Allgemeine Relativitätstheorie

Malte Henkel

[a]Laboratoire de Physique de Chimie Théoriques (CNRS UMR 7019),
Université de Lorraine **Nancy**, France

[b]Centro de Física Téorica e Computacional, Universidade de Lisboa, Portugal

E-Post/courriel: malte.henkel@univ-lorraine.fr

Vorlesung Wintersemester 2020/21, Université de Sarrebruck

Some further reading

L. Ryder, *General relativity*, Cambridge Univ. Press (2009)

T.P. Cheng, *Relativity, Gravitation and Cosmology*, 2^e éd., Oxford (2010)

S. Weinberg, *Gravitation and cosmology*, Wiley (1978)

A. Barrau, J. Grain *Relativité générale*, 2^e éd., Dunod (2016)

C.M. Will, *Confrontation between general relativity and experiments*, Liv. Rev. Relativity **9**, 3 (2006) & **17**, 4 (2014)

C.M. Will, *Theory and experiment in gravitational physics*, 2$^{\rm nd}$ ed, Cambridge (2018)

C.M. Will, N. Yunes, *Is Einstein still right ?*, Oxford (2020)

Überblick

Vorlesung VI: Kovariante Ableitung, Paralleltransport, Feldgleichungen
Vorlesung VII: Einsteins Feldgleichungen der Gravitation; Schwarzschild-Lösung
Vorlesung VIII: Experimentelle Prüfungen
Vorlesung IX: Schwarze Löcher; Effektives Potential; Innere Schwarzschild-Lösung
Vorlesung X: Relativistische Astrophysik; Weiße Zwerge und Neutronensterne
Vorlesung XI: Gravitationswellen

Vorlesung VI

Summary of previous results on curved geometry

distances are measured through the **metric tensor** $\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b$
the components of the metric tensor given by base vectors $\boldsymbol{e}_a$:

$$g_{ab} := \boldsymbol{e}_a \cdot \boldsymbol{e}_b = g_{ba}$$

geodesic: *shortest line between two fixed points of space*
derived from a 'Lagrangian' $L = g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\sigma}\frac{\mathrm{d}x^b}{\mathrm{d}\sigma}$ with σ: **arc length**
leads to **geodesic equation**, with $\dot{x}^\rho := \mathrm{d}x^\rho/\mathrm{d}\sigma$ and $A_{,\mu} := \partial A/\partial x^\mu$

$$\ddot{x}^\rho + \Gamma^\rho_{\kappa\lambda}\dot{x}^\kappa\dot{x}^\lambda = 0 \;\; , \;\; \Gamma^\rho_{\kappa\lambda} = \frac{1}{2}g^{\rho\mu}\left(g_{\mu\kappa,\lambda} + g_{\mu\lambda,\kappa} - g_{\kappa\lambda,\mu}\right) = \Gamma^\rho_{\lambda\kappa}$$

the $\Gamma^\rho_{\kappa\lambda}$ are the **Christoffel symbols**

Theorem: *Locally, one can find a new coordinate system $x \mapsto \bar{x}$ such that*

$$\bar{g}_{ab}(\bar{x}) = \delta_{ab} + \gamma_{abcd}\bar{x}^c\bar{x}^d + \ldots$$

☞ curvature effects are described by objects beyond the Christoffel symbols tensors have the most simple possible transformation behaviour under a coordinate change $x^\mu \mapsto x'^\mu$

a **contra-variant vector** transforms as $V^\mu \mapsto V'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu$

a **co-variant vector** transforms as $V_\mu \mapsto V'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} V_\nu$

a **tensor of level** $\binom{r}{s}$ transforms as

$$T'^{\mu_1,\ldots,\mu_r}_{\lambda_1\ldots,\lambda_s} = \frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}} \cdots \frac{\partial x'^{\mu_r}}{\partial x^{\rho_r}} \cdot \frac{\partial x^{\sigma_1}}{\partial x'^{\lambda_1}} \cdots \frac{\partial x^{\sigma_r}}{\partial x'^{\lambda_s}} T^{\rho_1\ldots\rho_r}_{\sigma_1\ldots\sigma_s}$$

vectors are tensors of level $\binom{1}{0}$ and $\binom{0}{1}$. The metric tensor $g_{\mu\nu}$ has level $\binom{0}{2}$.

If $D := \det \frac{\partial x'^\mu}{\partial x^\nu}$ is the jacobian, a **tensor density of weight** w transforms as

$$S'^{\mu_1,\ldots,\mu_r}_{\lambda_1\ldots,\lambda_s} = D^{-w} \frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}} \cdots \frac{\partial x'^{\mu_r}}{\partial x^{\rho_r}} \cdot \frac{\partial x^{\sigma_1}}{\partial x'^{\lambda_1}} \cdots \frac{\partial x^{\sigma_r}}{\partial x'^{\lambda_s}} S^{\rho_1\ldots\rho_r}_{\sigma_1\ldots\sigma_s}$$

for the metric tensor $g' = \det g'_{\mu\nu} = D^{-2} g$. With the **volume element** $\mathrm{d}V = \frac{1}{4!}\varepsilon_{\kappa\lambda\mu\nu}\mathrm{d}x^\kappa \mathrm{d}x^\lambda \mathrm{d}x^\mu \mathrm{d}x^\nu$ one has the invariance relation

($\varepsilon_{\kappa\lambda\mu\nu}$ is a tensor density with $w = -1$)

$$\sqrt{g'}\,\mathrm{d}V' = \sqrt{g}\,\mathrm{d}V$$

N.B.: in Minkowski space this becomes $\sqrt{-g'}\,\mathrm{d}V' = \sqrt{-g}\,\mathrm{d}V$.

3.6 Co-variant derivative

N.B.: ortho-normal basis vectors $\boldsymbol{e}_\mu = \boldsymbol{e}_\mu(x)$ are position-dependent !

Definition: *The* **connexion** *is given by* $\boldsymbol{e}_\mu \cdot \boldsymbol{e}^\nu = \delta_\mu^\nu$

$$\boxed{\partial_\nu \boldsymbol{e}_\mu = \gamma^\lambda_{\nu\mu} \boldsymbol{e}_\lambda \ \ , \ \ \partial_\nu \boldsymbol{e}^\mu = -\gamma^\mu_{\nu\lambda} \boldsymbol{e}^\lambda}$$

the signs are correct $0 = \partial_\nu \left(\boldsymbol{e}_\mu \cdot \boldsymbol{e}^\kappa\right) = \boldsymbol{e}_\mu \cdot \left(\partial_\nu \boldsymbol{e}^\kappa\right) + \left(\partial_\nu \boldsymbol{e}_\mu\right) \cdot \boldsymbol{e}^\kappa = \boldsymbol{e}_\mu \cdot (-1)\gamma^\kappa_{\nu\rho} \boldsymbol{e}^\rho + \gamma^\rho_{\nu\mu} \boldsymbol{e}_\rho \cdot \boldsymbol{e}^\kappa$

Example: for a vector $\boldsymbol{V}$, have components $V^\mu = \boldsymbol{e}^\mu \cdot \boldsymbol{V}$

$$\Rightarrow \quad \partial_\nu V^\mu = \underbrace{\boldsymbol{e}^\mu \cdot \left(\partial_\nu \boldsymbol{V}\right)}_{\text{transforms as a tensor}} + \underbrace{\boldsymbol{V} \cdot \left(\partial_\nu \boldsymbol{e}^\mu\right)}_{\text{spoils tensor properties}}$$

explicitly

$$\partial'_\nu V'^\mu = \underbrace{\frac{\partial x^\lambda}{\partial x'^\nu} \frac{\partial x'^\mu}{\partial x^\rho} \left(\partial_\lambda V^\rho\right)}_{\text{transforms as a tensor}} + \underbrace{\frac{\partial^2 x'^\mu}{\partial x'^\nu \partial x^\rho} V^\rho}_{\text{spoils tensor properties}}$$

N.B.: the $\gamma^\lambda_{\mu\nu}$ are *not* the components of a tensor

? how to correct the transformation properties of the derivative ?

(1) for a scalar Φ, all is well, since no basis vector is needed

$$\partial_\mu \Phi \mapsto \partial'_\mu \Phi' = \frac{\partial x^\lambda}{\partial x'^\mu} \partial_\lambda \Phi$$

(2) a vector $\boldsymbol{V}$ does not depend on a basis

$$\partial_\mu \boldsymbol{V} \mapsto \partial'_\mu \boldsymbol{V} = \frac{\partial x^\lambda}{\partial x'^\mu} \partial_\lambda \boldsymbol{V}$$

$$\boldsymbol{e}'^\nu \cdot \partial'_\mu \boldsymbol{V} = \boldsymbol{e}'^\nu \cdot \frac{\partial x^\lambda}{\partial x'^\mu} \partial_\lambda \boldsymbol{V} = \frac{\partial x'^\nu}{\partial x^\rho} \frac{\partial x^\lambda}{\partial x'^\mu} \boldsymbol{e}^\rho \cdot \partial_\lambda \boldsymbol{V}$$

$\Rightarrow$ $\boldsymbol{e}^\nu \cdot \partial_\mu \boldsymbol{V}$ transforms as a tensor.

Definition: *The* **co-variant derivative** *of a vector is defined as*

$$D_\mu V^\nu := \boldsymbol{e}^\nu \cdot \partial_\mu \boldsymbol{V}$$

rewrite as $D_\mu V^\nu = \partial_\mu V^\nu - \boldsymbol{V} \cdot \left(\partial_\mu \boldsymbol{e}^\nu\right) = \partial_\mu V^\nu + \gamma^\nu_{\mu\lambda} \boldsymbol{e}^\lambda \cdot \boldsymbol{V}$

in components, the co-variant derivative of a vector becomes finally

$$\boxed{D_\mu V^\nu = V^\nu{}_{;\mu} = V^\nu{}_{,\mu} + \gamma^\nu_{\mu\lambda} V^\lambda = \partial_\mu V^\nu + \gamma^\nu_{\mu\lambda} V^\lambda}$$

similarly, one write co-variant derivatives of more general tensors

contra-variant vector $D_\mu V^\nu = V^\nu{}_{;\mu} = V^\nu{}_{,\mu} + \gamma^\nu_{\mu\lambda} V^\lambda = \partial_\mu V^\nu + \gamma^\nu_{\mu\lambda} V^\lambda$

co-variant vector $D_\mu V_\nu = V_{\nu;\mu} = V_{\nu,\mu} - \gamma^\lambda_{\nu\mu} V_\lambda = \partial_\mu V_\nu - \gamma^\lambda_{\nu\mu} V_\lambda$

and for a general tensor of level $\binom{r}{s}$

$$\begin{aligned} T^{\mu_1\ldots\mu_r}_{\lambda_1\ldots\lambda_s}{}_{;\rho} &= T^{\mu_1\ldots\mu_r}_{\lambda_1\ldots\lambda_s}{}_{,\rho} \\ &+ \quad \gamma^{\mu_1}_{\nu_1\rho} T^{\nu_1\mu_2\ldots\mu_r}_{\lambda_1\ldots\lambda_s} + \quad \text{further } r-1 \text{ terms, one for each upper index} \\ &- \quad \gamma^{\kappa_1}_{\lambda_1\rho} T^{\mu_1\ldots\mu_r}_{\kappa_1\lambda_2\ldots\lambda_s} - \quad \text{further } s-1 \text{ terms, one for each lower index} \end{aligned}$$

* The metric tensor has the *important property*

$$\boxed{g_{\mu\nu;\rho} = 0}$$

Proof: start from $g_{\mu\nu} = \boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu$. Then

$$\begin{aligned} \partial_\rho g_{\mu\nu} &= \partial_\rho(\boldsymbol{e}_\mu \cdot \boldsymbol{e}_\nu) = (\partial_\rho \boldsymbol{e}_\mu) \cdot \boldsymbol{e}_\nu + \boldsymbol{e}_\mu \cdot (\partial_\rho \boldsymbol{e}_\nu) \\ &= \gamma^\kappa_{\rho\mu} \boldsymbol{e}_\kappa \cdot \boldsymbol{e}_\nu + \gamma^\kappa_{\rho\nu} \boldsymbol{e}_\mu \cdot \boldsymbol{e}_\kappa \end{aligned}$$

Hence, $g_{\mu\nu;\rho} = g_{\mu\nu,\rho} - \gamma^\kappa_{\rho\mu} g_{\kappa\nu} - \gamma^\kappa_{\rho\nu} g_{\mu\kappa} = 0.$ QED

Theorem: *One has the identity* $\boxed{\Gamma^\lambda_{\mu\nu} = \gamma^\lambda_{\mu\nu}}$

☞ The Christoffel symbols and the connexion are the same.

Proof: this is a consequence of the property $g_{\mu\nu;\lambda} = 0$. Write it three times

$$\begin{aligned}
g_{\mu\nu;\lambda} &= \partial_\lambda g_{\mu\nu} - \gamma^\rho_{\lambda\mu} g_{\rho\nu} - \gamma^\rho_{\lambda\nu} g_{\mu\rho} && (1)\\
g_{\lambda\mu;\nu} &= \partial_\nu g_{\lambda\mu} - \gamma^\rho_{\nu\lambda} g_{\rho\mu} - \gamma^\rho_{\mu\nu} g_{\lambda\rho} = \partial_\nu g_{\lambda\mu} - \gamma^\rho_{\nu\lambda} g_{\mu\rho} - \gamma^\rho_{\mu\nu} g_{\rho\lambda} && (2)\\
-g_{\nu\lambda;\mu} &= -\partial_\mu g_{\nu\lambda} + \gamma^\rho_{\mu\nu} g_{\rho\lambda} + \gamma^\rho_{\mu\lambda} g_{\nu\rho} = -\partial_\mu g_{\nu\lambda} + \gamma^\rho_{\mu\nu} g_{\lambda\rho} + \gamma^\rho_{\mu\lambda} g_{\rho\nu} && (3)
\end{aligned}$$

Take the sum of these three equations, and also recall symmetry $\gamma^\lambda_{\mu\nu} = \gamma^\lambda_{\nu\mu}$

$$\partial_\lambda g_{\mu\nu} + \partial_\nu g_{\lambda\mu} - \partial_\mu g_{\nu\lambda} - 2\gamma^\rho_{\lambda\nu} g_{\mu\rho} = 0$$

$$\Rightarrow \quad \gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\nu g_{\mu\rho} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\mu\nu}\right) = \Gamma^\lambda_{\mu\nu}$$

according to the definition of the Christoffel symbol. QED

From now on, we shall write for the co-variant derivatives

$$V_{\nu;\mu} = V_{\nu,\mu} - \Gamma^\lambda_{\nu\mu} V_\lambda \;\; , \quad V^\nu{}_{;\mu} = V^\nu{}_{,\mu} + \Gamma^\lambda_{\nu\mu} V_\lambda$$

and so on ...

Theorem (euclidean locality) *If at a point P, one has coordinates x^μ and the metric tensor $g_{\mu\nu}$, then there exists a transformation $x^\mu \mapsto \bar{x}^\mu$ such that*

$$\bar{g}_{\mu\nu}(\bar{x}) = \delta_{\mu\nu} + \gamma_{\mu\nu\alpha\beta}\bar{x}^\alpha \bar{x}^\beta + \dots$$

Proof: consider the transformation ($\bar{x}$ are called *'geodesic coordinates'*)

$$x^\mu = \bar{x}^\mu - \frac{1}{2}\Gamma^\mu_{\nu\lambda}\bar{x}^\nu \bar{x}^\lambda + \dots \quad , \quad \frac{\partial x^\mu}{\partial \bar{x}^\nu} = \delta^\mu_\nu - \Gamma^\mu_{\nu\lambda}\bar{x}^\lambda + \dots$$

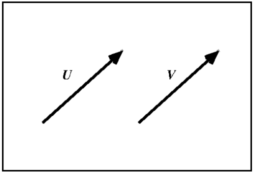
and also expand $g_{\mu\nu}(x) = g_{\mu\nu}(0) + g_{\mu\nu,\lambda}(0)x^\lambda + \dots$. Write the metric tensor

$$\begin{aligned}
\bar{g}_{\mu\nu}(\bar{x}) &= \frac{\partial x^\lambda}{\partial \bar{x}^\mu}\frac{\partial x^\rho}{\partial \bar{x}^\nu} g_{\lambda\rho}(x) \\
&\simeq \left(\delta^\lambda_\mu - \Gamma^\lambda_{\mu\alpha}\bar{x}^\alpha\right)\left(\delta^\rho_\nu - \Gamma^\rho_{\nu\beta}\bar{x}^\beta\right)\left(g_{\lambda\rho}(0) + g_{\lambda\rho,\gamma}\bar{x}^\gamma\right) + \dots \\
&= g_{\mu\nu}(0) + \left[g_{\mu\nu,\alpha}(0) - \Gamma^\lambda_{\mu\alpha}g_{\lambda\nu}(0) - \Gamma^\lambda_{\alpha\nu}g_{\mu\lambda}(0)\right]\bar{x}^\alpha + \dots \\
&= g_{\mu\nu}(0) + \underbrace{g_{\mu\nu;\alpha}(0)}_{=0}\ \bar{x}^\alpha + \dots
\end{aligned}$$

hence the first non-vanishing terms are quadratic in the $\bar{x}^\alpha$. Since the matrix $g_{\mu\nu}(0)$ is symmetric, it can be diagonalised via an orthogonal transformation. A change of scale in the $\bar{x}^\alpha$ achieves the form $\bar{g}_{\mu\nu}(0) = \delta_{\mu\nu}$. QED

3.7 Parallel transport

? when are two vectors $\boldsymbol{U}$, $\boldsymbol{V}$ parallel ?

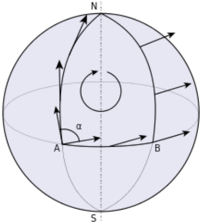


(a) euclidean plane

are the vectors $\boldsymbol{U}$, $\boldsymbol{V}$ parallel ?

to decide this, try to translate $\boldsymbol{V}$, without changing neither direction nor orientation, such that becomes identical to $\boldsymbol{U}$

☞ since this works, the plane is **flat**



Source: wikipedia

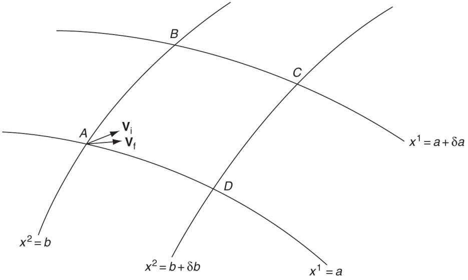
(b) sphere

after parallel transport, the initial vector $\boldsymbol{V}_i$ and final vectors $\boldsymbol{V}_f$, *after a round trip from A via N and B back to A*, need not be identical: $\boldsymbol{V}_i \neq \boldsymbol{V}_f$

☞ effect of **curvature** of sphere

$\boldsymbol{V}_f$ rotates by angle ε with respect to $\boldsymbol{V}_i \Rightarrow \varepsilon = K\sigma$ σ: surface
K: curvature

we now turn to a quantitative analysis of this phenomenon



Source: Ryder, *General Relativity*, (2009)

make a parallel round trip in the loop ABCD

observe the difference between the initial vector $\boldsymbol{V}_i$ and the final vector $\boldsymbol{V}_f$

(i) for a scalar

$$\frac{\mathrm{d}\Phi(x)}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{\Phi(x + \Delta x) - \Phi(x)}{\Delta x}$$

involves the difference of $\Phi(x)$ at 2 positions

$\Rightarrow$ *parallel transport of* **scalars** *will work*

(ii) for a vector, this is more complicated

$$\mathrm{d}V^\mu = \lim_{\Delta x \to 0} \big(V^\mu(x + \Delta x) - V^\mu(x) \big)$$

there are 2 contributions to the change of $V^\mu = \boldsymbol{e}^\mu \cdot \boldsymbol{V}$:

(α) $\boldsymbol{V}$ changes with the spatial position as for the scalar

(β) the frame $\boldsymbol{e}^\mu$ changes

$$\Delta V^\mu \big|_{\mathrm{total}} = \Delta V^\mu \big|_{\mathrm{true}} + \Delta V^\mu \big|_{\mathrm{coord.}}$$

$$\boxed{\mathrm{d}V^\mu = DV^\mu - \Gamma^\mu_{\nu\lambda} V^\nu \mathrm{d}x^\lambda}$$

the form of the second term follows from the definition of the connexion

Definition: *A* **parallel transport** *is such that* $\boxed{DV^\mu = 0}$.

for a parallel transport, the change in the components V^μ only comes from the coordinate changes

Consequence: consider parallel transport along a curve $x^\mu = x^\mu(\sigma)$

one has $0 = DV^\mu = \mathrm{d}V^\mu + \Gamma^\mu_{\nu\lambda} V^\nu \mathrm{d}x^\lambda$

take derivative with respect to arc length σ

$$
\begin{aligned}
0 = \frac{DV^\mu}{D\sigma} &= \frac{\mathrm{d}V^\mu}{\mathrm{d}\sigma} + \Gamma^\mu_{\nu\lambda} V^\nu \frac{\mathrm{d}x^\lambda}{\mathrm{d}\sigma} \\
&= \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}\right) + \Gamma^\mu_{\nu\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\sigma}\frac{\mathrm{d}x^\lambda}{\mathrm{d}\sigma}
\end{aligned}
$$

where we identified $V^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\sigma}$ with the velocity along the curve

☞ we recover the geodesic equation for the curve $x^\mu(\sigma)$

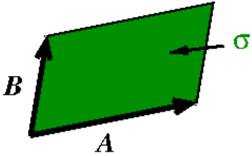
geodesic curves are not only the most short curves between two fixed points, but also the 'most straight' curves possible

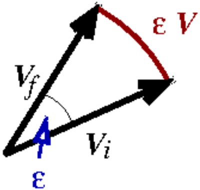
? Returning to a round trip along a loop, how to interpret the result of a parallel transport ?

