rho\kappa}\Gamma^{\rho}_{\mu\nu} - \Gamma^{\kappa}_{\rho\nu}\Gamma^{\rho}_{\mu\kappa}}_{=\mathrm{O}(h^2),\ \text{negligible}} \\
&= \frac{1}{2}\eta^{\kappa\sigma}\left(h_{\sigma\nu,\mu\kappa} + h_{\mu\kappa,\sigma\nu} - h_{\mu\nu,\sigma\kappa} - h_{\sigma\kappa,\mu\nu}\right) + \mathrm{O}(h^2)
\end{aligned}
$$

Concentrate on the component $\mu = \nu = 0$ (use the static approximation !):

$$
\begin{aligned}
R_{00} &\simeq \frac{1}{2}\eta^{\kappa\sigma}\Big(\underbrace{h_{\sigma 0,0\kappa} + h_{0\kappa,\sigma 0}}_{=0} - h_{00,\sigma\kappa} - \underbrace{h_{\sigma\kappa,00}}_{=0}\Big) \\
&= -\frac{1}{2}\eta^{\kappa\sigma}h_{00,\sigma\kappa} = -\frac{1}{2}\left(-\frac{1}{c^2}\underbrace{\frac{\partial^2}{\partial t^2}}_{=0} + \boldsymbol{\nabla}^2\right)h_{00} = -\frac{1}{2}\boldsymbol{\nabla}^2 h_{00}
\end{aligned}
$$

Next, in the newtonian limit, the energy-momentum tensor of matter is $T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$

Then $T = T^{\mu}_{\mu} = -\rho$ and $T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T = \begin{pmatrix} \rho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + \frac{\rho}{2}\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \frac{\rho}{2}\delta^{\mu\nu}$.

The field equation (4) now takes the form

$$R_{00} = -\frac{1}{2}\boldsymbol{\nabla}^2 h_{00} = \frac{1}{c^2}\boldsymbol{\nabla}^2\phi \overset{!}{=} \Lambda g_{00} - \kappa\frac{\rho}{2}\delta_{00} = -\frac{\kappa}{2}\rho - \Lambda$$

which gives the 'newtonian' form

$$\nabla^2\phi + \Lambda c^2 = -c^2\frac{\kappa}{2}\rho$$

For $\Lambda = 0$, this should reproduce the newtonian equation $\boldsymbol{\nabla}\phi = 4\pi G\rho$ from which the κ given in the problem statement (3) follows. The full field equation in the newtonian limit is

$$\boxed{\nabla^2\phi + \Lambda c^2 = 4\pi G\rho} \qquad (*)$$

N.B.: for the sake of comparison with the litterature, we have kept the c ...

(e) On phenomenological consequences: (*) is linear, hence just consider the extra term coming from $\nabla^2\phi_\Lambda = -\Lambda$.

$$\Rightarrow \ \frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\big(r\phi_\Lambda(r)\big) = -\Lambda \ \Rightarrow \ \frac{\mathrm{d}^2}{\mathrm{d}r^2}\big(r\phi_\Lambda(r)\big) = -\Lambda r \ \Rightarrow \ r\phi_\Lambda(r) = -\frac{1}{2\cdot 3}\Lambda r^3$$

such that finally $\phi_\Lambda(r) = -\frac{\Lambda}{6}r^2$. The corresponding *'cosmological force'* is $F_\Lambda(r) = -\frac{\partial\phi_\Lambda(r)}{\partial r} = \frac{1}{3}\Lambda r$.

Since $\Lambda \approx 10^{-52}[\mathrm{m}^{-2}]$, there is a corresponding length scale $\Lambda^{-1/2} \sim 10^{26}[\mathrm{m}]$. In astronomy, an often-used length scale is the light year, which is of order $1[\text{light year}] \sim 10^{16}[\mathrm{m}]$ (which is distance a photon travels in one year). Hence $\Lambda^{-1/2} \sim 10^{10}[\text{light year}]$ which is the order of magnitude of the radius of the visible universe. Effects of the 'cosmological force' should be seen in considerations of the behaviour (e.g. expansion) of the entire universe – and next, one might consider effects on the mouvement of clusters of galaxies which occur at scales at the order of $10^8[\text{light year}]$.

4. At the surface of a pseudo-sphere P^2 one has the metric

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = a^2 \left(\mathrm{d}\chi^2 + \sinh^2\chi \mathrm{d}\phi^2\right) \tag{6}$$

where a is a fixed length scale. Is P^2 a curved space ? Compute the Ricci scalar R as a function of a. Do you see a property in which the pseudo-sphere P^2 is distinct from the usual sphere S^2 ?
Hints: The Riemann tensor R and the Christoffel symbols Γ are given by

$$\begin{aligned} R^\kappa_{\lambda\mu\nu} &= \Gamma^\kappa_{\lambda\nu,\mu} - \Gamma^\kappa_{\lambda\mu,\nu} + \Gamma^\kappa_{\rho\mu}\Gamma^\rho_{\lambda\nu} - \Gamma^\kappa_{\rho\nu}\Gamma^\rho_{\lambda\mu} \\ \Gamma^\mu_{\nu\lambda} &= \frac{1}{2}g^{\mu\rho}\left(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho}\right) \end{aligned} \tag{7}$$

How many independent components the Riemann tensor does have for P^2 ? The Ricci tensor is $R_{\mu\nu} = R^\kappa_{\mu\kappa\nu}$ and the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu} = R^\mu_\mu$.