* for a sphere: angular excess $\varepsilon = K\sigma$, with σ: surface, K: curvature

* general case $\varepsilon = \frac{\mathrm{d}V}{V}$
$\Rightarrow \mathrm{d}V = \varepsilon V = KV\sigma$

gives the structure to look for





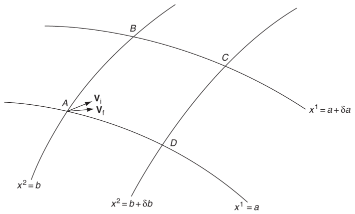
* tensor of a surface: in 2D have vector $\boldsymbol{\sigma} = \boldsymbol{A} \wedge \boldsymbol{B} \Rightarrow \sigma_k = \varepsilon_{ijk} A_i B_j$
write instead a tensor, as follows

$$\sigma^{ij} := \varepsilon^{ijk}\sigma_k = \varepsilon^{ijk}\varepsilon_{nmk}A^n B^m = \frac{1}{2}\left(A^i B^j - B^i A^j\right)$$

☞ extend this definition from 2D to any dimension

$$\boxed{\sigma^{\lambda\rho} = \frac{1}{2}\left(A^\lambda B^\rho - A^\rho B^\lambda\right)}$$

finally return to the round trip along the loop ABCD



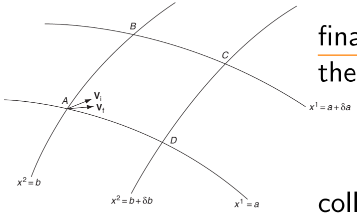
Source: Ryder, *General Relativity*, (2009)

the infinitesimal change of the vector is

$$\delta V^\kappa = -\Gamma^\kappa_{\lambda\mu} V^\lambda \delta x^\mu$$

collect the contributions of each segment

$$\begin{aligned}
V^\kappa(B) - V^\kappa(A) &= -\int_{x^2=b} \Gamma^\kappa_{\lambda\mu} V^\lambda \mathrm{d}x^\mu = -\int_{x^2=b} \Gamma^\kappa_{\lambda 1} V^\lambda \mathrm{d}x^1 \\
V^\kappa(C) - V^\kappa(B) &= -\int_{x^1=a+\delta a} \Gamma^\kappa_{\lambda\mu} V^\lambda \mathrm{d}x^\mu = -\int_{x^1=a+\delta a} \Gamma^\kappa_{\lambda 2} V^\lambda \mathrm{d}x^2 \\
V^\kappa(D) - V^\kappa(C) &= +\int_{x^2=b+\delta b} \Gamma^\kappa_{\lambda\mu} V^\lambda \mathrm{d}x^\mu = \int_{x^2=b+\delta b} \Gamma^\kappa_{\lambda 1} V^\lambda \mathrm{d}x^1 \\
V^\kappa(A) - V^\kappa(D) &= +\int_{x^1=a} \Gamma^\kappa_{\lambda\mu} V^\lambda \mathrm{d}x^\mu = \int_{x^1=a} \Gamma^\kappa_{\lambda 2} V^\lambda \mathrm{d}x^2
\end{aligned}$$



Source: Ryder, *General Relativity*, (2009)

finally: return to round trip along the loop ABCD
the infinitesimal change of the vector V^κ is

$$\delta V^\kappa = -\Gamma^\kappa_{\lambda\mu} V^\lambda \delta x^\mu$$

collect the contributions of each segment
Then the total change of V^κ becomes

$$\begin{aligned}
\Delta V^\kappa &= V^\kappa(A)_f - V^\kappa(A)_i \\
&= \left[-\int_{x^1=a+\delta a} + \int_{x^1=a} \Gamma^\kappa_{\lambda 2} V^\lambda \mathrm{d}x^2\right] + \left[\int_{x^2=b+\delta b} - \int_{x^2=b} \Gamma^\kappa_{\lambda 1} V^\lambda \mathrm{d}x^1\right] \\
&\simeq -\int_{x^1=a} \delta a\, \partial_1\left(\Gamma^\kappa_{\lambda 2} V^\lambda\right)\mathrm{d}x^2 + \int_{x^2=b} \delta b\, \partial_2\left(\Gamma^\kappa_{\lambda 1} V^\lambda\right)\mathrm{d}x^1 \\
&\simeq \delta a \delta b \left[-\partial_1\left(\Gamma^\kappa_{\lambda 2} V^\lambda\right) + \partial_2\left(\Gamma^\kappa_{\lambda 1} V^\lambda\right)\right] \qquad \text{since infinitesimal parallelogramme} \\
&= \delta a \delta b \left[\left(-\Gamma^\kappa_{\lambda 2,1} + \Gamma^\kappa_{\lambda 1,2}\right) V^\lambda - \Gamma^\kappa_{\lambda 2} V^\lambda{}_{,1} + \Gamma^\kappa_{\lambda 1} V^\lambda{}_{,2}\right] \\
&= \delta a \delta b \left[\Gamma^\kappa_{\lambda 1,2} - \Gamma^\kappa_{\lambda 2,1} + \Gamma^\kappa_{\lambda 2}\Gamma^\mu_{\lambda 1} - \Gamma^\kappa_{\lambda 1}\Gamma^\mu_{\lambda 2}\right] V^\lambda
\end{aligned}$$

Summary: the analysis of the round trip along the loop ABCD has shown that, in general

$$\begin{aligned}\Delta V^\kappa &= \frac{1}{2}\delta x^\mu \delta x^\nu \left[\Gamma^\kappa_{\lambda\mu,\nu} - \Gamma^\kappa_{\lambda\nu,\mu} + \Gamma^\kappa_{\rho\nu}\Gamma^\rho_{\lambda\mu} - \Gamma^\kappa_{\rho\mu}\Gamma^\rho_{\lambda\nu}\right] V^\lambda \\ &=: \frac{1}{2} R^\kappa{}_{\lambda\mu\nu} V^\lambda \sigma^{\mu\nu}\end{aligned}$$

where $R^\kappa{}_{\lambda\mu\nu}$ is the **Riemann tensor**

$$\boxed{R^\kappa{}_{\lambda\mu\nu} = \Gamma^\kappa_{\lambda\mu,\nu} - \Gamma^\kappa_{\lambda\nu,\mu} + \Gamma^\kappa_{\rho\nu}\Gamma^\rho_{\lambda\mu} - \Gamma^\kappa_{\rho\mu}\Gamma^\rho_{\lambda\nu}}$$

1. from the quotient theorem, $R^\kappa{}_{\lambda\mu\nu}$ is a tensor, of level $\binom{1}{3}$
2. $R^\kappa{}_{\lambda\mu\nu} \neq 0 \Leftrightarrow$ space is curved
3. R depends on $g_{\mu\nu}$ and its two first derivatives

☞ *R is the central quantity to study for curvatures*

Properties of the Riemann tensor

1. an alternative formulation $[D_\alpha, D_\beta]V^\mu = R^\mu{}_{\lambda\alpha\beta}V^\lambda$

2. symmetry relations

(α) $R^\kappa{}_{\lambda\mu\nu} = -R^\kappa{}_{\lambda\nu\mu}$ follows directly from definition

(β) rewrite as follows $R_{\kappa\lambda\mu\nu} = g_{\kappa\rho}R^\rho{}_{\lambda\mu\nu}$

we have seen above that one can always go into *'geodesic coordinates'* such that $\Gamma^\kappa_{\lambda\mu} = 0$

in geodesic coordinates $R_{\kappa\lambda\mu\nu} = \Gamma^\kappa_{\lambda\mu,\nu} - \Gamma^\kappa_{\lambda\nu,\mu}$

recall that $\Gamma^\kappa_{\lambda\nu,\mu} = \frac{1}{2}g^{\kappa\rho}(g_{\rho\lambda,\nu\mu} + g_{\rho\nu,\lambda\mu} - g_{\lambda\nu,\rho\mu})$

and one can show that

$$\boxed{R_{\kappa\lambda\mu\nu} = \frac{1}{2}\left(g_{\kappa\mu,\lambda\nu} - g_{\kappa\nu,\lambda\mu} + g_{\lambda\nu,\kappa\mu} - g_{\lambda\mu,\kappa\nu}\right)}$$

this gives the following **symmetry properties**

$$\begin{aligned} R_{\kappa\lambda\mu\nu} = -R_{\kappa\lambda\nu\mu} &= -R_{\lambda\kappa\mu\nu} = R_{\mu\nu\kappa\lambda} \\ 3R_{\kappa[\lambda\mu\nu]} := R_{\kappa\lambda\mu\nu} + R_{\kappa\mu\nu\lambda} + R_{\kappa\nu\lambda\mu} &= 0 \end{aligned}$$

N.B.: These are co-variant statements. They hold for geodesic coordinates.

$\Rightarrow$ hence the **symmetry properties** are valid for *all* coordinates.

Consequence of the symmetry relations:

* without any symmetry, the tensor R has $4^4 = 256$ independent components
* the antisymmetry in the first two and the last two indices, respectively:
  each of those blocks has 6 independent components
* rewrite R in terms of the blocks: $R_{\kappa\lambda\mu\nu} = R_{AB}$, with $A, B = 1, \dots, 6$.
  since furthermore $R_{AB} = R_{BA}$, R can be viewed as symmetric 6×6 matrix, which has $6 \cdot \frac{6+1}{2} = 21$ independent components.
* one further constraint from the last symmetry condition: $21 - 1 = 20$

in $d = 4$ dimensions, the Riemann tensor R has 20 independent components

Theorem: *In d dimensions, the Riemann tensor R has $\frac{1}{12} d^2(d^2 - 1)$ independent components.*

* if $d = 2$ ☞ one component p.ex. $R^1{}_{212}$
* if $d = 3$ ☞ 6 components
* if $d = 4$ ☞ 20 components

Definition: *(i) The* **Ricci tensor** *is* $R_{\mu\nu} := R^\rho{}_{\mu\rho\nu} = g^{\rho\sigma} R_{\sigma\mu\rho\nu} = R_{\nu\mu}$.
(ii) The **Ricci scalar** *is* $R := g^{\mu\nu} R_{\mu\nu} = R^\mu{}_\mu$.

Example 1: the $2D$ plane

(a) plane $\mathbb{R}^2$, polar coordinates $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\mathrm{d}\phi^2$, hence $x^1 = r$, $x^2 = \phi$

we already know that $\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}$, $\Gamma^1_{22} = -r$ and all other $\Gamma's$ vanish

Rappel: $R^\kappa{}_{\lambda\mu\nu} = \Gamma^\kappa_{\lambda\mu,\nu} - \Gamma^\kappa_{\lambda\nu,\mu} + \Gamma^\kappa_{\rho\nu}\Gamma^\rho_{\lambda\mu} - \Gamma^\kappa_{\rho\mu}\Gamma^\rho_{\lambda\nu}$

only independent component of Riemann tensor

$$\begin{aligned} R^1{}_{212} &= \Gamma^1_{22,1} - \Gamma^1_{21,2} + \Gamma^1_{i1}\Gamma^i_{22} - \Gamma^1_{i2}\Gamma^i_{21} \\ &= \frac{\partial}{\partial r}(-r) - \Gamma^1_{22}\Gamma^2_{21} \\ &= -1 - (-r)\frac{1}{r} = -1 + 1 = 0 \end{aligned}$$

indeed, the plane $\mathbb{R}^2$ is flat, as expected.

Example 2: the 2D sphere

(b) sphere S^2, spherical coordinates $\mathrm{d}s^2 = a^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)$,
hence $x^1 = \theta$, $x^2 = \phi$ — a: radius of the sphere

we already know that $\Gamma^1_{22} = -\sin\theta\cos\theta$, $\Gamma^2_{12} = \Gamma^2_{21} = \cot\theta$ ... all other Γ's vanish

$$\begin{aligned} R^1{}_{212} &= \Gamma^1_{22,1} - \Gamma^1_{21,2} + \Gamma^1_{i1}\Gamma^i_{22} - \Gamma^1_{i2}\Gamma^i_{21} \\ &= \frac{\partial}{\partial\theta}\left(-\sin\theta\cos\theta\right) - \Gamma^1_{22}\Gamma^2_{21} \\ &= -\cos^2\theta + \sin^2\theta + \cot\theta\sin\theta\cos\theta \;=\; \sin^2\theta \neq 0 \end{aligned}$$

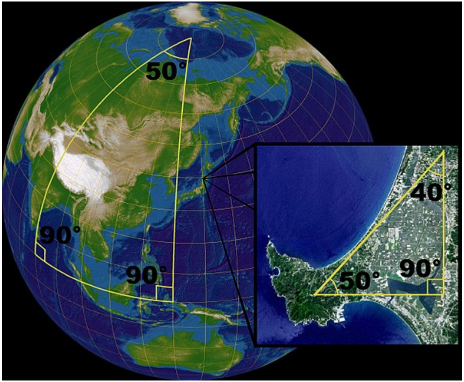
Ricci tensor:

$$\begin{aligned} R_{11} &= R^i{}_{1i1} = R^2{}_{121} = g^{22}g_{11}R^1{}_{212} = \frac{1}{a^2\sin^2\theta}a^2\cdot\sin^2\theta \;=\; 1 \\ R_{22} &= R^i{}_{2i2} = R^1{}_{212} \;=\; \sin^2\theta \\ R_{12} &= R_{21} \;=\; 0 \end{aligned}$$

Ricci scalar: $R = g^{11}R_{11} + g^{22}R_{22} = \frac{1}{a^2}\cdot 1 + \frac{1}{a^2\sin^2\theta}\cdot\sin^2\theta = \frac{2}{a^2} \neq 0$

$R = 2a^{-2}$ measures the curvature, independent of the coordinate choice.

Illustration: the $2D$ sphere S^2 is curved: (radius R)



for a large spherical triangle, with lengths $\approx R$, the sum of the three inner angles α, β, γ exceeds 180^o

but for a small triangle, with lengths $\ll R$, the euclidean statement on inner angles $\alpha + \beta + \gamma = 180^o$ holds true

☞ the small triangle becomes effectively euclidean

Source: https://www.businessinsider.com/triangles-in-elliptic-geometry-2014-6?IR=T finally, compare the metric tensor with the Ricci tensor

$$g_{\mu\nu} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \sin^2\theta \end{pmatrix} \quad , \quad R_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

such that one reads off $\boxed{R_{\mu\nu} = \frac{1}{a^2}\, g_{\mu\nu}}$ ☜

(i) both tensors are proportional !

(ii) the proportionality constant is position-independent

☞ conditions that *S^2 is a space of constant curvature*

Definition: *For a d-dimensional* **space of constant curvature** K, *one has*

$$\boxed{R_{\mu\nu} = (d-1)K\, g_{\mu\nu}}$$

where K is a position-independent constant.

Example: for the sphere S^2, one has $K = \frac{1}{a^2} = \frac{1}{2}R$ (Ricci scalar).

Definition: *An* **Einstein space** *has $R_{\mu\nu} = \lambda g_{\mu\nu}$ with $\lambda = \lambda(\boldsymbol{x})$.*

Theorem: (Besse) *In an Einstein space, a conformal transformation gives $\lambda = \mathrm{cste}$..*

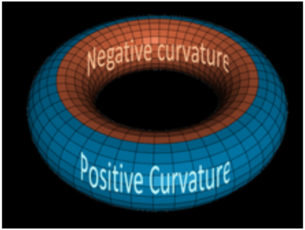
If $d = 2$ or $d = 3$, an Einstein space has constant curvature.

Example: the $2D$ **torus** $\mathbb{T}^2 := S^1 \times S^1$

parametric representation in $3D$ $\qquad \theta, \phi \in [0, 2\pi)$

$$x = \big(R + r\cos\theta\big)\cos\phi \ \ , \ \ y = \big(R + r\cos\theta\big)\sin\phi \ \ , \ \ z = r\sin\theta$$

The two constants R and r determine the size and the shape of the torus



For $R > r$, the torus has a positive curvature on the 'outside' and a negative curvature on the 'inside'

Source: `https://en.wikipedia.org/wiki/Gaussian_curvature`

$$\text{surface: } 4\pi rR \ \ , \quad \text{volume: } 2\pi^2 r^2 R$$

Cartesian equations to describe a torus include

$$\left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 = r^2 \iff \left(x^2 + y^2 + z^2 + R^2 - r^2\right)^2 = 4R^2\left(x^2 + y^2\right)$$

3.8 Field equations of gravitation in empty space

we are done with the mathematics !
Can now try to consider possible field equations of gravitation.
Begin with most simple case: field equations in empty space
what kind of equation do we want: co-variant equations of second order

* ? can we consider the equation $R^{\kappa}{}_{\lambda\mu\nu} = 0$? ☞ **NO**
20 equations for only 10 unknown components of $g_{\mu\nu}$ ☞ any non-trival solutions ?
such an equation would state that outside of massive bodies the time-space should be flat ☞ ! no gravitation !

* try something else, less restrictive: ? what about $R_{\mu\nu} = 0$?
  # gives 10 equations for the 10 unknown components of $g_{\mu\nu}$
  # $R^{\lambda}{}_{\mu\nu\kappa} \neq 0$ still possible
  # how can one know that this is a sensible physical choice ?
    ☞ look at non-relativistic (newtonian) limit !

Non-relativistic limit of equation $R_{\mu\nu} = 0$

should consider case of weak field,

when metric tensor $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ and h 'small' $h \ll 1$

$$R_{\mu\nu} = \Gamma^{\kappa}_{\mu\nu,\kappa} - \Gamma^{\kappa}_{\mu\kappa,\nu} + \Gamma^{\kappa}_{\rho\kappa}\Gamma^{\rho}_{\mu\nu} - \Gamma^{\kappa}_{\rho\nu}\Gamma^{\rho}_{\mu\kappa}$$

to leading order in h

$$\Gamma^{\kappa}_{\mu\nu} = \frac{1}{2}g^{\kappa\sigma}\left(g_{\sigma\mu,\nu} + g_{\sigma\nu,\mu} - g_{\mu\nu,\sigma}\right) \simeq \frac{1}{2}\eta^{\kappa\sigma}\left(h_{\sigma\mu,\nu} + h_{\sigma\nu,\mu} - h_{\mu\nu,\sigma}\right)$$

$$R_{\mu\nu} = \frac{1}{2}\eta^{\kappa\sigma}\left(h_{\sigma\nu,\mu\kappa} + h_{\mu\kappa,\sigma\nu} - h_{\mu\nu,\sigma\kappa} - h_{\sigma\kappa,\mu\nu}\right)$$

this is a static approximation, since $x^0 = t$ does not enter explicitly.

Consider in particular R_{00}: static case

$$\begin{aligned} R_{00} &= \frac{1}{2}\eta^{\kappa\sigma}\left(\underbrace{h_{\sigma 0,0\kappa}}_{=0} + \underbrace{h_{0\kappa,\sigma 0}}_{=0} - h_{00,\sigma\kappa} - \underbrace{h_{\sigma\kappa,00}}_{=0}\right) = -\frac{1}{2}\eta^{\kappa\sigma}h_{00,\sigma\kappa} \\ &= -\frac{1}{2}\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \boldsymbol{\nabla}^2\right)h_{00} = -\frac{1}{2}\boldsymbol{\nabla}^2 h_{00} \end{aligned}$$

If one introduces the gravitation potential $\phi = \phi(\boldsymbol{r})$, via $h_{00} = -\frac{2}{c^2}\phi$, we have

$R_{00} = 0$ *implies: the potential* $\phi(\boldsymbol{r})$ *obeys Laplace's equation* $\boldsymbol{\nabla}^2\phi = 0$.

Vorlesung VII

Rappel: had looked at curved spaces

central quantity: **Riemann tensor**

$$\boxed{R^{\kappa}{}_{\lambda\mu\nu} = \Gamma^{\kappa}_{\lambda\mu,\nu} - \Gamma^{\kappa}_{\lambda\nu,\mu} + \Gamma^{\kappa}_{\rho\nu}\Gamma^{\rho}_{\lambda\mu} - \Gamma^{\kappa}_{\rho\mu}\Gamma^{\rho}_{\lambda\nu}}$$

usefulness: $\boxed{R^{\kappa}{}_{\lambda\mu\nu} \neq 0 \Leftrightarrow \text{space is curved}}$

for many practical calculations, rather study

Ricci tensor: $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$ and **Ricci scalar**: $R = R^{\mu}{}_{\mu}$.

Examples: (i) the $2D$ plane $\mathbb{R}^2$ is flat,

(ii) the $2D$ sphere S^2 is curved

(since $R_{\mu\nu} = \frac{1}{a^2} g_{\mu\nu}$, S^2 has constant curvature $\frac{1}{a^2}$ everywhere)

physical importance: field equations are formulated with $R_{\mu\nu}$ and R.

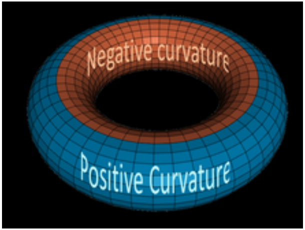
Example: field equations for empty space $R_{\mu\nu} = 0$.

Example: the $2D$ **torus** $\mathbb{T}^2 := S^1 \times S^1$

parametric representation in $3D$ $\qquad \theta, \phi \in [0, 2\pi)$

$$x = \big(R + r\cos\theta\big)\cos\phi \ , \quad y = \big(R + r\cos\theta\big)\sin\phi \ , \quad z = r\sin\theta$$

The two constants R and r determine the size and the shape of the torus



For $R > r$, the torus has a positive curvature on the 'outside' and a negative curvature on the 'inside'

Source: `https://en.wikipedia.org/wiki/Gaussian_curvature`

$$\text{surface: } 4\pi rR \ , \qquad \text{volume: } 2\pi^2 r^2 R$$

Cartesian equations to describe a torus include

$$\left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 = r^2 \iff \left(x^2 + y^2 + z^2 + R^2 - r^2\right)^2 = 4R^2\left(x^2 + y^2\right)$$

Rappel: how to obtain the newtonian limit (stationary, and $c \to \infty$)

The newtonian limit is a *weak-field limit* where one sets $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with h 'small'. In addition, as $c \to \infty$, one expects $\tau \simeq t$, $\frac{\mathrm{d}x^0}{\mathrm{d}\tau} \simeq c$, $\frac{\mathrm{d}x^i}{\mathrm{d}\tau} \simeq \frac{\mathrm{d}x^i}{\mathrm{d}t} = v^i \ll c$. Furthermore, this is a *static* approximation where the potentials are time-independent. The three spatial geodesic equations become

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + c^2 \Gamma^i_{00}\big(1 + \mathrm{O}(1/c)\big) = 0 \quad \Rightarrow \quad \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -c^2 \Gamma^i_{00} = a^i \quad \text{acceleration}$$

which begins to look like a newtonian equation of motion.

One must now work out, in the static approximation and to linear order in h:

$$\Gamma^i_{00} = \frac{1}{2} g^{i\nu}\big(\underbrace{2g_{\nu 0,0}}_{=0} - g_{00,\nu}\big) = -\frac{1}{2} g^{ik} g_{00,k} \simeq -\frac{1}{2}\eta^{ik} h_{00,k} + \mathrm{O}(h^2) = -\frac{1}{2}\nabla^i h_{00}$$

This gives the equation of motion $\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -c^2 \Gamma^i_{00} = \frac{c^2}{2}\nabla^i h_{00}$. Compare with the newtonian equation $\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\nabla^i \phi$. Identify the newtonian gravitational potential

$\boxed{h_{00} = -\frac{2}{c^2}\phi}$, or $g_{00} = -\big(1 + \frac{2}{c^2}\phi\big)$.

N.B.: herein, the mass of the test particle was set to $m = 1$

In order to find the newtonian limit of the field equation, consider again

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \Rightarrow \quad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathrm{O}(h^2)$$

[to see this: $g_{\mu\nu} g^{\nu\kappa} = (\eta_{\mu\nu} + h_{\mu\nu})(\eta^{\nu\kappa} - h_{\nu\kappa}) \simeq \eta_{\mu\nu}\eta^{\nu\kappa} - \eta_{\mu\nu}h_{\nu\kappa} + h_{\mu\nu}\eta^{\nu\kappa} + \mathrm{O}(h^2) = \delta^\kappa_\mu - h^\kappa_\mu + h^\kappa_\mu = \delta^\kappa_\mu$]

Then: $\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\rho}\left(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho}\right) \simeq \frac{1}{2}\eta^{\mu\rho}\left(h_{\rho\nu,\lambda} + h_{\rho\lambda,\nu} - h_{\nu\lambda,\rho}\right) + \mathrm{O}(h^2)$

which is of first order in h. Compute the Ricci tensor, as follows

$$R_{\mu\nu} = \Gamma^\kappa_{\mu\nu,\kappa} - \Gamma^\kappa_{\mu\kappa,\nu} + \underbrace{\Gamma^\kappa_{\rho\kappa}\Gamma^\rho_{\mu\nu} - \Gamma^\kappa_{\rho\nu}\Gamma^\rho_{\mu\kappa}}_{=\mathrm{O}(h^2),\ \text{negligible}} = \frac{1}{2}\eta^{\kappa\sigma}\left(h_{\sigma\nu,\mu\kappa} + h_{\mu\kappa,\sigma\nu} - h_{\mu\nu,\sigma\kappa} - h_{\sigma\kappa,\mu\nu}\right) + \mathrm{O}(h^2)$$

Concentrate on the component $\mu = \nu = 0$ (use the static approximation !):

$$\begin{aligned} R_{00} &\simeq \frac{1}{2}\eta^{\kappa\sigma}\big(\underbrace{h_{\sigma 0,0\kappa} + h_{0\kappa,\sigma 0}}_{=0} - h_{00,\sigma\kappa} - \underbrace{h_{\sigma\kappa,00}}_{=0}\big) \\ &= -\frac{1}{2}\eta^{\kappa\sigma} h_{00,\sigma\kappa} = -\frac{1}{2}\left(-\frac{1}{c^2}\underbrace{\frac{\partial^2}{\partial t^2}}_{=0} + \nabla^2\right) h_{00} = -\frac{1}{2}\nabla^2 h_{00} \end{aligned}$$

$\Rightarrow$ The vacuum field equation $R_{00} = 0$ reduces to Laplace's equation $\nabla^2\phi = 0$.

4. The Einstein field equations

4.1 Equivalence principle and general co-variance

Equivalence principle: *At each point of time-space with a gravitational field, one can find a local inertial frame such that the laws of physics are those of a non-accelerated cartesian frame.*

Principle of general co-variance:

A physical equation holds under the influence of gravity if

(1) it holds without gravitational field (and is consistent with special relativity)

(2) it is co-variant under an arbitrary coordinate change $\mathtt{x} \mapsto \mathtt{x}'$.

not required: that velocities, accelerations are eliminated from the equations (as in special relativity); one rather uses $g_{\mu\nu}$, $\Gamma^{\lambda}_{\mu\nu}$, ... to describe the effects of gravitation

this is a dynamic symmetry, rather than an invariance principle

analogy with gauge invariance in electromagnetism

⇒ look for equations of the form

$$A^{\mu_1\ldots\mu_r}{}_{\lambda_1\ldots\lambda_s} = B^{\mu_1\ldots\mu_r}{}_{\lambda_1\ldots\lambda_s}$$

A, B are tensors of level $\binom{r}{s}$ ☞ such equations are automatically co-variant !

4.2 Gravitational field equations

in empty space $R_{\mu\nu} = 0$, must find coupling with matter ? how ?

Example: take a cloud of slowly moving dust particles (no interactions).
At rest, this cloud has energy density $\rho_0 = m_0 n_0$, where m_0: mass of a dust grain; n_0: number density (# particles/volume) of dust.
If cloud is moving with velocity $\boldsymbol{v}$, find from Lorentz transformation

$$\left.\begin{cases} m_0 \mapsto m_0' = \gamma m_0 & \text{transformation of energy } E = m_0 c^2 \\ n_0 \mapsto n_0' = \gamma n_0 & \text{transformation of inverse volume} \\ & \to \text{Lorentz length contraction} \end{cases}\right\} \Rightarrow \rho_0 \mapsto \rho_0' = \gamma^2 \rho_0$$

☞ ρ_0 transforms as component T^{00} of a tensor $T^{\mu\nu}$

Definition: *The* **energy-momentum tensor** $T^{\mu\nu} = T^{\nu\mu}$ *is of the form*

$$T^{\mu\nu} = \begin{pmatrix} \rho_0 & s_x & s_y & s_z \\ \pi_x & G_{xx} & G_{xy} & G_{xz} \\ \pi_y & G_{yx} & G_{yy} & G_{yz} \\ \pi_z & G_{zx} & G_{zy} & G_{zz} \end{pmatrix}, \quad \begin{array}{lll} \rho_0 & : & \text{energy density} \\ \boldsymbol{s} & : & \text{energy current density} \\ \boldsymbol{\pi} & : & \text{momentum density} \\ G_{ij} & : & \text{momentum current density} \end{array}$$

if $G_{xx} = G_{yy} = G_{zz} = p$, then p is the **pressure**. $\boldsymbol{s} = \boldsymbol{\pi}$ Planck, Poincaré

conservation laws $\partial_\nu T^{\mu\nu} = 0$ give the energy- and momentum-conservation laws.

we restrict here to *clouds of non-interacting particles* ('**dust**' in astronomy).

Proposal: if $\mathbf{u} = \frac{1}{c}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau}$ is the four-velocity, and $\rho_0(\mathbf{x})$ the matter density

$$\boxed{T^{\mu\nu} = \rho_0 u^\mu u^\nu}$$

* Since $\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma = \left(1 - \frac{v^2}{c^2}\right)^{1/2}$, find $T^{00} = \rho_0 \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = \rho_0 \gamma^2 =: \rho$
hence T^{00} describes the matter density in a moving frame

* Similarly $T^{0i} = \rho_0 u^0 u^i = \frac{\rho_0}{c^2}\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\frac{\mathrm{d}x^i}{\mathrm{d}\tau} = \frac{\gamma^2\rho_0}{c} = \rho\frac{v^i}{c}$ and $\quad v^i = \frac{\mathrm{d}x^i}{\mathrm{d}t}$

$T^{ik} = \frac{\rho_0}{c^2}\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\mathrm{d}x^k}{\mathrm{d}\tau} = \rho\frac{v^i v^k}{c^2}$

☞ the energy-momentum tensor of dust has the form see exercices

$$T^{\mu\nu} = \rho \begin{pmatrix} 1 & \frac{v_x}{c} & \frac{v_y}{c} & \frac{v_z}{c} \\ \frac{v_x}{c} & \frac{v_x^2}{c^2} & \frac{v_x v_y}{c^2} & \frac{v_x v_z}{c^2} \\ \frac{v_y}{c} & \frac{v_x v_y}{c^2} & \frac{v_y^2}{c^2} & \frac{v_y v_z}{c^2} \\ \frac{v_z}{c} & \frac{v_x v_z}{c^2} & \frac{v_y v_z}{c^2} & \frac{v_z^2}{c^2} \end{pmatrix}$$

if N.R. limit $c \to \infty$: $T^{\mu\nu} \to T^{00} \simeq \rho \simeq \rho_0$ (the only non-vanishing component)

* let us verify & physically interpret the conservation law $\partial_\nu T^{\mu\nu} = T^{\mu\nu}{}_{,\nu} = 0$

for $\mu = 0$: this reads $T^{00}{}_{,0} + T^{0i}{}_{,i} = 0$ or

$$\begin{aligned} \frac{1}{c}\frac{\partial}{\partial t}(\rho) + \frac{1}{c}\frac{\partial}{\partial x^i}(\rho v^i) &= 0 \\ \Rightarrow \qquad \frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{v}) &= 0 \end{aligned}$$

☞ identify energy density ρ and energy current $\rho \boldsymbol{v}$

for $\mu = i$: this reads $T^{i0}{}_{,0} + T^{ij}{}_{,j} = 0$ or

$$\begin{aligned} \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\rho v^i\right) + \frac{1}{c^2}\frac{\partial}{\partial x^j}\left(\rho v^i v^j\right) &= 0 \\ \Rightarrow \qquad \frac{\partial}{\partial t}(\rho v^i) + \boldsymbol{\nabla}\cdot(\rho v^i \boldsymbol{v}) &= 0 \end{aligned}$$

☞ identify momentum density ρv^i and momentum current $\rho v^i \boldsymbol{v}$

continuity equation in a volume Ω: change in the conserved charge only through transport, via current across boundary $\partial\Omega$

☞ co-variant conservation law $\boxed{T^{\mu\nu}{}_{;\nu} = 0}$

holds at rest
is generally co-variant

Construction of the Einstein field equation

Lemma (Bianchi identity): *The Riemann tensor obeys the identity*

$$\boxed{R^{\mu}{}_{\nu\rho\sigma;\lambda} + R^{\mu}{}_{\nu\sigma\lambda;\rho} + R^{\mu}{}_{\nu\lambda\rho;\sigma} = 0} \qquad \text{(BI)}$$

(A) A first attempt: try the ansatz $\kappa = \text{cste.}$

$$R_{\mu\nu} = \kappa T_{\mu\nu}$$

cannot work, since $T^{\mu\nu}{}_{;\nu} = 0$ but $R^{\mu\nu}{}_{;\nu} \neq 0$
to see the last point, take in (BI) $\mu = \rho$ and contract

$$R_{\nu\sigma;\lambda} + R^{\mu}{}_{\nu\sigma\lambda;\mu} + \underbrace{R^{\mu}{}_{\nu\lambda\mu;\sigma}}_{=-R_{\nu\lambda;\sigma}} = 0$$

$$\Rightarrow \qquad R_{;\lambda} - R^{\mu}{}_{\lambda;\mu} - R^{\sigma}{}_{\lambda;\sigma} = 0 \qquad \text{after contraction with } g^{\nu\sigma}$$

$$\Rightarrow \qquad \delta^{\rho}{}_{\lambda} R_{;\rho} - 2R^{\rho}{}_{\lambda;\rho} = 0$$

$$\Rightarrow \qquad R^{\rho}{}_{\lambda;\rho} = \frac{1}{2} R_{;\lambda}$$

Definition: *The* **Einstein tensor** *is* $G^{\mu\nu} := R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R$. $\qquad G^{\mu\nu} = G^{\nu\mu}$

☞ co-variant conservation law $\boxed{G^{\mu\nu}{}_{;\nu} = 0}$.

(B) A second attempt: try the improved *ansatz* EINSTEIN 1915

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^2} T_{\mu\nu} \qquad \text{(E)}$$

with G: gravitational constant. These are EINSTEIN'S field equations.

Contrôles:

(i) set of 10 second-order PDEs for the 10 potentials in $g_{\mu\nu} = g_{\nu\mu}$

(ii) co-variant conservation law consistent $G^{\mu\nu}{}_{;\nu} = \frac{8\pi G}{c^2} T^{\mu\nu}{}_{;\nu} = 0$

(iii) does reproduce vacuum equation: $T_{\mu\nu} = 0 \Rightarrow R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$

but $g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} \underbrace{g^{\mu\nu} g_{\mu\nu}}_{=4} R = R - 2R = 0 \Rightarrow R = 0$ hence $R_{\mu\nu} = 0$.

(iv) **non-relativistic limit**, should reproduce Newton's theory:

as before, from (E): $R - 2R = \frac{8\pi G}{c^2} g^{\mu\nu} T_{\mu\nu}$, hence $R = -\frac{8\pi G}{c^2} g^{\mu\nu} T_{\mu\nu} =: -\frac{8\pi G}{c^2} T$

write alternative form of Einstein's field equations $T := g^{\mu\nu} T_{\mu\nu}$

$$R_{\mu\nu} = \frac{8\pi G}{c^2} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \qquad \text{(E')}$$

$$\boxed{R_{\mu\nu} = \frac{8\pi G}{c^2}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)} \qquad \text{(E')}$$

* to carry out the non-relativistic limit, have for 'dust'

$$T^{\mu\nu} \simeq \rho \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + \mathrm{O}(c^{-1})$$

therefore $T \simeq -\rho$ and $T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T \simeq \frac{\rho}{2}\delta^{\mu\nu}$

* On the other hand, consider weak-field case $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$

such that $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ and $g^{\mu\kappa}g_{\kappa\nu} = \delta^{\mu}_{\nu} - h^{\mu}_{\nu} + h^{\mu}_{\nu} + \mathrm{O}(h^2) = \delta^{\mu}_{\nu}$

* we have already seen before that $R_{00} \simeq -\frac{1}{2}\nabla^2 h_{00} = \frac{1}{c^2}\nabla^2\phi$
* finally, (E') for $\mu = \nu = 0$ reproduces Poisson's equation $\frac{1}{c^2}\nabla^2\phi = \frac{8\pi G}{c^2}\frac{\rho}{2}$

$$\boxed{\nabla^2\phi = 4\pi G\rho}$$

where $\phi = -\frac{1}{2}h_{00}$ is indeed the newtonian gravitational potential.
This justifies the choice of the constant in (E,E').

Gives the following scheme for field equations of gravitation

	Newton	Einstein
equation of motion	$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\boldsymbol{\nabla}^i \phi$	$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\nu\rho} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} = 0$
field equation	$\nabla^2 \phi = 4\pi G \rho$	$G_{\mu\nu} = \frac{8\pi G}{c^2} T_{\mu\nu}$
source	mass density	energy & momentum

the newtonian theory is the non-relativistic limit ($c \to \infty$) of Einstein's general relativity

"*Time-space tells matter how to move; matter tells time-space how to curve.*"

WHEELER 1973

N.B.: the equation of motion is the one of light 'test particles' which do not curve the time-space themselves.

4.3 Schwarzschild solution (1916)

gives the most simple of the non-trivial solutions of the Einstein equation

look for solution of the vacuum equation $R_{\mu\nu} = 0$, around a gravitating spherical shell at rest $\rightarrow$ sun at rest in the centre of the solar system

static system: $g_{\mu\nu}$ independent of x^0

hence $\mathrm{d}s^2$ invariant under $x^0 \mapsto -x^0 \Rightarrow g_{0i} = g_{i0} = 0$.

because of spherical symmetry, have ansatz

$$\mathrm{d}s^2 = -U(r)c^2\mathrm{d}t^2 + V(r)\mathrm{d}r^2 + W(r)r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)$$

where $U(r)$, $V(r)$, $W(r)$ are to be found.

* in general, *can always arrange for $W(r) = 1$*.

to see this: general co-variance $\Rightarrow$ *coordinate r is just a radial parameter !*

set $W(r)r^2 =: \rho^2 \Rightarrow \rho = r\sqrt{W} \Rightarrow \frac{\mathrm{d}\rho}{\mathrm{d}r} = \sqrt{W}\left(1 + \frac{r}{2W}\frac{\mathrm{d}W}{\mathrm{d}r}\right)$

$$V(r)\mathrm{d}r^2 = \frac{V}{W}\left(1 + \frac{r}{2W}\frac{\mathrm{d}W}{\mathrm{d}r}\right)^{-2}\mathrm{d}\rho^2 =: \bar{V}(\rho)\mathrm{d}\rho^2 \quad \text{and} \quad U(r) =: \bar{U}(\rho)$$

at the end, relabel: $\rho \mapsto r$, $\bar{V}(\rho) \mapsto V(r)$, $\bar{U}(\rho) \mapsto U(r)$.

the ansatz has been reduced to the form

$$\mathrm{d}s^2 = -U(r)c^2\mathrm{d}t^2 + V(r)\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)$$

* new notation: $U(r) = e^{2\nu(r)}$ and $V(r) = e^{2\lambda(r)}$. The ansatz becomes

$$\mathrm{d}s^2 = -e^{2\nu(r)}c^2\mathrm{d}t^2 + e^{2\lambda(r)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)$$

have *diagonal metric* with metric tensor

$$\begin{aligned} g_{\rho\sigma} &= \mathrm{diag}\left(-e^{2\nu}, e^{2\lambda}, r^2, r^2\sin^2\theta\right) \\ g^{\rho\sigma} &= \mathrm{diag}\left(-e^{-2\nu}, e^{-2\lambda}, r^{-2}, r^{-2}\sin^{-2}\theta\right) \end{aligned}$$

Herein, the two functions $\nu = \nu(r)$ and $\lambda = \lambda(r)$ are to be derived from the Einstein equation $R_{\rho\sigma} = 0$.

$$g_{\rho\sigma} = \mathrm{diag}\,\left(-e^{2\nu}, e^{2\lambda}, r^2, r^2\sin^2\theta\right)$$
$$g^{\rho\sigma} = \mathrm{diag}\,\left(-e^{-2\nu}, e^{-2\lambda}, r^{-2}, r^{-2}\sin^{-2}\theta\right)$$

* work out *Christoffel symbols* from diagonal metric tensor
in general: $\Gamma^{\kappa}_{\rho\sigma} = \frac{1}{2}g^{\kappa\iota}\left(g_{\iota\rho,\sigma} + g_{\iota\sigma,\rho} - g_{\rho\sigma,\iota}\right)$

$\nu' = \frac{\partial\nu}{\partial r}$

see exercice

$$\Gamma^1_{00} = \frac{1}{2}g^{1\iota}\left(2g_{0\iota,0} - g_{00,\iota}\right) = -\frac{1}{2}g^{11}g_{00,1} = -\frac{1}{2}e^{-2\lambda}\frac{\partial}{\partial r}\left(-e^{2\nu}\right) = \nu' e^{2\nu-2\lambda}$$
$$\Gamma^0_{01} = \Gamma^0_{10} = \nu'$$
$$\Gamma^1_{11} = \lambda'\ ,\ \Gamma^1_{22} = -re^{-2\lambda}\ ,\ \Gamma^1_{33} = -r\sin^2\theta e^{-2\lambda}$$
$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$$
$$\Gamma^2_{33} = -\sin\theta\cos\theta$$
$$\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta$$

and all other $\Gamma = 0$.

* the Einstein equations are   do not confuse the index ν with the function $\nu = \nu(r)$

$$R_{\mu\nu} = \Gamma^{\kappa}_{\mu\nu,\kappa} - \Gamma^{\kappa}_{\mu\kappa,\nu} + \Gamma^{\kappa}_{\rho\kappa}\Gamma^{\rho}_{\mu\nu} - \Gamma^{\kappa}_{\rho\nu}\Gamma^{\rho}_{\mu\kappa} = 0$$

for example

$$\begin{aligned} R_{00} &= \Gamma^{\kappa}_{00,\kappa} - \Gamma^{\kappa}_{0\kappa,0} + \Gamma^{\kappa}_{\rho\kappa}\Gamma^{\rho}_{00} - \Gamma^{\kappa}_{\rho 0}\Gamma^{\rho}_{0\kappa} \\ &= \Gamma^{1}_{00,1} + \Gamma^{\kappa}_{1\kappa}\Gamma^{1}_{00} - \left(\Gamma^{1}_{\rho 0}\Gamma^{\rho}_{01} + \Gamma^{0}_{\rho 0}\Gamma^{\rho}_{00}\right) \\ &= \frac{\partial}{\partial r}\left(\nu' e^{2\nu-2\lambda}\right) + \nu' e^{2\nu-2\lambda}\left(\nu' + \lambda' + \frac{2}{r}\right) - 2\left(\nu'\right)^2 e^{2\nu-2\lambda} \\ &= e^{2\nu-2\lambda}\left(\nu'' + \nu'^2 - \nu'\lambda' + 2\nu'/r\right) \end{aligned}$$

* this gives the components of the Ricci tensor & Einstein field equations

$$R_{00} = 0 \quad \Rightarrow \quad \nu'' + \nu'^2 - \nu'\lambda' + \frac{2}{r}\nu' = 0 \qquad \text{(A)}$$

$$R_{11} = 0 \quad \Rightarrow \quad -\nu'' + \nu'\lambda' + \frac{2}{r}\lambda' - \nu'^2 = 0 \qquad \text{(B)}$$

$$R_{22} = 0 \quad \Rightarrow \quad -1 - r\nu' + r\lambda' + e^{2\lambda} = 0 \qquad \text{(C)}$$

and also $R_{33} = R_{22}\sin^2\theta$ and all other $R_{\rho\sigma} = 0$ for $\rho \neq \sigma$.

☞ have **three** equations (A,B,C) for **two** functions ν, λ

have three independent equations

$$R_{00} = 0 \quad \Rightarrow \quad \nu'' + \nu'^2 - \nu'\lambda' + \frac{2}{r}\nu' = 0 \qquad \text{(A)}$$

$$R_{11} = 0 \quad \Rightarrow \quad -\nu'' + \nu'\lambda' + \frac{2}{r}\lambda' - \nu'^2 = 0 \qquad \text{(B)}$$

$$R_{22} = 0 \quad \Rightarrow \quad -1 - r\nu' + r\lambda' + e^{2\lambda} = 0 \qquad \text{(C)}$$

* solution of these equations:

add (A) and (B): $\frac{2}{r}\left(\lambda' + \nu'\right) = 0 \Rightarrow \lambda(r) + \nu(r) = \mathrm{cste.}$

for $r \to \infty$ expect return to Minkowski metric, hence $\lambda(r), \nu(r) \to 0 \Rightarrow \mathrm{cste.} = 0$

$\Rightarrow$ have $\lambda(r) = -\nu(r)$

from (C): $\left(1 + 2r\nu'\right)e^{2\nu} = 1 \Rightarrow \left(re^{2\nu}\right)' = 1 \Rightarrow re^{2\nu} = r - 2m$

with the final form $\boxed{e^{2\nu} = 1 - \frac{2m}{r}}$. $\qquad m = \mathrm{cste.}$

inject this solution into (A,B) and check that it also solves these.

* have for the metric tensor $g_{00} = -\left(1 - \frac{2m}{r}\right)$ and $g_{11} = \left(1 - \frac{2m}{r}\right)^{-1}$.

however, for *weak gravitational fields*, we know already $g_{00} = -\left(1 - \frac{2GM}{c^2}\frac{1}{r}\right)$

comparison gives $m = \frac{GM}{c^2} = \frac{1}{2}\mathscr{R}$ $\qquad$ with $\mathscr{R}$: Schwarzschild radius the final result gives the (outer) **Schwarzschild metric**

$$\mathrm{d}s^2 = -\left(1 - \frac{\mathscr{R}}{r}\right) c^2 \mathrm{d}t^2 + \left(1 - \frac{\mathscr{R}}{r}\right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2 \mathrm{d}\phi^2\right)$$

∗ depends on the length $\mathscr{R} = \frac{2GM}{c^2}$, M: mass of central object

∗ *exact solution* of Einstein equations, valid at *exterior* of central body ($r > \mathscr{R}$)

∗ the (newtonian) weak-field solution is exact as well

∗ any details of the mass distribution in the centre do not enter

∗ large-distance (flat) and weak-field (newtonian) boundary conditions

N.B.: the auxiliary assumption of a static, time-independent, solution is not really required

Theorem (Birkhoff): *Any spherically symmetric solution of $R_{\mu\nu} = 0$ is static, and hence given by the Schwarzschild metric.*

Example: A spherically symmetric star with radial pulsations still produces the static Schwarzschild metric.