Solution:
One has the metric, with $(x^1, x^2) = (\chi, \phi)$: $\mathrm{d}s^2 = a^2\left(\mathrm{d}\chi^2 + \sinh^2\chi\,\mathrm{d}\phi^2\right)$. The metric tensor is diagonal, hence using previous exercices one readily has the non-vanishing Christoffel symbols

$$\begin{aligned}
\Gamma^1_{22} &= -\frac{1}{2}\frac{1}{g_{11}}\frac{\partial}{\partial\chi}g_{22} = -\frac{1}{2}\cdot 1\cdot 2\sinh\chi\cosh\chi = -\sinh\chi\cosh\chi \\
\Gamma^2_{21} &= \Gamma^2_{12} = \frac{1}{2}\frac{1}{g_{22}}\frac{\partial}{\partial\chi}g_{22} = \frac{1}{2}\frac{2\sinh\chi\cosh\chi}{\sinh^2\chi} = \coth\chi
\end{aligned}$$

Since P^2 is two-dimensional, there is a single independent component of the Riemann tensor, for example

$$\begin{aligned}
R^1_{212} &= \Gamma^1_{22,1} - \Gamma^1_{21,2} + \Gamma^1_{i1}\Gamma^i_{22} - \Gamma^1_{i2}\Gamma^i_{21} \\
&= \frac{\partial}{\partial\chi}\left(-\sinh\chi\cosh\chi\right) - \Gamma^1_{22}\Gamma^2_{21} \\
&= -\cosh^2\chi - \sinh^2\chi - \left(-\sinh\chi\cosh\chi\right)\coth\chi \\
&= -\cosh^2\chi - \sinh^2\chi + \cosh^2\chi = -\sinh^2\chi \neq 0
\end{aligned}$$

N.B.: all terms not spelled out explicitly here vanish !

Since $R^1_{212} \neq 0 \Rightarrow$ *P^2 is curved*

$$R^2_{121} = g^{22} g_{11} R^1_{212} = \frac{1}{a^2 \sinh^2 \chi} a^2 \left(-\sinh^2 \chi\right) = -1 \neq 0$$

N.B.: one might have found R^2_{121} directly as well

$$\begin{aligned} R^2_{121} &= \Gamma^2_{11,2} - \Gamma^2_{12,1} + \Gamma^2_{i2} \Gamma^i_{11} - \Gamma^2_{i1} \Gamma^i_{12} \\ &= -\frac{\partial}{\partial \chi} \left(\coth \chi\right) - \Gamma^2_{21} \Gamma^2_{12} \\ &= -\frac{\sinh^2 \chi - \cosh^2 \chi}{\sinh^2 \chi} - \coth^2 \chi \\ &= -1 + \coth^2 \chi - \coth^2 \chi \; = \; -1 \end{aligned}$$

One computes the Ricci tensor

$$\begin{aligned} R_{11} &= R^i_{1i1} = R^2_{121} = -1 \\ R_{22} &= R^i_{2i2} = R^1_{212} = -\sinh^2 \chi \end{aligned}$$

N.B.: for illustration: $R_{12} = R^i_{1i2} = R^1_{112} + R^2_{122}$ and

$$\begin{aligned} R^1_{112} &= \Gamma^1_{12,1} - \Gamma^1_{11,2} + \Gamma^1_{i1} \Gamma^i_{12} - \Gamma^1_{i2} \Gamma^i_{11} \; = \; 0 \\ R^2_{122} &= \Gamma^2_{12,2} - \Gamma^2_{12,2} + \Gamma^2_{i2} \Gamma^i_{12} - \Gamma^2_{i2} \Gamma^i_{12} \; = \; 0 \end{aligned}$$

as one might have anticipated from the symmetries of the Riemann tensor. $\Rightarrow R_{12} = 0$.

Finally, the Ricci scalar is

$$R = g^{11}R_{11} + g^{22}R_{22} = \frac{1}{a^2}(-1) + \frac{1}{a^2\sinh^2\chi}\left(-\sinh^2\chi\right) = -\frac{2}{a^2} < 0 \qquad \text{(R)}$$

R is non-vanishing, of dimension $[\mathrm{length}^{-2}]$ and negative. This last property is distinct from the usual sphere S^2, where $R = +\frac{2}{a^2} > 0$ (see lectures).

We also note the matrix forms of the metric and Ricci tensors:

$$g_{\mu\nu} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2\sinh^2\chi \end{pmatrix} \;,\; R_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & -\sinh^2\chi \end{pmatrix}$$

Hence, $\boxed{R_{\mu\nu} = -\frac{1}{a^2}g_{\mu\nu}}$. *The pseudo-sphere P^2 is a space of constant curvature K.*

N.B.: Indeed, it is known from a previous exercice that in $2D$ spaces with constant curvature $K = \frac{1}{2}R$ – in agreement with the explicit results found here, where $K = -\frac{1}{a^2} = \frac{1}{2}R$, see eq. (R).

⚠ the calculations here are for geometric spaces and **not** for Minkowski time-spaces !

Theorem: (Huygens) *The surface area of P^2 is $4\pi a^2$ and its volume $\frac{2}{3}\pi a^3$.*

Theorem: (Hilbert) *It is impossible to embed P^2 into the euclidean $\mathbb{R}^3$.*

Theorem: (Whitney) *For any manifold with $\dim \mathscr{M} = m \leq \frac{n}{2}$ there is an embedding $f : \mathscr{M} \to \mathbb{R}^n$.*