Analogue of the derivation of Newton's potential $V(r) = -G\frac{M}{r}$ of gravitation.

the test mass was scaled to $m = 1$

Experimental test I: gravitational red-shift

now describe a set of experimental tests, all based on the (outer) Schwarzschild metric

metric tensor of Schwarzschild solution

$$g_{\mu\nu} = \mathrm{diag}\left(-\left(1-\frac{\mathscr{R}}{r}\right), \left(1-\frac{\mathscr{R}}{r}\right)^{-1}, r^2, r^2\sin^2\theta\right)$$

metric independent of $x^0 = ct \Rightarrow$ t is *universal time*

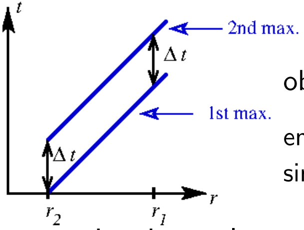
consider proper times between two events at a fixed space point (Schwarzschild metric)

$$\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2 = g_{00}c^2\mathrm{d}t^2 \ \ , \ \ |g_{00}(r)| = 1 - \frac{2GM}{c^2}\frac{1}{r} = 1 - \frac{\mathscr{R}}{r} < 1$$

hence $\boxed{\mathrm{d}\tau = \sqrt{-g_{00}}\,\mathrm{d}t < \mathrm{d}t}$.

☞ *time passes more slowly in a gravitational field*.

in order to measure this, compare time-dilation effects in two distinct places under influence of a spherical gravitational field, of total mass M (planet, star,...)



observe at place r_1 a light signal emitted at place r_2

emission of two wave maxima, with time difference Δt at r_2
since t is universal time, have time difference Δt at r_1, too

proper time intervals are related to frequencies

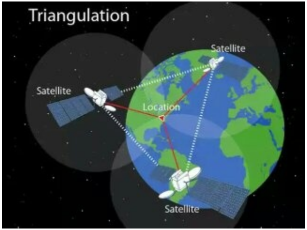
$$\left.\begin{array}{l} \mathrm{d}\tau_2 = \frac{1}{\nu_2} = \frac{\lambda_2}{c} = \Delta t\sqrt{-g_{00}(r_2)} \\ \mathrm{d}\tau_1 = \frac{1}{\nu_1} = \frac{\lambda_1}{c} = \Delta t\sqrt{-g_{00}(r_1)} \end{array}\right\} \Rightarrow \boxed{\frac{\nu_1}{\nu_2} = \sqrt{\frac{g_{00}(r_2)}{g_{00}(r_1)}} \simeq 1 - \frac{1}{2}\left(\frac{\mathscr{R}}{r_2} - \frac{\mathscr{R}}{r_1}\right)}$$

same as heuristic result in the introduction → Pound-Snider-Rebka experiment

no distinction between different theories of gravitation: does test the equivalence principle

Vessot-Levine Experiment (1976): send ‘hydrogene maser clock’ by rocket to altitude $10^4[\mathrm{km}]$ and compare with frequency of identical clock on Earth. Confirm GR-EP prediction with relative precision $< 2 \cdot 10^{-4}$. ‘Gravity Probe A’

Technological application: the GPS



principle of GPS:

localise a position on Earth by triangulation with several satellites

distances found from time-delay measurements

Source: https://www.quora.com/Why-does-your-phones-GPS-need-Einsteins-General-relativity-to-work

for a precision of $\approx 15[\mathrm{m}]$, need accuracy of $\approx 50[\mathrm{ns}]$ in time measurement

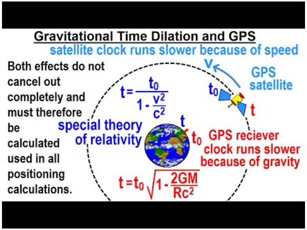
* rotating frames $\rightarrow$ Sagnac effect

* non-spherical form of the Earth

* time-dilation effects are large:

$46[\mu\mathrm{s}]$	gravit. red shift
$-7[\mu\mathrm{s}]$	special relativity
$39[\mu\mathrm{s}]$	$10^3 \cdot$ required accuracy

N. Ashby, Phys. Today **55** (May 2002), 41 (2002); Living Rev. Relat. **6**, 1 (2003)



https://www.youtube.com/watch?v=91BvUr2wcdw

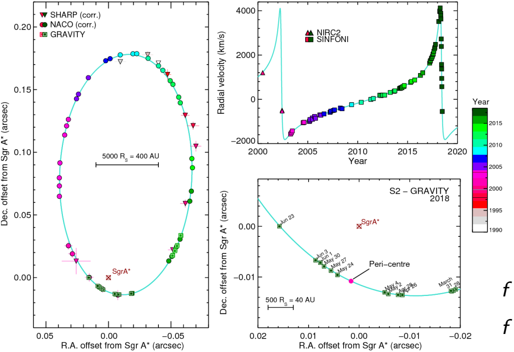
N.B.: precise enough to observe directly motion of tectonic plates, velocities up to $\approx 10[\mathrm{cm/year}]$

A new kind of test: strong gravitational fields I

Nobel prize 2020

astron. observation: Sgr A* compact, extremely massive object immobile at galaxy centre

infra-red observations (interferometers & adaptive optics): cluster of stars orbiting Sgr A*



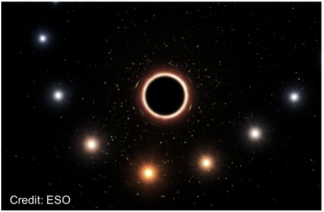
star S2 passes close to centre

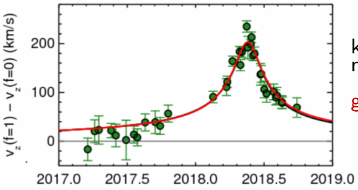
high velocity $v \approx 7650[\mathrm{km/h}] = 0.026c$
keplerian orbit plus relativistic corrections
redshift, extra $v \approx 200[\mathrm{km/h}]$ at pericentre

comparison parameter f, such that $f_{GR} = 1$

$f = 1.04 \pm 0.05$ Genzel *et al.* A&A **615**, L15 (2018) A&A **636**, L5 (2020)

$f = 0.88 \pm 0.17$ Ghez *et al.* Science **365**, 664 (2019)





keplerian orbit ruled out near pericentre (gray line)

general relativity fit

Sources: `https://www.eso.org/public/images/eso1825b/`; Genzel *et al.*, A&A **636**, L5 (2020)

4.5 Geodesics in Schwarzschild time-space

work out the mouvement of test particles in Schwarzschild time-space

in a given metric $g_{\mu\nu}$, have geodesic equation

$$\boxed{\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau} = 0}$$

need list of non-vanishing Christoffel symbols **Rappel**: $\Gamma^\kappa_{\rho\sigma} = \frac{1}{2} g^{\kappa\iota} (g_{\iota\rho,\sigma} + g_{\iota\sigma,\rho} - g_{\rho\sigma,\iota})$

$$\Gamma^0_{10} = \Gamma^0_{01} = \frac{\frac{1}{2}\mathscr{R}}{r(r-\mathscr{R})} \ , \ \Gamma^1_{00} = -\frac{1}{2}\frac{\mathscr{R}}{r}\left(1-\frac{\mathscr{R}}{r}\right)\frac{1}{r}$$

$$\Gamma^1_{11} = -\frac{\frac{1}{2}\mathscr{R}}{r\left(r-\mathscr{R}\right)} \ , \ \Gamma^1_{22} == -\left(r-\mathscr{R}\right) \ , \ \Gamma^1_{33} = -\left(r-\mathscr{R}\right)\sin\theta$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \ , \ \Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta$$

Notation: have $x^0 = ct$ and $\dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau}$ etc. **Have 4 eqs. for variables (t, r, θ, ϕ).**

now write down the geodesic equations. For $\mu = 0$, have
$\ddot{t} + \frac{\mathscr{R}}{r\left(r-\mathscr{R}\right)}\dot{t}\dot{r} = 0 \Leftrightarrow \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\left(1-\frac{\mathscr{R}}{r}\right)\dot{t}\right] = 0$ which becomes

$$\left(1-\frac{\mathscr{R}}{r}\right)\dot{t} = b = \mathrm{cste.} \tag{G0}$$

For $\mu = 2$, we have

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0 \tag{G2}$$

For $\mu = 3$, we have

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi} = 0 \tag{G3}$$

Instead of writing the second-order equation for $\mu = 1$, consider the invariant

$$\mathrm{d}s^2 = -\left(1-\frac{\mathscr{R}}{r}\right)c^2\mathrm{d}t^2 + \left(1-\frac{\mathscr{R}}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)$$

distinguish two cases: if particles $\mathrm{d}s^2 = -c^2\mathrm{d}\tau^2 \neq 0$, if light $\mathrm{d}s^2 = 0$. Then

$$\left(1-\frac{\mathscr{R}}{r}\right)\dot{t}^2 - \left(1-\frac{\mathscr{R}}{r}\right)^{-1}\frac{\dot{r}^2}{c^2} - \frac{r^2}{c^2}\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) = \begin{cases} 1 & \text{particles} \\ 0 & \text{light} \end{cases} \tag{G1}$$

* *The mouvement is in a plane*

to see this: consider geodesic, on equator $\theta = \frac{\pi}{2}$, tangential at the plane $\dot{\theta} = 0$

(G2) $\Rightarrow \ddot{\theta} = 0 \Rightarrow$ for all τ have $\dot{\theta} = 0$ $\Rightarrow$ $\boxed{\theta = \frac{\pi}{2} = \mathrm{cste.}}$

hence a mouvement restricted to a plane is admissible, as for newtonian gravitation.

(G3) $\Rightarrow \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\dot{\phi}\right) = 0$ $\Rightarrow$ $\boxed{r^2\dot{\phi} = a = \mathrm{cste.}}$

this is the *conservation of angular momentum*, as for newtonian gravitation.

Case distinction: from now on, consider motion of **particles**.

Insert angular momentum conservation and the first integral (G0), namely $\left(1 - \frac{\mathscr{R}}{r}\right)\dot{t} = b$, into (G1); recall as well $\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\phi}\dot{\phi} = \frac{\mathrm{d}r}{\mathrm{d}\phi}\frac{a}{r^2}$

$$1 = \left(1 - \frac{\mathscr{R}}{r}\right)\dot{t}^2 - \left(1 - \frac{\mathscr{R}}{r}\right)^{-1}\frac{\dot{r}^2}{c^2} - \frac{r^2}{c^2}\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right)$$

* *The mouvement is in a plane*

to see this: consider geodesic, on equator $\theta = \frac{\pi}{2}$, tangential at the plane $\dot{\theta} = 0$

(G2) $\Rightarrow \ddot{\theta} = 0 \Rightarrow$ for all τ have $\dot{\theta} = 0$ $\qquad \Rightarrow \boxed{\theta = \frac{\pi}{2} = \text{cste.}}$

hence a mouvement restricted to a plane is admissible, as for newtonian gravitation.

(G3) $\Rightarrow \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\dot{\phi}\right) = 0$ $\qquad \Rightarrow \boxed{r^2\dot{\phi} = a = \text{cste.}}$

this is the *conservation of angular momentum*, as for newtonian gravitation.

Case distinction: from now on, consider motion of **particles**.

Insert angular momentum conservation and the first integral (G0),

namely $\left(1 - \frac{\mathscr{R}}{r}\right)\dot{t} = b$, into (G1); recall as well $\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\phi}\dot{\phi} = \frac{\mathrm{d}r}{\mathrm{d}\phi}\frac{a}{r^2}$

$$1 = \left(1 - \frac{\mathscr{R}}{r}\right)\dot{t}^2 - \left(1 - \frac{\mathscr{R}}{r}\right)^{-1}\frac{\dot{r}^2}{c^2} - \frac{r^2}{c^2}\left(\underbrace{\dot{\theta}^2}_{=0} + \underbrace{\sin^2\theta}_{=1}\,\dot{\phi}^2\right)$$

* *The mouvement is in a plane*

to see this: consider geodesic, on equator $\theta = \frac{\pi}{2}$, tangential at the plane $\dot{\theta} = 0$

(G2) $\Rightarrow \ddot{\theta} = 0 \Rightarrow$ for all τ have $\dot{\theta} = 0$ $\Rightarrow$ $\boxed{\theta = \frac{\pi}{2} = \text{cste.}}$

hence a mouvement restricted to a plane is admissible, as for newtonian gravitation.

(G3) $\Rightarrow \ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} = 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\dot{\phi}\right) = 0$ $\Rightarrow$ $\boxed{r^2\dot{\phi} = a = \text{cste.}}$

this is the *conservation of angular momentum*, as for newtonian gravitation.

Case distinction: from now on, consider motion of **particles**.

Insert angular momentum conservation and the first integral (G0), namely $\left(1 - \frac{\mathscr{R}}{r}\right)\dot{t} = b$, into (G1); recall as well $\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\phi}\dot{\phi} = \frac{\mathrm{d}r}{\mathrm{d}\phi}\frac{a}{r^2}$

$$1 = \left(1 - \frac{\mathscr{R}}{r}\right)^{-1} b^2 - \frac{1}{c^2}\left(1 - \frac{\mathscr{R}}{r}\right)^{-1}\frac{a^2}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 - \frac{1}{c^2}\frac{a^2}{r^2}$$

this has now turned into an equation for the **orbit** $r = r(\phi)$

$$1 = \left(1 - \frac{\mathscr{R}}{r}\right)^{-1} b^2 - \frac{1}{c^2}\left(1 - \frac{\mathscr{R}}{r}\right)^{-1} \frac{a^2}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 - \frac{1}{c^2}\frac{a^2}{r^2}$$

now remember that $\frac{1}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = \left(\frac{\mathrm{d}}{\mathrm{d}\phi}\frac{1}{r}\right)^2$:

$$\frac{1}{a^2}\left(1 - \frac{\mathscr{R}}{r}\right) = \frac{b^2}{a^2} - \frac{1}{c^2}\left(\frac{\mathrm{d}}{\mathrm{d}\phi}\frac{1}{r}\right)^2 - \frac{1}{c^2}\frac{1}{r^2}\left(1 - \frac{\mathscr{R}}{r}\right)$$

$$\Rightarrow \left(\frac{\mathrm{d}}{\mathrm{d}\phi}\frac{1}{r}\right)^2 + \frac{1}{r^2} = \frac{c^2\left(b^2 - 1\right)}{a^2} + \frac{\mathscr{R}}{r}\frac{c^2}{a^2} + \frac{\mathscr{R}}{r^3}$$

take another derivative with respect to ϕ:

$$2\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{1}{r}\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\left(\frac{1}{r}\right)\right) + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{1}{r}\right) = \frac{\mathscr{R}c^2}{a^2}\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{1}{r}\right) + \frac{3\mathscr{R}}{r^2}\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{1}{r}\right)$$

If $\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{1}{r}\right) = 0$, have a perfect circle. Otherwise $\frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{1}{r}\right) \neq 0$. Then, in general

$$\boxed{\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{2}\frac{\mathscr{R}c^2}{a^2} + \frac{3}{2}\frac{\mathscr{R}}{r^2}}$$

This is the relativistic generalisation of **Binet's formula** for the orbit.

Have found the relativistic generalisation of Binet's formula, for particles

$$\boxed{\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{2}\frac{\mathscr{R}c^2}{a^2} + \frac{3}{2}\frac{\mathscr{R}}{r^2}} \quad , \quad \text{particle} \tag{B}$$

and similarly for light

exercice

$$\boxed{\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{3}{2}\frac{\mathscr{R}}{r^2}} \quad , \quad \text{light} \tag{B'}$$

Have found the relativistic generalisation of Binet's formula, for particles

$$\boxed{\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{1}{2}\frac{\mathscr{R}c^2}{a^2}+\frac{3}{2}\frac{\mathscr{R}}{r^2}} \quad , \quad \text{particle} \qquad \text{(B)}$$

and similarly for light

$$\boxed{\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{3}{2}\frac{\mathscr{R}}{r^2}} \quad , \quad \text{light} \qquad \text{(B')}$$

with respect to the newtonian cases, have exact relativistic corrections.

N.B.: for light, recover for $\frac{\mathscr{R}}{r} \to 0$ straight line, since with $u = \frac{1}{r}$ have $u'' + u = 0$ **!**

Eqs. (B,B') are the requested equations for the orbits of particles/light, around a spherical mass M.

solving Binet's formula gives the solution of the *relativistic one-body problem*

4.6 Experimental test II: perihelion precession

Rappel:

for a newtonian gravitational potential $\sim \frac{1}{r}$, have *standard Binet's formula*

$$\boxed{\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{p}} \quad , \qquad \text{with } p = \frac{a^2}{GM} = \frac{2a^2}{\mathscr{R}c^2} \text{ `parameter'}$$

find **conic** $\frac{1}{r} = \frac{1}{p}\big(1 + e\cos(\phi - \phi_0)\big)$ an ellipse if $0 < e < 1$, choose $\phi_0 = 0$

maximal/minimal distance from centre: $r_{\substack{\max\\ \min}} = p/\big(1 \mp e\big)$, and $p = \bar{a}((1-e^2)$, $\bar{a}$: major half-axis

* here, must study the relativistic corrections
  since $\frac{3}{2}\frac{\mathscr{R}}{r^2}/\frac{1}{r} \approx 10^{-7} \ll 1$, a perturbative treatment is sufficient
  use newtonian solution to insert into relativistic Binet's formula

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} &\simeq \frac{1}{2}\frac{\mathscr{R}c^2}{a^2} + \frac{3}{2}\mathscr{R}\frac{\mathscr{R}c^4}{4a^4}\big(1 + 2e\cos\phi + \ldots\big) \\
&\simeq \frac{1}{2}\frac{\mathscr{R}c^2}{a^2} + \frac{3}{4}\frac{\mathscr{R}^3c^4}{a^2}e\cos\phi + \ldots
\end{aligned}$$

The solution is found as follows, to leading order

$$
\begin{aligned}
\frac{1}{r} &\simeq \frac{\mathscr{R}c^2}{2a^2}\left(1 + e\cos\phi\right) + \frac{3}{8}\frac{\mathscr{R}^3c^4}{a^4}e\phi\sin\phi + \ldots \\
&= \frac{\mathscr{R}c^2}{2a^2}\left(1 + e\cos\phi + \frac{3}{4}\frac{\mathscr{R}^2c^2}{a^2}e\phi\sin\phi + \ldots\right) \\
&\simeq \left[1 + e\cos\left(\phi\left(1 - \frac{3}{4}\frac{\mathscr{R}^2c^2}{a^2}\right)\right)\right] + \mathrm{O}((\mathscr{R}/r)^2)
\end{aligned}
$$

this implies that the axis of the ellipse is not stationary, but rotates !
After a period, the angular shift is EINSTEIN 1915/16

$$
\Delta\phi = \frac{2\pi}{1 - \frac{3}{4}\frac{\mathscr{R}^2c^2}{a^2}} - 2\pi \simeq \frac{3\pi}{2}\frac{\mathscr{R}^2c^2}{a^2}
$$

N.B.: absolute prediction, without any free parameter !

N.B.: comes about since Binet's formula not only has $\frac{1}{r}$-potential, but $\frac{1}{r^2}$-contributions as well.

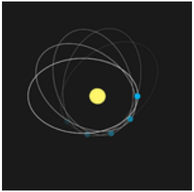
Classic example: perihelion shift of planet Mercury

for orbit around sun $\mathscr{R} = \frac{2GM_\odot}{c^2}$, $M_\odot = 1.99 \cdot 10^{30}[\mathrm{kg}]$
orbit of Mercury: $\bar{a} \simeq 5.8 \cdot 10^{10}[\mathrm{m}]$, $e \simeq 0.21$

leads to a predicted rotation angle (**perihelion shift**)

$$\Delta\phi = 43''/[\mathrm{century}]$$

N.B.: this is **not** observed directly !



Source: https://de.wikipedia.org/wiki/Tests_der_allgemeinen_Relativitätstheorie

a long-time study (decades !) of many astronomers gives the following

$\Delta\phi$ $(''/[\mathrm{century}])$	
574.103 ± 0.65	observed total precession from gravity effects
532.3100 ± 0.0015	predicted from newtonian theory, including all perturbations from other planets (Venus, Jupiter, Earth,..)
42.9799 ± 0.0009	from Schwarzschild metric

The residual difference is in spectacular agreement with general relativity !

historically, was the first test of the Einstein equations

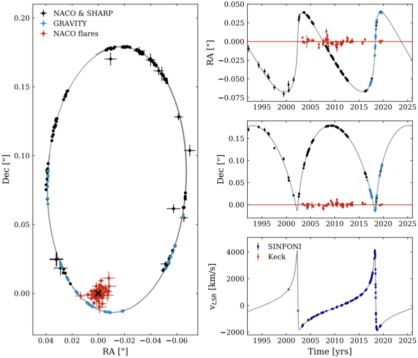
R.S. Park *et al.*, Astron. J. **153**, 121 (2017); G.M. Clemence, Rev. Mod. Phys. **19**, 361 (1947)

A new kind of test: strong gravitational fields II

Nobel prize 2020

astron. observation: Sgr A* compact, extremely massive object immobile at galaxy centre

infra-red observations (interferometers & adaptive optics): cluster of stars orbiting Sgr A*



star S2 passes close to centre

high velocity $v \approx 7650[\mathrm{km/h}] = 0.026c$

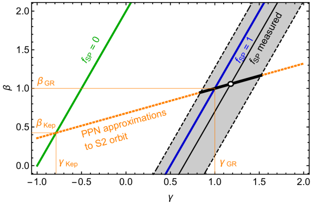
keplerian orbit plus relativistic corrections

precession of pericentre

$$\Delta\phi\Big|_{\rm orbit} = 3f\,\frac{3\mathscr{R}}{a^2(1-e^2)} = f\cdot 12.1'$$

comparison parameter f, such that $f_{GR} = 1$

$f = 1.10 \pm 0.19$ Genzel *et al.* A&A **636**, L5 (2020)



first 'post-newtonian' correction to acceleration

$$\boldsymbol{a} = -\frac{GM}{r^3}\boldsymbol{r} + \frac{GM}{c^2 r^2}\left[\left(2(\gamma+\beta)\frac{GM}{r} - \gamma\boldsymbol{v}^2\right)\frac{\boldsymbol{r}}{r} + 2(1+\gamma)\dot{r}\boldsymbol{v}\right]$$

blue: prediction of GR ($\beta = \gamma = 1$), green: keplerian ($\beta = \gamma = 0$)

observation: $\beta = 1.05 \pm 0.11$ and $\gamma = 1.18 \pm 0.34$

not as precise as in solar system, but for much more strong fields

Vorlesung VIII

Rappel: Einstein's proposal of field equation of gravitation with sources

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^2} T_{\mu\nu}} \quad \Leftrightarrow \quad \boxed{R_{\mu\nu} = \frac{8\pi G}{c^2}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)}$$

with $T_{\mu\nu}$: energy-momentum tensor of matter, G: Newton's gravitational constant

most simple solution of physical interest: gravitational field outside of a mass point at rest (exact solution) (Schwarzschild)

$$\boxed{\mathrm{d}s^2 = -\left(1 - \frac{\mathscr{R}}{r}\right) c^2 \mathrm{d}t^2 + \left(1 - \frac{\mathscr{R}}{r}\right)^{-1} \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2 \mathrm{d}\phi^2\right)}$$

can derive geodesic curves $\Rightarrow$ orbits of freely falling test masses

☞ derive relativistic extension of Binet's formula for the orbit $r = r(\phi)$

$$\boxed{\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{2}\frac{\mathscr{R} c^2}{a^2} + \frac{3}{2}\frac{\mathscr{R}}{r^2}} \quad , \quad \textbf{particle}$$

Rappel: EINSTEIN's proposal of field equation of gravitation with sources

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^2} T_{\mu\nu}} \quad \Leftrightarrow \quad \boxed{R_{\mu\nu} = \frac{8\pi G}{c^2}\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right)}$$

with $T_{\mu\nu}$: energy-momentum tensor of matter, G: Newton's gravitational constant

most simple solution of physical interest: gravitational field outside of a mass point at rest (exact solution) (SCHWARZSCHILD)

$$\boxed{\mathrm{d}s^2 = -\left(1 - \frac{\mathscr{R}}{r}\right) c^2 \mathrm{d}t^2 + \left(1 - \frac{\mathscr{R}}{r}\right)^{-1} \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2 \mathrm{d}\phi^2\right)}$$

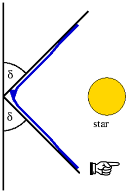
can derive geodesic curves ⇒ orbits of freely falling test masses

☞ derive relativistic extension of Binet's formula for the orbit $r = r(\phi)$

$$\boxed{\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \qquad \frac{3}{2}\frac{\mathscr{R}}{r^2}} \quad , \quad \textbf{light}$$

Experimental test III: deviation of light

the orbit of a light ray is also given by a Binet formula exercice



$$\boxed{\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{3}{2}\frac{\mathscr{R}}{r^2}} \quad , \quad \text{light} \qquad \text{(B')}$$

$$\Rightarrow \quad \frac{1}{r}=\frac{1}{r_0}\cos\phi+\frac{\mathscr{R}}{2r_0^2}\left(1+\sin^2\phi\right)$$

☞ remarkable: light rays do **not** follow a straight line !

N.B.: curved orbits of light predicted by Cavendish (unpublished 1784) & Soldner (1804)

look at asymptotics for $r\to\infty$: (a) if $\mathscr{R}=0$, have $\phi\to\pm\frac{\pi}{2}$

(b) if $\mathscr{R}>0$, have $\phi\to\pm\left(\frac{\pi}{2}+\delta\right)$ such that $-\frac{1}{r_0}\sin\delta+\frac{\mathscr{R}}{2r_0^2}\left(1+\cos^2\delta\right)\stackrel{!}{=}0$

$\Rightarrow \delta\simeq\frac{\mathscr{R}}{r_0}$. The angle of light deviation is, with numbers for deviation at border of sun

$$\boxed{\Delta=2\delta\simeq\frac{4M_\odot G}{R_\odot c^2}=\frac{2\mathscr{R}_\odot}{R_\odot}\simeq 1.75''}$$

curved orbits & numerical value spectacularly confirmed by EDDINGTON 1919

present values: $\boxed{\Delta=\left(0.99992\pm0.00023\right)\Delta_{GR}}$

have seen three spectacular confirmations of general relativity:

the so-called **classical tests**

I) test of the equivalence principle via gravitational red shift

☞ Pound-Snider-Rebka and Vessot-Levine experiments

☞ also confirmed in strong fields: Sirius B and stars close to Sgr A*

☞ **!** necessary ingredient for proper functioning of the GPS **!**

II) perihelion shift of planets in solar system

☞ general relativity explains extra rotation left *unexplained by newtonian celestial mechanics* for more than 50 years

☞ also seen in strong fields for stars in close orbits around Sgr A*

III) deviation of light rays in gravitational fields

☞ *light does not follow a straight line* under the influence of gravitation clear contradiction with well-established newtonian physics and first evidence for a new paradigm also noticed by larger public

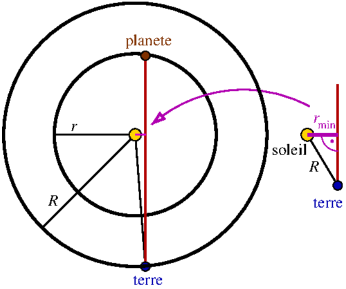
N.B.: these confirmations are about **parameter-free predictions**, no data fitting possible

4.6 Experimental test IV: radar echo

first example of a new class of experimental tests

1960s new technology : use radar echos to measure better distances of planets

☞ revision of the astronomical unit (radius of Earth's orbit $\simeq 150 \cdot 10^6 [\mathrm{km}]$) by $\sim 9.3 \cdot 10^4 [\mathrm{km}]$



measured quantity: time of passage of a radar signal Earth - Planet -Earth

waiting time until return of signal

$$T = 2\big(t(R, r_{\mathrm{mim}}) + t(r, r_{\min})\big)$$

R: radius of Earth orbit,

r: radius of planet's orbit

echo returns so fast that planets' & Earth's motion is neglected

herein $r_{\min}$ is the minimal distance of the radar's orbit from the sun

non-relativistic calculation

$$t_{\mathrm{NR}}(R, r_{\min}) = \frac{1}{c}\sqrt{R^2 - r_{\min}^2}$$

VOLUME 13, NUMBER 26 PHYSICAL REVIEW LETTERS 28 DECEMBER 1964

FOURTH TEST OF GENERAL RELATIVITY

Irwin I. Shapiro
Lincoln Laboratory,* Massachusetts Institute of Technology, Lexington, Massachusetts

(B) prediction of the Schwarzschild metric $\dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau}$ etc.

radar echos are light-like $\Rightarrow$ $\mathrm{d}s^2 = 0$ and mouvement is planar $\Rightarrow$ $\mathrm{d}\theta = 0$

$$0 = -\left(1 - \frac{\mathscr{R}}{r}\right) + \frac{1}{c^2}\left[\left(1 - \frac{\mathscr{R}}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2\right]$$

Rappel: in calculation of orbit, had $\left(1 - \frac{\mathscr{R}}{r}\right)\dot{t} = b = \mathrm{cste.}$ and $r^2\dot{\phi} = a = \mathrm{cste.}$

$$\Rightarrow \quad \frac{a}{b} = \frac{r^2 \frac{\mathrm{d}\phi}{\mathrm{d}\tau}}{\left(1 - \frac{\mathscr{R}}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau}} = \frac{r^2}{\left(1 - \frac{\mathscr{R}}{r}\right)}\frac{\mathrm{d}\phi}{\mathrm{d}t} =: B = \mathrm{cste.}$$

which implies that $\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{B}{r^2}\left(1 - \frac{\mathscr{R}}{r}\right)$. Insert this into the metric above

$$0 = -\left(1 - \frac{\mathscr{R}}{r}\right) + \frac{1}{c^2}\left[\left(1 - \frac{\mathscr{R}}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + \frac{B^2}{r^2}\left(1 - \frac{\mathscr{R}}{r}\right)^2\right]$$

at $r = r_{\min}$ the distance r is minimal, hence $\left.\frac{\mathrm{d}r}{\mathrm{d}t}\right|_{r=r_{\min}} \stackrel{!}{=} 0$.

$\Rightarrow$ this fixes $B^2 = c^2 r_{\min}^2 \left(1 - \frac{\mathscr{R}}{r_{\min}}\right)^{-1}$.

$$\Rightarrow \quad \frac{\mathrm{d}r}{\mathrm{d}t} = c\left(1-\frac{\mathscr{R}}{r}\right)\sqrt{1-\frac{r_{\min}^2}{r^2}\frac{1-\frac{\mathscr{R}}{r}}{1-\frac{\mathscr{R}}{r_{\min}}}}$$

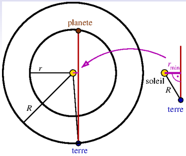
and integration leads to

$$\begin{aligned} t(r, r_{\min}) &= \frac{1}{c}\int_{r_{\min}}^{r}\mathrm{d}r'\,\left(1-\frac{\mathscr{R}}{r'}\right)^{-1}\left(1-\frac{r_{\min}^2}{r'^2}\frac{1-\frac{\mathscr{R}}{r'}}{1-\frac{\mathscr{R}}{r_{\min}}}\right)^{-1/2} \\ &\simeq \frac{1}{c}\int_{r_{\min}}^{r}\mathrm{d}r'\,\frac{r'}{\left(r'^2-r_{\min}^2\right)^{1/2}}\left(1+\frac{\mathscr{R}}{r'}+\frac{1}{2}\frac{\mathscr{R}\,r_{\min}}{r'\left(r'+r_{\min}\right)}+\dots\right) \\ &= \frac{1}{c}\left(\sqrt{r^2-r_{\min}^2}+\mathscr{R}\ln\left(\frac{r+\sqrt{r^2-r_{\min}^2}}{r_{\min}}\right)+\frac{\mathscr{R}}{2}\sqrt{\frac{r-r_{\min}}{r+r_{\min}}}\right) \end{aligned}$$

where the first term is the non-relativistic contribution.
Since both Earth & planet are far from the sun, one may simplify further

$$t(r, r_{\min}) \simeq \frac{1}{c}\left(\sqrt{r^2-r_{\min}^2}+\mathscr{R}\ln\left(\frac{2r}{r_{\min}}\right)+\frac{\mathscr{R}}{2}\right)$$

and the final time of passage (A/R) is   SHAPIRO 1964



$$T = 2\big(t(R, r_{\min}) + t(r, r_{\min})\big) \simeq \frac{2}{c}\left(R + r + \mathscr{R}\left[\ln\left(\frac{4Rr}{r_{\min}^2}\right) + 1\right]\right)$$

Numerical illustration: planet Mars $r = 1.52[\mathrm{AU}] = 2.28 \cdot 10^{11}[\mathrm{m}]$

planet Earth $R = 1[\mathrm{AU}] = 1.49 \cdot 10^{11}[\mathrm{m}]$

dominant contribution $T_0 = \frac{2}{c}(R + r) \simeq 2.52 \cdot 10^3[\mathrm{s}] \simeq 42[\mathrm{min}]$

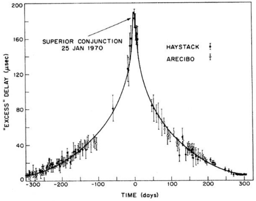
maximal size of relativistic correction if $r_{\min} = R_\odot =$ solar radius

$\frac{4Rr}{R_\odot^2} \simeq 2.82 \cdot 10^5 \Rightarrow \ln \frac{4Rr}{R_\odot^2} \simeq 12.6$

additional time from leading relativistic correction

$$\Delta T \simeq \frac{2\mathscr{R}}{c}\left[1 + \ln\frac{4Rr}{R_\odot^2}\right] \simeq 2.66 \cdot 10^{-4}[\mathrm{s}] = 266[\mu\mathrm{s}]$$

☞ required to measure T with relative precision better than 10^{-7}
atomic clocks achieve accuracies of order 10^{-12} ☞ feasible in principle



radar echo reflected at surface of planet Venus

shown is *excess time delay*, as a function of time
comparison with general relativity works up to $\lesssim 10\%$
precision limited by surface roughness of planet

I.M. Shapiro *et al.*, Phys. Rev. Lett. **26**, 1132 (1971)

practical comparison:

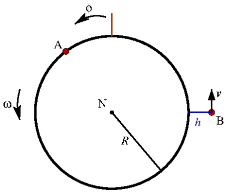
direct echo from planets	$\sim 10\%$
space crafts	$< 1\%$
space craft on planet (Viking)	$\sim 0.1\%$
space craft Cassini in Saturn orbit	$\lesssim 2.3 \cdot 10^{-5}$

4.6 Experimental test V: time delay

thought experiment: take two identical clocks, synchronise them
clock A stays in the labo (at the equator), clock B travels around the earth
? what time-difference should one measure **?**

from the Schwarzschild metric at equator $\theta = \frac{\pi}{2}$, height fixed $r = \mathrm{cste}$

$$\mathrm{d}s^2 = -\left(1 - \frac{\mathscr{R}}{r}\right) c^2 \mathrm{d}t^2 + r^2 \mathrm{d}\phi^2 = -c^2 \mathrm{d}\tau^2$$



schematic view onto the north pole of the earth,

angular velocity of rotation ω
angle ϕ measured with respect to point of reference
positions of the clocks at A,B

we evaluate the proper times for both clocks

clock A: which remains fixed (and rotates with the earth)

because of the earth's rotation, with angular velocity ω: $\mathrm{d}\phi = \omega \mathrm{d}t$

$$\mathrm{d}\tau_A^2 = \left[\left(1 - \frac{\mathscr{R}}{r}\right) - \frac{R^2\omega^2}{c^2}\right]\mathrm{d}t^2$$

$$\mathrm{d}\tau_A \simeq \left(1 - \frac{GM}{Rc^2} - \frac{R^2\omega^2}{2c^2}\right)\mathrm{d}t$$

with M: earth's mass, since $\frac{\mathscr{R}}{R} \ll 1$ and $\frac{R^2\omega^2}{C^2} \ll 1$.

clock B: which travels around the earth (direction east)

with the velocity $\simeq v + (R+h)\omega$ with respect to the static metric

$$\mathrm{d}\tau_B^2 = \left[\left(1 - \frac{\mathscr{R}}{R+h}\right) - \left(\frac{(R+h)\omega + v}{c}\right)^2\right]\mathrm{d}t^2$$

$$\mathrm{d}\tau_B \simeq \left(1 - \frac{GM}{(R+h)c^2} - \frac{R^2\omega^2 + 2R\omega v + v^2}{2c^2}\right)\mathrm{d}t$$

the relative deviation becomes

$$\boxed{\Delta := \frac{\mathrm{d}\tau_A - \mathrm{d}\tau_B}{\mathrm{d}\tau_A} \simeq -\frac{GM}{R^2c^2}h + \frac{2R\omega v + v^2}{c^2}}$$

flight in western direction: replace $\boldsymbol{v} \to -\boldsymbol{v}$

Numerical illustration:

(a) flight in eastern direction:

take $h \simeq 10[\mathrm{km}] = 10^4[\mathrm{m}]$, $v = 300[\mathrm{m/s}]$, $\frac{GM}{R^2} = g = 9.81[\mathrm{m/s^2}]$.

N.B.: these are typical estimates for a commercial air-plane flight

$\frac{gh}{c^2} \simeq 1.09 \cdot 10^{-12}$, $2R\omega \simeq 931[\mathrm{m/s^2}]$, $\frac{(2R\omega+v)v}{2c^2} \simeq 2.1 \cdot 10^{-12}$

☞ this gives $\Delta_{\mathrm{east}} \simeq 1.0 \cdot 10^{-12}$

(b) flight in western direction:

re-use $h \simeq 10[\mathrm{km}] = 10^4[\mathrm{m}]$, $v = -300[\mathrm{m/s}]$, $\frac{GM}{R^2} = g = 9.81[\mathrm{m/s^2}]$.

a change occurs for $\frac{-(2R\omega-v)v}{2c^2} \simeq -1.05 \cdot 10^{-12}$

☞ this gives $\Delta_{\mathrm{west}} \simeq -2.1 \cdot 10^{-12}$

! *these values are within reach of the precision of atomic clocks* **!**

. . . even in the early 1970s . . .

☞ instead of applying for money for an expensive satellite mission, and wait patiently many years for approval, *just put your atomic clock into a civil air-plane and fly* around the earth !

$\Rightarrow$ that is what Hafele & Keating did ...

cost 8000\$, 95% for flight tickets (4 persons, incl. 2× 'Mr. Clock')

J.C. Hafele, R.E. Keating, Science **177**, 166 & 168 (1972); and https://en.wikipedia.org/wiki/Hafele-Keating_experiment

they give the table (all times in $[\mathrm{ns}]$)

	grav. (GR)	kinem. (SRT)	total	measured
east	$+144 \pm 14$	-184 ± 18	-40 ± 23	-59 ± 10
west	$+179 \pm 18$	$+96 \pm 10$	$+275 \pm 21$	$+273 \pm 7$

main source of error: precise schedule of the flights

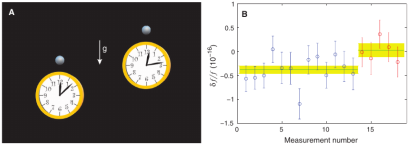
this inexpensive (!) experiment works at the level of signal precision $\sim 10^{-12}$

has been repeated several times, with increasing precision. For example:

(1) signal precision $\simeq 3 \cdot 10^{-13}$ S. Iijima, K. Fujiwara, Ann. Tokyo Observatory **17**, 68 (1978)

(2) signal precision $\sim 4 \cdot 10^{-16}$ C.W. Chou *et al.*, Science **329**, 1630 (2010)

N.B.: height differences of $33[\mathrm{cm}]$ gravitationally detected !



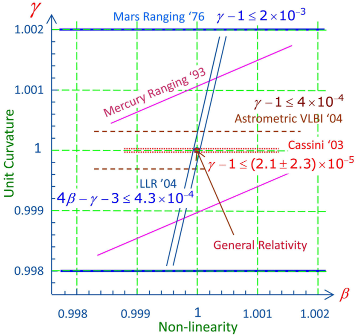
4.10 Post-newtonian parameters

it has become common to express the results of precision tests on general relativity in terms of certain parameters - the **post-newtonian parameters**

for example, it is common to consider the following *ad hoc* extension of the outer Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{\mathscr{R}}{r} - \frac{\beta-\gamma}{2}\frac{\mathscr{R}^2}{r^2}\right)c^2\mathrm{d}t^2 + \left(1 - \gamma\frac{\mathscr{R}}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

such that EINSTEIN's theory corresponds to $\beta = \gamma = 1$.



combined results for estimates on the *'post-newtonian parameters'* β, γ

the joint experiments constrain β, γ more than any single experiment could achieve alone

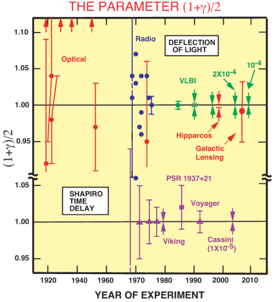
$$\Rightarrow \boxed{\gamma - 1 = (-0.3 \pm 2.5)\cdot 10^{-5}} \text{ and}$$

$$\boxed{\beta - 1 = (0.2 \pm 2.5)\cdot 10^{-5}}$$

S.G. Turychev, Proc. IAU Symposium **261** (2009); LLR: **L**unar **L**aser **R**anging
C.M. Will *Theory and Experiment in Gravitation*, (Cambridge ^{2}2018)

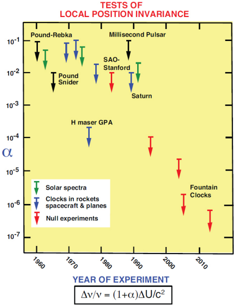
the following graphs illustrate the experimental improvements realised

Source & Refs.: C.M. Will, *Theory and experiment in gravitational physics*, 2^e éd. Cambridge (2018)



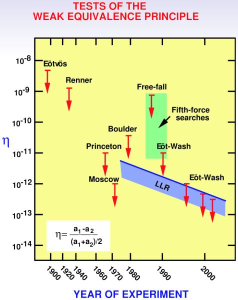
experimental measurements of the post-newtonian parameter γ

any disagreement with general relativity $< 10^{-3}\%$

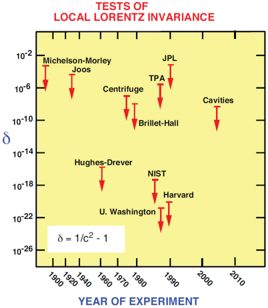


experimental test of position-invariance (frequency shift of light in gravitation field) the post-newtonian parameter α is defined via

$$\frac{\Delta\nu}{\nu} = (1+\alpha)\frac{\Delta U}{c^2}$$



experimental tests of the (weak) principle of equivalence (Eötvös experiment)



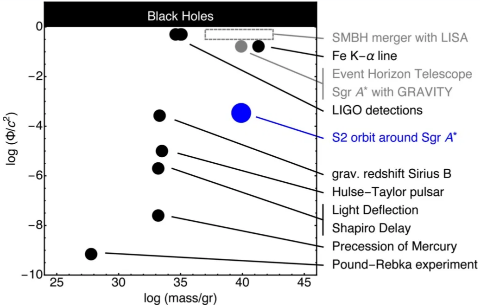
experimental tests of Lorentz invariance

$$\delta = 1 - \frac{c_0^2}{c^2}$$

where c: speed of electromagnetic radium in vacuum
c_0: limiting speed of test particles of unbroken Lorentz invariance

☞ *illustrates the degree to which fundamental assumptions of relativity are experimentally supported*

a comment on the context of these experimental tests



Source: GRAVITY collab., Genzel *et al.* A&A **615**, L15 (2018)

can distinguish five classes of experiments

1. small mass, weak field: Pound-Snider-Rebka experiment
2. medium mass, weak field: all classical tests and binary pulsar (1970s)
3. medium mass, medium field: gravitational redshift Sirius B (since 2018)
4. medium mass, strong field: coalescence of two black holes (since 2016)
5. large mass, medium field: astrophysics of black holes (since 2018)

⇒ do expect more & exciting news from future telescopes:

(1) **E**vent **H**orizon **T**elesope & (2) LISA (space-based gravitational waves)

4.11 The cosmological constant

there is a 'minimal extension' of EINSTEIN's field equations

$$\boxed{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^2} T_{\mu\nu}} \qquad (\bar{E})$$

herein, Λ is a new constant of Nature, of dimension $[\mathrm{Length}^{-2}]$.
Definition: Λ *is called the* **cosmological constant**.

presently accepted value (Planck coll. 2018): $\boxed{\Lambda \simeq (1.11 \pm 0.02) \cdot 10^{-52} [\mathrm{m}^{-2}]} > 0$

$\Rightarrow$ length scale $\Lambda^{-1/2} \sim 10^{26} [\mathrm{m}] \sim$ radius of the universe

Is the most natural term to add to EINSTEIN's field equation, but not a second derivative. Historically introduced, by EINSTEIN in 1917, in order to achieve non-expanding solutions of his field equations for the entire world. The true distances of galaxies were only found later (so-called *'great debate'* SHAPLEY-CURTIS in 1920, solved by observations of cepheïds in the Andromeda galaxy by HUBBLE in 1924). The general expansion of the universe was predicted in 1927 by LEMAÎTRE (with $\Lambda > 0$) and found through HUBBLE's law as late as 1929. In 1917, EINSTEIN required $\Lambda < 0$ for his stationary (unstable !) solution.

N.B.: Λ cannot be much smaller than its observed value, otherwise its effects would be unobservable even at the length scale of the entire universe

(a) special case without massive sources: $T_{\mu\nu} = 0$

$$\Rightarrow g^{\mu\nu} R_{\mu\nu} - \frac{1}{2} g^{\mu\nu} g_{\mu\nu} + \Lambda g^{\mu\nu} g_{\mu\nu} = 0 \;\Rightarrow\; R - \frac{1}{2} \cdot 4 \cdot R + 4\Lambda = 0 \;\Rightarrow\; R = 4\Lambda$$

hence $R_{\mu\nu} = \Lambda g_{\mu\nu}$ $\Rightarrow$ *if $\Lambda \neq 0$, time-space is space of constant curvature*, and $\Lambda^{-1/2}$ describes the curvature radius.

(b) **Schwarzschild-de Sitter solution**, at the exterior of a spherically symmetric mass M

$$\boxed{\mathrm{d}s^2 = -\left(1 - \frac{\mathscr{R}}{r} - \frac{\Lambda}{3} r^2\right) \mathrm{d}t^2 + \left(1 - \frac{\mathscr{R}}{r} - \frac{\Lambda}{3} r^2\right)^{-1} + r^2 \mathrm{d}\Omega^2}$$

with $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2$ and $\mathscr{R} = \frac{2GM}{c^2}$ is the Schwarzschild radius.

(c) non-relativistic limit: transform field equation $(\bar{E})$

$$g^{\mu\nu}R_{\mu\nu}-\frac{1}{2}g^{\mu\nu}g_{\mu\nu}+\Lambda g^{\mu\nu}g_{\mu\nu}=\frac{8\pi G}{c^2}g^{\mu\nu}T_{\mu\nu}\ \Rightarrow\ -R+4\Lambda=\frac{8\pi G}{c^2}T\ \Rightarrow\ R=4\Lambda+\frac{8\pi G}{c^2}T$$

then $(\bar{E})$ implies: $$R_{\mu\nu}=\Lambda g_{\mu\nu}+\frac{8\pi G}{c^2}\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right)$$

repeat the steps for carrying out the classical limit, identify $g_{00}=-\left(1+\frac{2}{c^2}\phi\right)$ with the newtonian potential ϕ, which obeys a modified Poisson's equation

$$\boxed{\nabla^2\phi+\Lambda=4\pi G\rho}$$

For a point mass M fixed the origin $r=0$, the solution is exercice

$$\boxed{\phi=-\frac{GM}{r}-\frac{\Lambda}{6}r^2}$$

in this setting, *Newton's law of gravitation is not exact*.

N.B.: in the *Principia*, Newton clearly states that he neglects unobservable effects find gravitational force on light test body, non-relativistic limit, mass m

$$\frac{1}{m}\boldsymbol{F} = -\boldsymbol{\nabla}\phi = -\frac{GM}{r^2}\boldsymbol{e}_r + \frac{\Lambda}{3}\boldsymbol{r}$$

Λ generates a *'cosmological force'* which tears objects apart.

? can one observe effects of Λ in the solar system ?

Answer: **NO**, and this will remain so forever !

C. Lämmerzahl *et al.*, Phys. Lett. **B634**, 465 (2006)

effect	bound on Λ, in $[\mathrm{m}^{-2}]$
light deflection	no effect
gravitational time delay	$\lesssim 6 \cdot 10^{-24}$
gravitational red shift	$\lesssim 10^{-27}$
shift of perihelia	$\lesssim 10^{-41}$
cosmology	$\approx 10^{-52}$

Λ can only be measured at the scale of the whole universe, or at the scale of clusters of galaxies

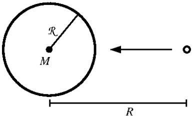
* since 2019: ? is there just a 'single cosmological constant' ?

* since a while: ?? can one understand Λ from quantum field theory ?? present tries off by a factor 10^{120}

4.12 Singularities and black holes

in the Schwarzschild metric, there is a singularity at radius $r = \mathscr{R}$

? what happens if a particle crosses the Schwarzschild radius $\mathscr{R}$ **?**



particle starts at rest, at a distance $r = R$ from the centre of large spherical mass M, and then falls centrally into it $\qquad \mathscr{R} = \frac{2GM}{c^2}$

(a) radial mouvement, seen by an external observer who uses universal time t

$$\mathrm{d}s^2 = -\left(1 - \frac{\mathscr{R}}{r}\right) c^2 \mathrm{d}t^2 + \left(1 - \frac{\mathscr{R}}{r}\right)^{-1} \mathrm{d}r^2 = -c^2 \mathrm{d}\tau^2$$

notations: $\dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau}$, $\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\tau}$, hence $\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t}\dot{t}$

$$\left(1 - \frac{\mathscr{R}}{r}\right)\dot{t}^2 - \left(1 - \frac{\mathscr{R}}{r}\right)^{-1} \frac{\dot{r}^2}{c^2} = 1 \;\Rightarrow\; \left[\frac{c^2(r - \mathscr{R})}{r} - \frac{r}{r - \mathscr{R}}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\right]\dot{t}^2 = c^2 \qquad (*)$$

the initial condition $\left.\frac{\mathrm{d}r}{\mathrm{d}t}\right|_{r=R} \stackrel{!}{=} 0$ implies with (*) $\left.\frac{\mathrm{d}t}{\mathrm{d}\tau}\right|_{r=R} = \left(\frac{R}{R-\mathscr{R}}\right)^{1/2}$.

next, recall from (G0) that $\left(1 - \frac{\mathscr{R}}{r}\right)\dot{t} = b = \mathrm{cste}$.

use this at $r = R$: $b = \left(1 - \frac{\mathscr{R}}{R}\right) \frac{\mathrm{d}t}{\mathrm{d}\tau}\Big|_{r=R} = \left(\frac{R-\mathscr{R}}{R}\right)^{1/2}$. Since $b = \mathrm{cste}$.

$$\dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{r}{r-\mathscr{R}} b = \frac{r}{r-\mathscr{R}} \left(\frac{R-\mathscr{R}}{R}\right)^{1/2} \qquad (**)$$

and finally, combining (*) and (**), it is easy to see that

$$\frac{\mathrm{d}r}{\mathrm{d}t} = -c\frac{r-\mathscr{R}}{r} \left(\frac{\mathscr{R}}{r}\right)^{1/2} \left(\frac{R-r}{R-\mathscr{R}}\right)^{1/2} \qquad (***)$$

one takes the *negative* solution, since the particle falls *into* the centre.

We want to find the time t_f the particle needs to fall from its initial radius R to a smaller radius $r < R$. Need to integrate (***)

$$t = t_f(r) = -\frac{1}{c}\left(\frac{R-\mathscr{R}}{\mathscr{R}}\right)^{1/2} \int_R^r \mathrm{d}\varrho \, \frac{\varrho^{3/2}}{\left(\varrho - \mathscr{R}\right)\left(R-\varrho\right)^{1/2}}$$

clearly, *the falling time* $t_f(r)$ *diverges, when* $r \to \mathscr{R}$.

had falling time $t_f(r)$, from radius R at $t=0$ to $r<R$

$$t_f(r) = -\frac{1}{c}\left(\frac{R-\mathscr{R}}{\mathscr{R}}\right)^{1/2}\int_R^r \mathrm{d}\varrho\,\frac{\varrho^{3/2}}{\left(\varrho-\mathscr{R}\right)\left(R-\varrho\right)^{1/2}}$$

In order to analyse the singularity, consider $\varrho = \mathscr{R}+\varepsilon$ with $\varepsilon \ll 1$ such that

$$t_f(r) = -\frac{1}{c}\left(\frac{R-\mathscr{R}}{\mathscr{R}}\right)^{1/2}\int_{R-\mathscr{R}}^{r-\mathscr{R}}\frac{\mathrm{d}\varepsilon}{\varepsilon}\,\frac{\left(\mathscr{R}+\varepsilon\right)^{3/2}}{\left(R-\mathscr{R}-\varepsilon\right)^{1/2}} \simeq -\frac{\mathscr{R}}{c}\int_{R-\mathscr{R}}^{r-\mathscr{R}}\frac{\mathrm{d}\varepsilon}{\varepsilon} = -\frac{\mathscr{R}}{c}\ln\left(\frac{r-\mathscr{R}}{R-\mathscr{R}}\right)$$

Interpretation: *for an observer far away from the centre, the distance of the falling particle with respect to the Schwarzschild radius decreases as*

$$\boxed{r-\mathscr{R} = \left(R-\mathscr{R}\right)e^{-ct/\mathscr{R}}}$$

* very rapid slowing-down of apparent fall, on time scale $\mathscr{R}/c$.
  Numerical example: for the sun $\mathscr{R}_\odot \approx 3[\mathrm{km}]$, so $\mathscr{R}_\odot/c \approx 10^{-5}[\mathrm{s}]$
* the **event horizon** at $r=\mathscr{R}$ is never reached ☞ **frozen particle**

(b) mouvement seen by the particle itself

who uses proper time τ

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = -c\frac{r-\mathscr{R}}{r}\left(\frac{\mathscr{R}}{r}\right)^{1/2}\left(\frac{R-r}{R-\mathscr{R}}\right)^{1/2} \cdot \frac{r}{r-\mathscr{R}}\left(\frac{R-\mathscr{R}}{R}\right)^{1/2}$$

where (***) and (**) were used.

(b) mouvement seen by the particle itself

who uses proper time τ

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = -c\frac{r-\mathscr{R}}{r}\left(\frac{\mathscr{R}}{r}\right)^{1/2}\left(\frac{R-r}{R-\mathscr{R}}\right)^{1/2}\cdot\frac{r}{r-\mathscr{R}}\left(\frac{R-\mathscr{R}}{R}\right)^{1/2}$$

where (***) and (**) were used.

(b) mouvement seen by the particle itself

who uses proper time τ

$$\frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\tau} = -c\left(\frac{\mathscr{R}}{R}\right)^{1/2}\left(\frac{R}{r} - 1\right)^{1/2}$$

where (***) and (**) were used.
The proper time $\tau = \tau_f(r)$ for the fall from radius R to radius r is then

$$\tau_f(r) = -\frac{1}{c}\left(\frac{R}{\mathscr{R}}\right)^{1/2}\int_R^r \frac{\mathrm{d}r'}{\left(R/r' - 1\right)^{1/2}} = \frac{1}{c}\left(\frac{R^3}{\mathscr{R}}\right)^{1/2}\left(\sqrt{\frac{r}{R}\left(1 - \frac{r}{R}\right)} + \arccos\sqrt{\frac{r}{R}}\right)$$

For a freely falling observer, the falling time until arrival at the centre is

$$\boxed{\tau_f(0) = \frac{\pi}{2c}\sqrt{\frac{R^3}{\mathscr{R}}}}$$

* this falling time is *finite* !

Numerical example: for the sun, starting at sun's radius $R_\odot = 7\cdot 10^8[\mathrm{m}] \Rightarrow \tau_f(0) \approx 100[\mathrm{s}]$.

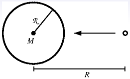
* nothing special happens at $r = \mathscr{R}$, from the point of view of a freely falling observer.

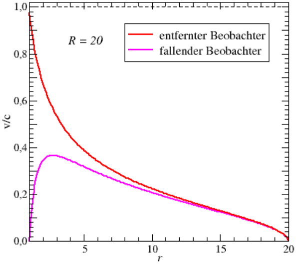
(provided $R < \infty$ is finite)

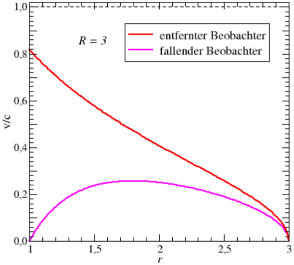
for further illustration: the velocity of the particle falling into the centre

measured with proper time: as seen by the in-falling particle itself

measured with universal time: as seen by a remote observer







radial velocity $v = v(r)$ is given in units of the speed of light c

the distance r of the particle from the centre and the initial distance R } are in units of the Schwarzschild radius $\mathscr{R}$

radius-dependent velocity $v = v(r)$ evolves differently for both observers, and also depends on the initial distance R, although $v(R) = 0$ always

N.B.: for R finite, one always has $v(r) < c$ for all $r > \mathscr{R}$

N.B'.: in the examples shown R is still quite close to $\mathscr{R}$!

depending on the place of the observer, very different results were obtained:

* an observer far away sees the particle freeze very rapidly at radius $r \gtrsim \mathscr{R}$. The particle *never* appears to arrive at the centre.
* a freely falling observer find nothing special at $r = \mathscr{R}$ and arrives after a *finite* time at the centre.
* falling-in velocity $v = v(r)$ behaves very differently for both observers

☞ *nice illustration of the relativity of time*

for radii $r > \mathscr{R}$, this picture can be used to describe the behaviour of the outer layers of a collapsing stars: seen from the outside, the outer layers rapidly 'freeze' at a radius $r \approx \mathscr{R}$ (**frozen star**) and it takes them an infinite time to cross the Schwarzschild radius. Seen from the outside, the star *never* collapses to a point. For the freely falling stellar matter, however, it will have arrived after a finite time right at the centre and nothing occurred at a radius $\mathscr{R}$.

☞ the region with radii $< \mathscr{R}$ seems to decouple completely from the regions far away from the centre.

Definition: *A* **black hole** *is a massive body with radius $R < \mathscr{R}$.*

N.B.: this requires extremely high mass densities !

Example: for the sun $R_\odot \simeq 7 \cdot 10^5[\mathrm{km}] \gg \mathscr{R}_\odot \simeq 3[\mathrm{km}]$. It is not possible, however, to reach $r = \mathscr{R}$ by going deep into the sun, since the mass $M(r) = 4\pi \int_0^r \mathrm{d}r\, r^2 \rho(r) \to 0$ as $r \to 0$.

N.B.: a non-rotating black hole is described exactly by the Schwarzschild metric

☞ *near to $r \gtrsim \mathscr{R}$ extreme and unintuitive effects may arise*

? where are the 'real' singularities of the Schwarzschild metric **?**

Definition: *The* **Kretschmann invariant** *is* $\mathscr{K} := R^{\mu\nu\kappa\lambda} R_{\mu\nu\kappa\lambda}$.

Theorem: (Kretschmann) *The invariant $\mathscr{K}$ is independent of the choice of coordinates.*

Example: For the Schwarzschild metric, one has $\mathscr{K} = 12\left(\frac{\mathscr{R}}{r}\right)^2 \frac{1}{r^4}$.

This means that at $r = \mathscr{R}$, the corresponding time-space of a black hole is non-singular, but there does exist a singularity at the centre $r = 0$.

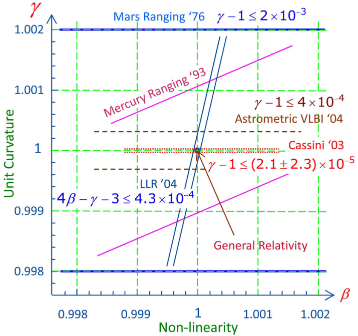
☞ There are many different coordinate systems in which the Schwarzschild metric is non-singular at $r = \mathscr{R}$.

Vorlesung IX

Rappel: use of **post-newtonian parameters** for characterisation of experiments often used: *ad hoc* extension of the outer Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{\mathscr{R}}{r} - \frac{\beta - \gamma}{2}\frac{\mathscr{R}^2}{r^2}\right) c^2\mathrm{d}t^2 + \left(1 - \gamma\frac{\mathscr{R}}{r}\right)^{-1} \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

such that EINSTEIN's theory corresponds to $\beta = \gamma = 1$.



combined results for estimates on the *'post-newtonian parameters'* β, γ

$$\Rightarrow \boxed{\gamma - 1 = (-0.3 \pm 2.5) \cdot 10^{-5}} \text{ and } \boxed{\beta - 1 = (0.2 \pm 2.5) \cdot 10^{-5}}$$

S.G. Turychev, Proc. IAU Symposium **261** (2009); LLR: **L**unar **L**aser **R**anging
C.M. Will *Theory and Experiment in Gravitation*, (Cambridge ^{2}2018)

although spectacular good confirmations of EINSTEIN's field equations, does not exclude possibilities for generalisations/extensions

Example: cosmological constant $\Lambda \simeq (1.11 \pm 0.02) \cdot 10^{-52}[\mathrm{m}^{-2}]$

Rappel: analysed mouvement in time-space with Schwarzschild metric

$$ds^2 = -\left(1 - \frac{\mathscr{R}}{r}\right) c^2 dt^2 + \left(1 - \frac{\mathscr{R}}{r}\right)^{-1} dr^2 + d\Omega^2$$

? existence of a physical singularity at the Schwarzschild radius $r = \mathscr{R}$?

Definition: *A* ***black hole*** *is a massive body with radius $R < \mathscr{R}$.*

N.B.: this requires extremely high mass densities !

Example: for the sun $R_\odot \simeq 7 \cdot 10^5[\mathrm{km}] \gg \mathscr{R}_\odot \simeq 3[\mathrm{km}]$. It is not possible, however, to reach $r = \mathscr{R}$ by going deep into the sun, since the mass $M(r) = 4\pi \int_0^r dr\, r^2 \rho(r) \to 0$ as $r \to 0$.

N.B.: a non-rotating black hole is described exactly by the Schwarzschild metric

☞ *near to r $\gtrsim \mathscr{R}$ extreme and unintuitive effects may arise*

? where are the 'real' singularities of the Schwarzschild metric ?

Definition: *The* **Kretschmann invariant** *is $\mathscr{K} := R^{\mu\nu\kappa\lambda} R_{\mu\nu\kappa\lambda}$.*

Theorem: (Kretschmann) *The invariant $\mathscr{K}$ is independent of chosen coordinates.*

Example: For the Schwarzschild metric, one has $\mathscr{K} = 12\left(\frac{\mathscr{R}}{r}\right)^2 \frac{1}{r^4}$.

This means that at $r = \mathscr{R}$, the corresponding time-space of a black hole is non-singular, but there does exist a singularity at the centre $r = 0$.

☞ many different, physcially equivalent, coordinate systems without a singularity at $r = \mathscr{R}$.

(c) orbital mouvement around a black hole

use here the Schwarzschild metric in a more general form $\quad \mathbf{x} = (ct, r, \theta, \phi)$

$$\mathrm{d}s^2 = -h(r)c^2\mathrm{d}t^2 + \frac{1}{h(r)}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2 \quad , \quad h(r) = 1 - \frac{\mathscr{R}}{r}$$

geodesic equations: $\quad \ddot{x}^\mu + \Gamma^\mu_{\kappa\lambda}\dot{x}^\kappa\dot{x}^\lambda = 0, \quad \Gamma^\mu_{\kappa\lambda} = \frac{1}{2}g^{\mu\rho}\left(g_{\rho\kappa,\lambda} + g_{\rho\lambda,\kappa} - g_{\kappa\lambda,\rho}\right)$

the non-vanishing Christoffel symbols are, with $c = 1$ $\quad h'(r) = \frac{\mathrm{d}h(r)}{\mathrm{d}r}$

$$\begin{aligned}
\Gamma^0_{10} &= \Gamma^0_{01} = \frac{h'}{2h} \\
\Gamma^1_{00} &= -\tfrac{1}{2}hh' \;,\; \Gamma^1_{11} = -\frac{h'}{2h} \;,\; \Gamma^1_{22} = -rh \;,\; \Gamma^1_{33} = -rh'\sin^2\theta \\
\Gamma^2_{12} &= \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \;,\; \Gamma^2_{33} = -\sin\theta\cos\theta \\
\Gamma^3_{23} &= \Gamma^3_{32} = \cot\theta
\end{aligned}$$

* mouvement in a plane, can fix coordinates such that $\theta = \frac{\pi}{2}$ $\quad \dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau}$ etc.

N.B.: velocities with respect to proper time $\Rightarrow$ observer co-moving with particule in orbit mouvement in a plane, fixed coordinates such that $\theta = \frac{\pi}{2}$ $\dot{t} = \frac{\mathrm{d}t}{\mathrm{d}\tau}$ etc.

Write down geodesic equations explicitly. Cast them all as first integrals:

$\mu = 0$: have $\ddot{t} + \frac{h'}{h}\dot{t}\dot{r} = 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}\tau}\left(h\dot{t}\right) = 0$ $\Rightarrow \boxed{h\dot{t} = b = \mathrm{cste}}$

N.B.: is the analogue of (G0) treated before

Interpretation of b: in case without interactions (e.g. for $r \to \infty$), expect $h(r) \to 1$.

For special relativity, and using the proper time τ as parameter, have

$$b = \frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma = \frac{E}{m}$$

with E: energy of particle, m: mass of particle.

This first integral is interpreted as the *conservation of energy*

$$\boxed{E = mh\,\dot{t}} \qquad \text{(T0)}$$

$\mu = 2$: can take over (G2) from earlier treatment of the orbit:

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0 \qquad \text{(G2=T2)}$$

is taken care of by fixing $\theta = \frac{\pi}{2}$

$\mu = 3$: can take over (G3) from earlier treatment of the orbit:

have $\ddot\phi + \frac{2}{r}\dot r\dot\phi + 2\cot\theta\,\dot\theta\dot\phi = 0 \overset{\theta=\pi/2}{\Rightarrow} \frac{\mathrm{d}}{\mathrm{d}\tau}\left(r^2\dot\phi\right) = 0 \Rightarrow r^2\dot\phi = a = \mathrm{cste}$

This is the *conservation of angular momentum*. We shall write $L = \frac{\ell}{m}$

$$\boxed{r^2\dot\phi = L} \qquad \text{(T3)}$$

$\mu = 1$: we use the metric instead, since this gives the conservation law directly (here for particles)

$$1 = h\dot t^2 - h^{-1}\dot r^2 - r^2\left(\dot\theta^2 + \sin^2\theta\,\dot\phi^2\right)$$

fixing $\theta = \frac{\pi}{2}$ and using the conservation laws (T0), (T2) gives

$$1 = h\left(\frac{E}{mh}\right)^2 - \frac{1}{h}\dot r^2 - r^2\left(\frac{L}{r^2}\right)^2 = \frac{E^2}{m^2}\frac{1}{h} - \frac{1}{h}\dot r^2 - \frac{L^2}{r^2} \quad \Rightarrow \quad \dot r^2 = \frac{E^2}{m^2} - h - h\frac{L^2}{r^2}$$

so that we find

a *second* analogy of energy conservation

$$\left(m\dot r\right)^2 = E^2 - m^2 h - m^2 h\frac{L^2}{r^2} = E^2 - \left(1 + \frac{L^2}{r^2}\right) m^2 h(r) \qquad \text{(T1)}$$

Interpretation: $m\dot r$ is the radial component of the particle's four-momentum $\mathtt{p}$

writing $p = m\dot{r}$, find relativistic energy-momentum relation $p^2 = E^2 - \left(1 + \frac{L^2}{r^2}\right) m^2 h$.

Definition: *The* **effective potential** $V_{\text{eff}}(r)$ *for a massive particle is defined from the energy-momentum relation*

$$\boxed{E^2 = p^2 + V_{\text{eff}}^2(r) \quad , \quad \text{with} \quad V_{\text{eff}}^2(r) := m^2 h(r) \left(1 + \frac{L^2}{r^2}\right)}$$

photons or other massless particles can be treated similarly

Definition: *The* **effective potential** $V_{\text{eff}}(r)$ *for a photon is defined from the energy-momentum relation*

$$\boxed{E_\gamma^2 = p_\gamma^2 + V_{\text{eff}}^2(r) \quad , \quad \text{with} \quad V_{\text{eff}}^2(r) := h(r) \frac{L^2}{r^2}}$$

☞ V_{eff} permits a clear qualitative discussion of possible movements around a black hole (for either particles or photons)

[for completeness, the derivation of the effective potential for photons is provided:

$\mu = 0$: had seen that $h\dot{t} = b$. In order to interpret b, return to special relativity in Minkowski space, have $b = \frac{\mathrm{d}t}{\mathrm{d}\tau} = E_\gamma = \text{photon's energy}$. Therefore

$$\boxed{E_\gamma = h\,\dot{t}} \tag{P0}$$

$\mu = 2$, $\mu = 3$: is analogous to the case of a particle.

$\mu = 1$: from the metric find directly the conservation law

$$0 = h\dot{t}^2 - h^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\,\dot{\phi}^2$$

fixing $\theta = \frac{\pi}{2}$ and using the conservation laws (P0), (T3), find $c = 1$

$$0 = h\frac{E_\gamma^2}{h^2} - \frac{1}{h}\dot{r}^2 - r^2\frac{L^2}{r^4} = \frac{E_\gamma^2}{h} - \frac{\dot{r}^2}{h} - \frac{L^2}{r^2}$$

such that now $\dot{r}^2 = E_\gamma^2 - h\frac{L^2}{r^2}$. For a photon, one now interprets $p_\gamma = c\dot{r} = \dot{r}$ as the radial component of the photon's four-momentum p_γ. This gives the energy-momentum relation $p_\gamma^2 = E_\gamma^2 - h\frac{L^2}{r^2}$ which motivates the given definition for photons.]

Special case: the Schwarzschild metric $h(r) = 1 - \frac{\mathscr{R}}{r}$ units $c = 1$

(a): particles of mass m

$$V_{\text{eff}}^2(r) = m^2\left(1 - \frac{\mathscr{R}}{r}\right)\left(1 + \frac{L^2}{r^2}\right) = m^2\left(1 - \frac{\mathscr{R}}{r} + \frac{L^2}{r^2} - \frac{\mathscr{R}L^2}{r^3}\right)$$

Comment: in the non-relativistic limit (here achieved for $r \to \infty$)

$$V_{\text{eff}}(r) = m\sqrt{1 - \frac{\mathscr{R}}{r} + \frac{L^2}{r^2} - \frac{\mathscr{R}L^2}{r^3}} \simeq m\left(1 \underbrace{- \frac{1}{2}\frac{\mathscr{R}}{r} + \frac{1}{2}\frac{L^2 - \mathscr{R}^2/4}{r^2}}_{V_{\text{eff,cl}}(r)} \underbrace{- \frac{1}{4}\frac{L^2 + \mathscr{R}^2/4}{r^3}}_{\text{relat. correct.}} + \dots\right)$$

one has, besides the rest energy mc^2, the classical effective potential (up to a shift in the angular momentum) and further relativistic corrections.

Discussion of the shape of $V^2_{\text{eff}}(r)$: one has $V^2_{\text{eff}}(\mathscr{R}) = 0$ and $\lim_{r\to\infty} V^2_{\text{eff}}(r) = m^2$

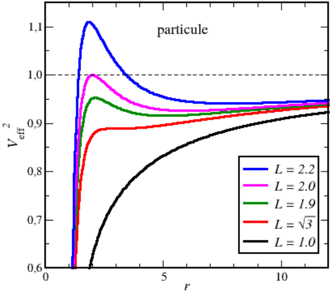
parameters $\mathscr{R}$ and m merely define scales of length and energy, respectively

☞ shape of $V^2_{\text{eff}}(r)$ only determined by angular momentum $L/\mathscr{R}$

extremal points $r_\pm$: maximum at r_-, minimum at r_+, where

$$r_\pm = \frac{L^2}{\mathscr{R}} \pm \frac{L^2}{\mathscr{R}}\sqrt{1 - \frac{3\mathscr{R}^2}{L^2}} \quad \Rightarrow \quad \text{critical value} \quad \boxed{L_c = \sqrt{3}\,\mathscr{R}}$$

in the plot, r, L are in units of $\mathscr{R}$, and V^2_{eff} is in units of m^2



real maximum/minimum for $L > L_c$

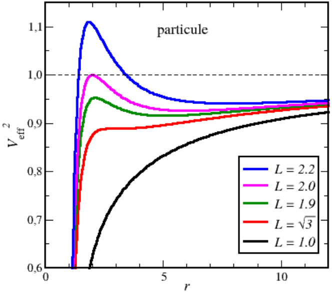
maximum at r_-, minimum at r_+ for $L > L_c$

saddle point for $L = L_c$ at $r = 3\mathscr{R}$

for $L \to \infty$, find $r_- \to \frac{3}{2}\mathscr{R}$

if $L = 2\mathscr{R}$, then $V^2_{\text{eff}}(r_-) = m^2$

☞ physical behaviour depends on two parameters: angular momentum L and energy E



in the plot, r, L are in units of $\mathscr{R}$, and V_{eff}^2 is in units of m^2

real maximum/minimum for $L > L_c = \sqrt{3}\,\mathscr{R}$

maximum at r_-, minimum at r_+ for $L > L_c$

saddle point for $L = L_c$ at $r = 3\mathscr{R}$

for $L \to \infty$, find $r_- \to \frac{3}{2}\mathscr{R}$

if $L = 2\mathscr{R}$, then $V_{\text{eff}}^2(r_-) = m^2$

NR limit: $L_c = \frac{1}{2}\sqrt{4 + \sqrt{21}}\,\mathscr{R} \simeq 1.46\mathscr{R}$, qualitatively similar

* if $L < L_c$, $E^2 < m^2$: confined but unstable orbit $\Rightarrow$ particle falls back into centre
* if $L \leq L_c$, $E^2 \geq m^2$: unbounded motion, $\Rightarrow$ particle escapes/falls into centre
* if $L = L_c = \sqrt{3}\,\mathscr{R}$, $E^2 < m^2$: confined unstable orbit $\Rightarrow$ particle falls back into centre

for $r = 3\mathscr{R}$ there is a marginally unstable orbit

* if $L_c < L < 2\mathscr{R}$ and $V_{\text{eff}}^2(r_+) < E^2 < V_{\text{eff}}^2(r_-) < m^2$: stable bound orbit
* if $L_c < L < 2\mathscr{R}$ and $V_{\text{eff}}^2(r_-) < E^2 < m^2$: confined unstable orbit $\Rightarrow$ particle falls into centre
* if $L_c < L < 2\mathscr{R}$ and $m^2 < E^2$: unbounded motion $\Rightarrow$ particle escapes/falls into centre
* if $L > 2\mathscr{R}$ and $V_{\text{eff}}^2(r_+) < E^2 < V_{\text{eff}}^2(r_-)$: stable bound orbit
* if $L > 2\mathscr{R}$ and $m^2 < V_{\text{eff}}^2(r_-) < E^2$: unbounded motion $\Rightarrow$ particle escapes/falls into centre

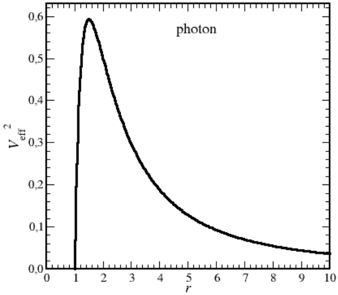
$\Rightarrow$ no stable orbits for finite distance from event horizon $r < \frac{3}{2}\mathscr{R}$

Special case: the Schwarzschild metric $h(r) = 1 - \frac{\mathscr{R}}{r}$ units $c = 1$

(b): photons
$$V^2_{\text{eff}}(r) = \frac{L^2}{r^2}\left(1 - \frac{\mathscr{R}}{r}\right) = \frac{L^2}{r^2} - \frac{L^2\mathscr{R}}{r^3}$$

Discussion of the shape of $V^2_{\text{eff}}(r)$: one has $V^2_{\text{eff}}(\mathscr{R}) = 0$ and $\lim_{r\to\infty} V^2_{\text{eff}}(r) = 0$

in the plot, r, L are in units of $\mathscr{R}$

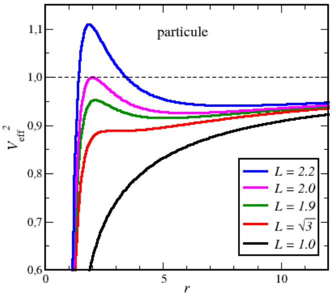


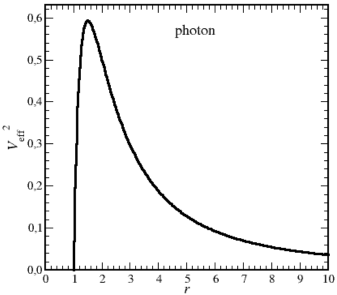
shape of $V^2_{\text{eff}}(r)$ does not depend on L

a single maximum at $r_0 = \frac{3}{2}\mathscr{R}$
gives an unstable circular orbit
$$V_0^2 := V^2_{\text{eff}}(r_0) = \frac{4}{27}\left(\frac{L}{\mathscr{R}}\right)^2$$

* if $E^2 < V_0^2$: unbound orbit, incoming particle reflected at potential barrier
* if $E^2 > V_0^2$: particle falls directly into centre

☞ absence of bound states for photons





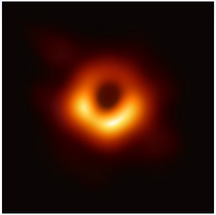
shapes of effective potentials for mouvement around black holes very different for massive particles and photons

$\Rightarrow$ study of effective potential V_{eff}^2 gives useful insight

☞ particle orbits have minimal radius $r_{\min}$ such that no bound stable orbits possible for $r < r_{\min}$ (according to criterion $r_{\min} = \frac{3}{2}\mathscr{R}$ or $3\mathscr{R}$)

☞ no bound orbits at all for photons, but scattering is possible

black holes occur very frequently in Nature: **4 examples**



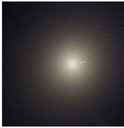
The first direct picture of a black hole in the centre of the galaxy M87, of mass $6.5 \cdot 10^9 M_\odot$

the black disk in the centre has a diameter $\approx 2.5\mathscr{R}$

hot gas emits radiation (the jet !) before falling into the black hole

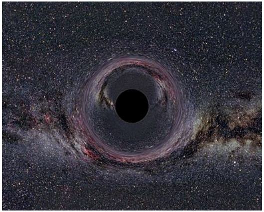
`Event Horizon Telescope` (2019)

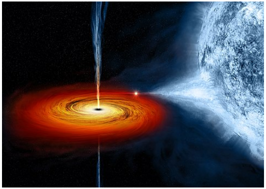
Sources: `https://de.wikipedia.org/wiki/Schwarzes_Loch` `https://de.wikipedia.org/wiki/Messier_87`



Simulation of the distortion of time-space by a non-rotating black hole with mass $M = 10M_\odot$, seen from a distance $r = 600[\mathrm{km}]$ before the background of our own galaxy.

Source: `https://de.wikipedia.org/wiki/Schwarzes_Loch`

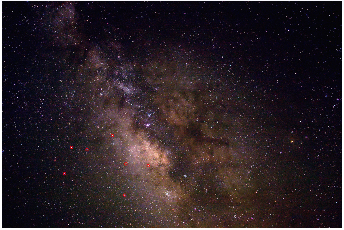


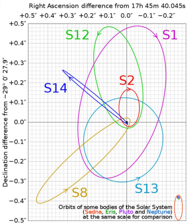


The first ever identified black hole: the stellar system Cyg X-1. The 'companion' is a blue super-giant, of mass $\approx 27M_\odot$, radius $\approx 32R_\odot$ temperature $3 \cdot 10^4[\mathrm{K}]$, luminosity $2 \cdot 10^5 L_\odot$. The black hole has a mass $\approx 16M_\odot$. The period of the system is $5.60[\mathrm{days}]$. Cyg X-1 is the source of intensive X-ray radiation, with X-luminosity $5 \cdot 10^{24}[\mathrm{W}] \sim 10^4 L_{\odot,X}$ (eruptions up to $10^{31}[\mathrm{W}]$.)

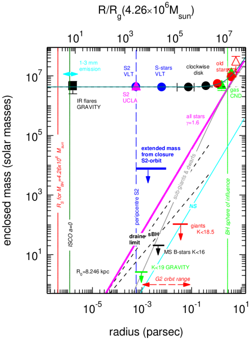
Source: `https://de.wikipedia.org/wiki/Cygnus_X-1`

vues sur le centre de la Voie Lactée, *constellation Sagittaire*, avec son amas central d'étoiles





source: `https://en.wikipedia.org/wiki/Galactic_Center`



tout au centre: source très compacte de rayonnement intense: Sgr A*

les orbites d'étoiles (S2 etc.) donnent la masse de Sgr A*:

$M_\bullet = (4.154 \pm 0.014) \cdot 10^6 M_\odot$ $\qquad \mathscr{R}_\bullet \approx 0.08[\mathrm{UA}]$

images directes préliminaires de la Collaboration GRAVITY:

diamètre disque $2R_{\mathrm{Sgr\,A}^*} \approx (12.3 \pm 4.3)\mathscr{R}$ (mars 2019)

très probable que Sgr A* est un trou noir super-massif

observations aussi fittées par trou noir Schwarzschild, dernier orbite stable $\approx (1.2 \pm 0.3) R_{\min}$

également possible: trou noir en rotation rapide $\Rightarrow$ métrique de Kerr

$1[\mathrm{UA}] = 4.85 \cdot 10^{-6}[\mathrm{pc}]$

source: GRAVITY collab, Genzel *et al.* A&A, **618**, L10 (2018) and **636**, L5 (2020)

5. White dwarfs and neutron stars

5.1 The inner Schwarzschild solution

analyse gravitational & relativistic effects in the presence of sources

good physical example: **stars**

* formed by contraction out of a gas cloud, under the influence of gravitation
* stabilised by internal nuclear fusion, according to $p + p \to d + e^+ + \nu + \gamma$

• stars are stationary ! (indeed, well, most stars are *not* variable)

• stars are spherically symmetric (at least, well, if not rotating very fast)

$\Rightarrow$ use ansatz for metric *à la* Schwarzschild ☞ star's centre at rest

$$\mathrm{d}s^2 = -e^{2\nu(r)} c^2 \mathrm{d}t^2 + e^{2\lambda(r)} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2 \right) \qquad \text{(AM)}$$

in order to describe interior of star, must now solve full field equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^2} T_{\mu\nu}$$

(and also provide link with outer solution, from $R_{\mu\nu} = 0$)

(I) for the geometry:

can recall results from previous calculations of the Ricci tensor rappel: $g_{\mu\nu}$ diagonal

$$\begin{aligned} R^0_0 &= g^{00} R_{00} + \underbrace{g^{0j}}_{=0} R_{j0} \\ &= -e^{-2\nu} R_{00} = e^{-2\lambda}\left(-\nu'' - \nu'^2 + \nu'\lambda' - \frac{2}{r}\nu'\right) \\ R^1_1 &= g^{11} R_{11} = e^{-2\lambda}\left(\nu'' - \nu'^2 + \nu'\lambda' + \frac{2}{r}\lambda'\right) \\ R^2_2 &= R^3_3 = e^{-2\lambda}\left(-\frac{1}{r^2} - \frac{\nu'}{r} + \frac{\lambda'}{r}\right) + \frac{1}{r^2} \end{aligned}$$

and all other $R^\mu_\nu = 0$. $\nu'(r) = \frac{\mathrm{d}\nu(r)}{\mathrm{d}r}$ etc.

finally, find the Ricci scalar

$$\begin{aligned} R &= R^\mu_\mu = R^0_0 + R^1_1 + R^2_2 + R^3_3 \\ &= e^{-2\lambda}\left(-2\nu'' - 2\nu'^2 + 2\nu'\lambda' - \frac{4}{r}(\nu' - \lambda') - \frac{2}{r^2}\right) + \frac{2}{r^2} \end{aligned}$$

(II) for the matter:

good choice for interior of stars: **perfect fluid** with density ρ and pressure p

(1) both viscosity and thermal conduction are disregarded $\Rightarrow$ mouvement of fluid is adiabatic

(2) stars are hot gas balls under enormous pressure, deep in coexistence regime of liquids and gases

the energy-momentum tensor of a perfect fluid reads (*a priori* position-dependent)

$$\boxed{T^{\mu\nu} = \frac{p}{c^2} g^{\mu\nu} + \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu} \qquad (*)$$

with $\mathbf{u}$: four-velocity of small volume of fluid

○ (*) is generally co-variant; at rest & in cartesian coordinates, reduces to

$$T^{\mu\nu}_{(\mathrm{cart})} = \mathrm{diag}\,\left(\rho, \frac{p}{c^2}, \frac{p}{c^2}, \frac{p}{c^2}\right)$$

○ rest frame correct choice for star, since centre of gravitation at rest as well

○ but need $T_{\mu\nu}$ in spherical coordinates $\qquad \mathbf{x} = (ct, r, \theta, \phi)$

$\Rightarrow$ from the ansatz (AM) for the metric, in spherical coordinates (*) leads to

$$T^{\mu\nu}_{(\mathrm{spher})} = \begin{pmatrix} \rho e^{-2\nu} & & & \\ & \frac{p}{c^2} e^{-2\lambda} & & \\ & & \frac{p}{c^2}\frac{1}{r^2} & \\ & & & \frac{p}{c^2}\frac{1}{r^2 \sin^2\theta} \end{pmatrix}$$

recast field equations: $R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}R = \frac{8\pi G}{c^2} T^{\mu}{}_{\nu}$.
require, again in spherical coordinates, for a perfect fluid

$$T^{\mu}{}_{\nu} = \begin{pmatrix} -\rho & & & \\ & \frac{p}{c^2} & & \\ & & \frac{p}{c^2} & \\ & & & \frac{p}{c^2} \end{pmatrix}$$

Then, the EINSTEIN field equations read, with $G^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}R$

$$G^{0}{}_{0} = -e^{-2\lambda}\left(\frac{2\lambda'}{r} - \frac{1}{r^2}\right) - \frac{1}{r^2} \overset{!}{=} -\frac{8\pi G}{c^2}\rho = \frac{8\pi G}{c^2} T^{0}{}_{0} \quad \text{(a)}$$

$$G^{1}{}_{1} = e^{-2\lambda}\left(\frac{2\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} \overset{!}{=} \frac{8\pi G}{c^2}\frac{p}{c^2} = \frac{8\pi G}{c^2} T^{1}{}_{1} \quad \text{(b)}$$

$$G^{2}{}_{2} = -e^{-2\lambda}\left(\nu'' + \nu'^2 - \nu'\lambda' + \frac{\nu'}{r} - \frac{\lambda'}{r}\right) \overset{!}{=} \frac{8\pi G}{c^2}\frac{p}{c^2} = \frac{8\pi G}{c^2} T^{2}{}_{2} \quad \text{(c)}$$

equations for $\mu = 2$ and $\mu = 3$ are the same, all others are trivial

- This must be completed by an **equation of state** $p = p(\rho)$.

◦ For stars, a good choice is a **polytrope equation**

$$\frac{p}{c^2} = K\rho^{\gamma} \quad \text{(p)}$$

where γ is the *polytrope exponent*.

Eqs. (a,b,c,p) are the complete system to find the metric.

$$e^{-2\lambda}\left(\frac{2\lambda'}{r}-\frac{1}{r^2}\right)+\frac{1}{r^2} = \frac{8\pi G}{c^2}\rho \qquad \text{(a)}$$

$$e^{-2\lambda}\left(\frac{2\nu'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2} = \frac{8\pi G}{c^2}\frac{p}{c^2} \qquad \text{(b)}$$

$$-e^{-2\lambda}\left(\nu''+\nu'^2-\nu'\lambda'+\frac{\nu'}{r}-\frac{\lambda'}{r}\right) = \frac{8\pi G}{c^2}\frac{p}{c^2} \qquad \text{(c)}$$

$$\frac{p}{c^2}=K\rho^\gamma \qquad \text{(p)}$$

☞look for the two functions $\lambda=\lambda(r)$, $\nu=\nu(r)$

to simplify the integration: assume $\rho=\mathrm{cste.}$ – result will be valid more generally

$$(a) \;\Rightarrow\; \frac{\mathrm{d}}{\mathrm{d}r}\left(re^{-2\lambda}\right)=1-\frac{8\pi G}{c^2}\rho r^2$$

$$\Rightarrow\; e^{-2\lambda}\overset{\rho=\mathrm{cste}}{=}1-\frac{8\pi G}{3c^2}\rho r^2+\frac{C}{r}=:1-Ar^2 \quad ; \quad \boxed{A=\frac{8\pi G}{3c^2}\rho}$$

N.B.: no singularity at $r=0$ admissible $\Rightarrow$ fix $C=0$.

Eqs. (a,b,c,p) are the complete system to find the metric.

$$e^{-2\lambda}\left(\frac{2\lambda'}{r}-\frac{1}{r^2}\right)-\frac{1}{r^2} = \frac{8\pi G}{c^2}\rho \qquad \text{(a)}$$

$$e^{-2\lambda}\left(\frac{2\nu'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2} = \frac{8\pi G}{c^2}\frac{p}{c^2} \qquad \text{(b)}$$

$$-e^{-2\lambda}\left(\nu''+\nu'^2-\nu'\lambda'+\frac{\nu'}{r}-\frac{\lambda'}{r}\right) = \frac{8\pi G}{c^2}\frac{p}{c^2} \qquad \text{(c)}$$

$$\frac{p}{c^2}=K\rho^\gamma \qquad \text{(p)}$$

☞still look for the function $\nu=\nu(r)$, with assumption $\rho=\mathrm{cste.}$

Derive (b) with respect to r and insert ν'' from (c), to find

$$\frac{8\pi G}{c^2}\frac{p'}{c^2}=-\frac{2e^{-2\lambda}}{r}\nu'\left(\nu'+\lambda'\right)$$

adding eqs. (a) and (b) gives: $\frac{2e^{-2\lambda}}{r}\left(\nu'+\lambda'\right)=\frac{8\pi G}{c^2}\left(\rho+\frac{p}{c^2}\right)$

Eqs. (a,b,c,p) are the complete system to find the metric.

$$e^{-2\lambda}\left(\frac{2\lambda'}{r}-\frac{1}{r^2}\right)-\frac{1}{r^2} = \frac{8\pi G}{c^2}\rho \qquad \text{(a)}$$

$$e^{-2\lambda}\left(\frac{2\nu'}{r}+\frac{1}{r^2}\right)-\frac{1}{r^2} = \frac{8\pi G}{c^2}\frac{p}{c^2} \qquad \text{(b)}$$

$$-e^{-2\lambda}\left(\nu''+\nu'^2-\nu'\lambda'+\frac{\nu'}{r}-\frac{\lambda'}{r}\right) = \frac{8\pi G}{c^2}\frac{p}{c^2} \qquad \text{(c)}$$

$$\frac{p}{c^2}=K\rho^\gamma \qquad \text{(p)}$$

☞still look for the function $\nu=\nu(r)$, with assumption $\rho=\mathrm{cste}$.

Derive (b) with respect to r and insert ν'' from (c), to find

$$\frac{8\pi G}{c^2}\frac{p'}{c^2}=(-\nu')\cdot\frac{2e^{-2\lambda}}{r}(\nu'+\lambda')$$

adding eqs. (a) and (b) gives: $\frac{2e^{-2\lambda}}{r}(\nu'+\lambda')=\frac{8\pi G}{c^2}\left(\rho+\frac{p}{c^2}\right)$

$$\Rightarrow \left(\rho+\frac{p}{c^2}\right)'=\frac{p'}{c^2}=-\nu'\left(\rho+\frac{p}{c^2}\right) \overset{\rho=\mathrm{cste}}{\Rightarrow} \boxed{\rho+\frac{p}{c^2}=De^{-\nu(r)}} \qquad (*)$$

where D is a constant

so far have found: $\rho + \frac{p}{c^2} = De^{-\nu}$ and $e^{-2\lambda} = 1 - Ar^2$. Furthermore

$$\frac{2e^{-2\lambda}}{r}\left(\nu' + \lambda'\right) = \frac{8\pi G}{c^2}\left(\rho + \frac{p}{c^2}\right)$$

so far have found: $\rho + \frac{p}{c^2} = De^{-\nu}$ and $e^{-2\lambda} = 1 - Ar^2$. Furthermore

$$\frac{2e^{-2\lambda}}{r}\left(\nu' + \lambda'\right) = \frac{8\pi G}{c^2}\left(\rho + \frac{p}{c^2}\right)$$

so far have found: $\rho + \frac{p}{c^2} = De^{-\nu}$ and $e^{-2\lambda} = 1 - Ar^2$. Furthermore

$$\frac{2}{r}\left(\nu' e^{-2\lambda} + \lambda' e^{-2\lambda}\right) = \frac{2e^{-2\lambda}}{r}\left(\nu' + \lambda'\right) = \frac{8\pi G}{c^2} De^{-\nu}$$

so far have found: $\rho + \frac{p}{c^2} = De^{-\nu}$ and $e^{-2\lambda} = 1 - Ar^2$. Furthermore

$$\frac{2}{r}\left(\nu'\left(1 - Ar^2\right) + \frac{\mathrm{d}}{\mathrm{d}r}\left(-\tfrac{1}{2}e^{-2\lambda}\right)\right) = \frac{8\pi G}{c^2}De^{-\nu}$$

so far have found: $\rho + \frac{p}{c^2} = De^{-\nu}$ and $e^{-2\lambda} = 1 - Ar^2$. Furthermore

$$\frac{2}{r}\left(\nu'\left(1 - Ar^2\right) + \frac{\mathrm{d}}{\mathrm{d}r}\left(-\tfrac{1}{2}e^{-2\lambda}\right)\right) = \frac{8\pi G}{c^2}De^{-\nu}$$

so far have found: $\rho + \frac{p}{c^2} = De^{-\nu}$ and $e^{-2\lambda} = 1 - Ar^2$. Furthermore

$$\frac{2}{r}\left(\nu'\left(1 - Ar^2\right) + \left(-\frac{1}{2}\left(-2Ar\right)\right)\right) = \frac{8\pi G}{c^2} De^{-\nu}$$

so far have found: $\rho + \frac{p}{c^2} = De^{-\nu}$ and $e^{-2\lambda} = 1 - Ar^2$. Furthermore

$$\frac{2}{r}\left(\nu'\left(1 - Ar^2\right) + Ar\right) = \frac{8\pi G}{c^2} De^{-\nu}$$

so far have found: $\rho + \frac{p}{c^2} = De^{-\nu}$ and $e^{-2\lambda} = 1 - Ar^2$. Furthermore

$$2e^{\nu}\left(\nu'\left(1 - Ar^2\right) + Ar\right) = \frac{8\pi G}{c^2} D r$$

so far have found: $\rho + \frac{p}{c^2} = De^{-\nu}$ and $e^{-2\lambda} = 1 - Ar^2$. Furthermore

$$2e^{\nu}\left(\nu'\left(1 - Ar^2\right) + Ar\right) = \frac{8\pi G}{c^2} Dr$$

or equivalently

$$2\nu' e^{\nu}\left(1 - Ar^2\right) + 2Ar\, e^{\nu} = \frac{8\pi G}{c^2} Dr$$

which is a linear differential equation in $\gamma(r) := e^{\nu(r)}$, $\gamma' = \nu' e^{\nu}$

$$\Rightarrow \quad e^{\nu(r)} = \gamma(r) = \frac{4\pi G}{c^2}\frac{D}{A} - B\left(1 - Ar^2\right)^{1/2}$$

Still must fix constants B, D. Specified by physical boundary conditions ☜

(i) vanishing pressure at the stellar border, $p(R) \stackrel{!}{=} 0$ R: stellar radius

$$(*) \Rightarrow \rho = De^{-\nu(R)} \quad \Rightarrow \quad D = \rho e^{\nu(R)} = 2\rho B\left(1 - AR^2\right)^{1/2}$$

$$e^{\nu(r)} = B\left[3\left(1 - AR^2\right)^{1/2} - \left(1 - Ar^2\right)^{1/2}\right]$$

so far have found: $\rho + \frac{p}{c^2} = De^{-\nu}$ and $e^{-2\lambda} = 1 - Ar^2$. Furthermore

$$2e^{\nu}\left(\nu'\left(1 - Ar^2\right) + Ar\right) = \frac{8\pi G}{c^2} D\, r$$

or equivalently

$$\gamma'(r)\left(1 - Ar^2\right) + Ar\,\gamma(r) = \frac{4\pi G}{c^2} D\, r$$

a linear inhomogeneous differential equation in $\gamma(r) := e^{\nu(r)}$, $\qquad \gamma' = \nu' e^{\nu}$

$$\Rightarrow \quad e^{\nu(r)} = \gamma(r) = \frac{4\pi G}{c^2}\frac{D}{A} - B\left(1 - Ar^2\right)^{1/2} \qquad A = \frac{8\pi G}{3c^2}\rho$$

Still must fix constants B, D. Specified by physical boundary conditions ☜

(i) vanishing pressure at the stellar border, $p(R) \stackrel{!}{=} 0$ $\qquad$ R: stellar radius

$$(*) \Rightarrow \rho = De^{-\nu(R)} \quad \Rightarrow \quad D = \rho e^{\nu(R)} = 2\rho B\left(1 - AR^2\right)^{1/2}$$

$$e^{\nu(r)} = B\left[3\left(1 - AR^2\right)^{1/2} - \left(1 - Ar^2\right)^{1/2}\right]$$

(ii) *at stellar border $r = R$, the metric should meet continuously the outer Schwarzschild solution* (SO): we had

$$-g_{00}^{(\mathrm{SO})}(r) = e^{2\nu_{\mathrm{SO}}(r)} = 1 - \frac{\mathscr{R}}{r} \quad , \quad g_{11}^{(\mathrm{SO})}(r) = e^{2\lambda_{\mathrm{SO}}(r)} = \left(1 - \frac{\mathscr{R}}{r}\right)^{-1}$$

- gives first matching condition

$$\left(1 - AR^2\right)^{-1} = e^{2\lambda(R)} \overset{!}{=} e^{2\lambda_{\mathrm{SO}}(R)} = \left(1 - \frac{\mathscr{R}}{R}\right)^{-1}$$

$\Rightarrow AR^2 \overset{!}{=} \frac{\mathscr{R}}{R}$ which reduces to the standard relation for the radius $R = \left(\frac{3}{4\pi}\frac{M}{\rho}\right)^{\frac{1}{3}}$

- the second matching condition $e^{2\nu(R)} \overset{!}{=} e^{2\nu_{\mathrm{SO}}(R)}$ leads to $B = \frac{1}{2}$.

Final result: the (inner) **Schwarzschild solution**, with $A = \frac{8\pi G}{3c^2}\rho$

$$\mathrm{d}s^2 = -\left[\frac{3}{2}\left(1 - AR^2\right)^{1/2} - \frac{1}{2}\left(1 - Ar^2\right)^{1/2}\right] c^2 \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - Ar^2} + r^2 \mathrm{d}\Omega^2$$

describes the interior of a spherical star, with constant density ρ

Vorlesung X

Rappel: mouvement in the gravitational field of a black hole

use a slight generalisation of the Schwarzschild metric

$$\mathrm{d}s^2 = -h(r)c^2\mathrm{d}t^2 + \frac{1}{h(r)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)$$

Qualitative studies via effective potentials:

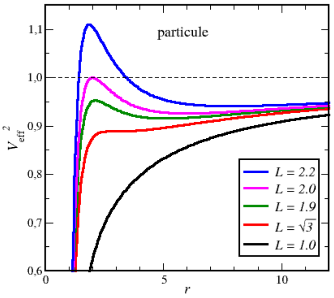
(a) for massive particles have

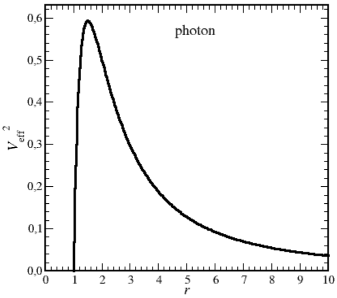
$$\boxed{E^2 = p^2 + V_{\rm eff}^2(r) \;\;,\;\; \text{with} \quad V_{\rm eff}^2(r) := m^2 h(r)\left(1 + \frac{L^2}{r^2}\right)}$$

(b) for photons (or massless particles) have

$$\boxed{E_\gamma^2 = p_\gamma^2 + V_{\rm eff}^2(r) \;\;,\;\; \text{with} \quad V_{\rm eff}^2(r) := h(r)\frac{L^2}{r^2}}$$

⋆ Schwarzschild metric: $h(r) = 1 - \frac{\mathscr{R}}{r}$





shapes of effective potentials for mouvement around black holes very different for massive particles and photons

$\Rightarrow$ study of effective potential $V_{\rm eff}^2$ gives useful insight

☞ particle orbits have minimal radius $r_{\rm min}$ such that no bound stable orbits possible for $r < r_{\rm min}$ (according to criterion $r_{\rm min} = \frac{3}{2}\mathscr{R}$ or $3\mathscr{R}$)

☞ no bound orbits at all for photons, but scattering is possible

Rappel: solve full field equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^2} T_{\mu\nu}$,

with $T^{\mu\nu} = \frac{p}{c^2} g^{\mu\nu} + \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu$ for a perfect fluid, with density ρ, pressure p

- interior of spherical star (with $\rho = \mathrm{cste}$) given by inner Schwarzschild metric

$$\mathrm{d}s^2 = -\left[\frac{3}{2}\left(1 - AR^2\right)^{1/2} - \frac{1}{2}\left(1 - Ar^2\right)^{1/2}\right] c^2 \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - Ar^2} + r^2 \mathrm{d}\Omega^2$$

with $A = \frac{8\pi G}{3c^2}\rho$.

only requirements: stationary solution, spherical symmetry.

also required continuity with outer Schwarzschild metric at stellar radius R

please do **not** confuse with Ricci scalar !

This was derived from the field equations of a stationary spherically symmetric field

$$\mathrm{d}s^2 = -c^2 e^{2\nu(r)}\mathrm{d}t^2 + e^{2\lambda(r)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)$$

with a perfect fluid as source

$$\begin{aligned}
e^{-2\lambda}\left(\frac{2\lambda'}{r} - \frac{1}{r^2}\right) - \frac{1}{r^2} &= \frac{8\pi G}{c^2}\rho && \text{(a)}\\
e^{-2\lambda}\left(\frac{2\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} &= \frac{8\pi G}{c^2}\frac{p}{c^2} && \text{(b)}\\
-e^{-2\lambda}\left(\nu'' + \nu'^2 - \nu'\lambda' + \frac{\nu'}{r} - \frac{\lambda'}{r}\right) &= \frac{8\pi G}{c^2}\frac{p}{c^2} && \text{(c)}
\end{aligned}$$

and the equation of state of the perfect fluid is taken as a polytrope

$$\frac{p}{c^2} = K\rho^\gamma \qquad \text{(p)}$$

☞ Eqs. (a,b,c,p) are the complete system to find the metric.

5.2 Tolman-Oppenheimer-Volkoff equation

next step: obtain an equation of state for the star equilibrium, at rest

the equation of state $p = p(\rho)$ relates density and pressure at a single space point, must understand the radial dependence $p = p(r)$ in the stellar interior

☞ reconsider the relationship between ρ and p.

starting point:

conservation law for the energy-momentum tensor of a perfect fluid at rest

$$0 = T^{\mu\nu}{}_{;\nu} = \left[\frac{p}{c^2}g^{\mu\nu} + \left(\rho + \frac{p}{c^2}\right)u^\mu u^\nu\right]_{;\nu} = \frac{1}{c^2}\frac{\partial p}{\partial x^\nu}g^{\mu\nu} + \left[\left(\rho + \frac{p}{c^2}\right)u^\mu u^\nu\right]_{;\nu}$$

for any tensor of level $\binom{2}{0}$ one has $S^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{-g}}\partial_\alpha\left(\sqrt{-g}\,S^{\mu\alpha}\right) + \Gamma^\mu_{\alpha\nu}S^{\alpha\nu}$

$$\begin{aligned} 0 \quad = \quad & T^{\mu\nu}{}_{;\nu} = \frac{1}{c^2}\frac{\partial p}{\partial x^\nu}g^{\mu\nu} + \underbrace{\frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}\,\left(\rho + \frac{p}{c^2}\right)u^\mu u^\nu\right)} \\ & + \Gamma^\mu_{\nu\alpha}\left(\rho + \frac{p}{c^2}\right)u^\alpha u^\nu \end{aligned}$$

at rest, underlined term vanishes for $\nu \neq 0$ since $u^\nu = 0$ if $\nu \neq 0$. The derivative $\partial_0(\cdots) = 0$ since star at equilibrium. Hence, the underlined term also vanishes for $\nu = 0$.

Again, **at rest** have the relation $\frac{1}{c^2}\frac{\partial p}{\partial x^\alpha} = -g_{\mu\alpha}\Gamma^\mu_{00}\left(\rho + \frac{p}{c^2}\right)\left(u^0\right)^2$

Since $g_{\mu\alpha}\Gamma^\mu_{00}\left(u^0\right)^2 = -\frac{1}{2}g_{\mu\alpha}g^{\mu\sigma}g_{00,\sigma}\left(g_{00}\right)^{-1} = \partial_\alpha\left(\ln\sqrt{-g_{00}}\,\right) = \partial_\alpha \ln\sqrt{e^{2\nu(r)}} = \nu'(r)\delta_{\alpha,1}$,

for $\alpha = 1$ this leads to

$$\frac{\partial p}{\partial r} = -\frac{1}{c^2}\left(\rho c^2 + p\right)\nu' \qquad \text{(t)}$$

Next, take the difference of (a) and (b)

$$e^{-2\lambda}\left(\frac{2\lambda'}{r} - \frac{2\nu'}{r} - \frac{2}{r^2}\right) + \frac{2}{r^2} = \frac{8\pi G}{c^2}\left(\rho - \frac{p}{c^2}\right)$$

$$\Rightarrow \quad 1 - e^{-2\lambda}\left(1 + r\nu'\right)) + r\lambda' e^{-2\lambda} = \frac{4\pi G}{c^2}r^2\left(\rho - \frac{p}{c^2}\right) \qquad (\#)$$

Also, re-write (a) in the form & use inner Schwarzschild metric

$$e^{-2\lambda}\frac{\lambda'}{r} = \frac{1}{2}\left[\frac{e^{-2\lambda}}{r^2} - \frac{1}{r^2} + \frac{8\pi G}{c^2}\rho\right] \quad \Rightarrow \quad r\lambda' e^{-2\lambda} = \frac{4\pi G}{c^2}\rho r^2 - \frac{1}{2}\frac{\mathscr{R}}{r}$$

Insert this into (#) and also (t). This gives

$$1 - \left(1 - \frac{\mathscr{R}}{r}\right)\left(1 - \frac{rp'}{p + \rho c^2}\right) + \frac{4\pi G}{c^2}\rho r^2 - \frac{1}{2}\frac{\mathscr{R}}{r} = \frac{4\pi G}{c^2}r^2\left(\rho - \frac{p}{c^2}\right)$$

which simplifies into the **Tolman-Oppenheimer-Volkoff equation** $\rho = \mathrm{cste}$

$$\boxed{\frac{\mathrm{d}p}{\mathrm{d}r} = -4\pi G\frac{\left(\rho + \frac{p}{c^2}\right)\left(\frac{\rho}{3} + \frac{p}{c^2}\right)r^2}{r - \mathscr{R}}}$$

Discussion: Tolman-Oppenheimer-Volkoff equation, for $\rho = \mathrm{cste}$

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -4\pi G \frac{\left(\rho + \frac{p}{c^2}\right)\left(\frac{\rho}{3} + \frac{p}{c^2}\right) r^2}{r - \mathscr{R}}$$

to be completed by **equation of state** $p(r) = p(\rho(r))$

difficult non-linear differential equation for pressure profile $p = p(r)$

(i) in the newtonian limit ($c \to \infty$ and $r \gg \mathscr{R}$)

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{4\pi G}{3} \rho^2 \, r$$

which is the standard fundamental equation of newtonian hydrostatics

(ii) obviously, increasing the density ρ leads to an increased pressure gradient, directed towards the centre ⇒ contributes to collapse.

remarkably, *increasing the pressure p has the* same *effect* !

☞ qualitatively different from newtonian hydrodynamics.

Tolman-Oppenheimer-Volkoff equation for space-dependent density $\rho = \rho(r)$

$$r^2 \frac{\mathrm{d}p}{\mathrm{d}r} = -G\mathscr{M}(r)\rho(r)\left(1+\frac{p(r)}{\rho(r)}\right)\left(1+\frac{4\pi r^3 p(r)}{\mathscr{M}(r)}\right)\left(1+\frac{2G\mathscr{M}(r)}{r}\right)^{-1}$$

where $\mathscr{M}(r) := \int_0^r \mathrm{d}r'\, 4\pi r'^2 \rho(r')$. $c = 1$

This must be completed by an *equation of state* $p(r) = p\big(\rho(r)\big)$.

This is a system of equations for a star *at* equilibrium.

? What about the stability of the solutions ?

Theorem 1: *A star made from a perfect fluid of constant chemical* composition and with entropy/nucleon s constant, goes from stability under *a radial perturbation $\delta\rho = \delta\rho(t,r)$ to instability at a point of the central density $\rho(0)$ where*

$$\frac{\partial U\big(\rho(0), s, \ldots\big)}{\partial\rho(0)} = 0 \ , \quad \frac{\partial N\big(\rho(0), s, \ldots\big)}{\partial\rho(0)} = 0$$

where U is the equilibrium energy and N is the number of nucleons.

shows where a transition from stability to instability can occur at all

Theorem 2: *A star, with constant entropy/nucleon s and constant chemical composition, satisfies the Tolman-Oppenheimer-Volkoff (TOV) equations if and only if the total stellar mass*

$$M := \mathscr{M}(\infty) = \int_0^\infty \mathrm{d}r \, 4\pi r^2 \, \rho(r)$$

$\rho(r) = 0$ for $r > R$

is stationnary under all radial variations of $\rho(r)$ which conserve the total baryon number

$n(r)$ is the baryon number density

$$N = \int_0^\infty \mathrm{d}r \, 4\pi r^2 \, n(r) \underbrace{\left(1 - \frac{2G\mathscr{M}(r)}{r}\right)^{-1/2}}$$

This equilibrium is stable if and only if M (or the total energy U) is minimal with respect to such variations.

gives a variational characterisation (with a constraint) of stellar equilibrium

furnishes a clear illustration of the physical conditions for the validity of TOV

underlined term comes from gravitational length contraction (☞ inner Schwarzschild metric)

N.B.: nothing is said yet on the mechanisms the star uses to equilibrate

Proof: use a Lagrange multiplicator λ to write the constrained variation as

$$\delta M - \lambda \delta N = 0$$

Explicitly, this is spelled out as follows

$$\begin{aligned}\delta M - \lambda\delta N &= \int_0^\infty \mathrm{d}r\, 4\pi r^2\, \delta\rho(r) - \lambda \int_0^\infty \mathrm{d}r\, 4\pi r^2 \left(1 - \frac{2G\mathscr{M}(r)}{r}\right)^{-1/2} \delta n(r) \\ &\quad -\lambda G \int_0^\infty \mathrm{d}r\, 4\pi r \left(1 - \frac{2G\mathscr{M}(r)}{r}\right)^{-3/2} n(r)\delta\mathscr{M}(r)\end{aligned}$$

Rappel: **[** thermodynamics $\mathrm{d}U + p\mathrm{d}V = \frac{1}{T}\mathrm{d}S$

U: internal energy p: pressure
V: volume S: entropy

internal energy of a star $U = M - m_N N$

m_N: mass of a nucleon

for the densities this reads $u(r) = \rho(r) - m_N n(r)$. Thermodynamics for the densities

$$\boxed{\frac{1}{T}\mathrm{d}s = \mathrm{d}\left(\frac{\rho}{n} - m_N\right) + p\mathrm{d}\left(\frac{1}{n}\right)}$$

]

For an isentropic variation $0 \stackrel{!}{=} \delta s = \mathrm{d}\left(\frac{\rho}{n}\right) + p\mathrm{d}\left(\frac{1}{n}\right) \;\Rightarrow\; \delta n(r) = \frac{n(r)}{p(r)+\rho(r)}\delta\rho(r)$

in addition: $\delta\mathscr{M}(r) = \int_0^r \mathrm{d}r'\, 4\pi r'^2\, \delta\rho(r')$. Insertion gives

$$\begin{aligned} \delta M - \lambda\delta N &= \int_0^\infty \mathrm{d}r\, 4\pi r^2 \left\{ 1 - \frac{\lambda n(r)}{p(r)+\rho(r)} \left(1 - \frac{2G\mathscr{M}(r)}{r}\right)^{-1/2} \right. \\ &\qquad \left. -\lambda G \int_0^\infty \mathrm{d}r'\, 4\pi r' \left(1 - \frac{2G\mathscr{M}(r')}{r'}\right)^{-3/2} n(r') \right\} \delta\rho(r) \end{aligned}$$

This variation is stationary, if $\{\cdots\} = 0$. This implies a certain equation for the (constant) Lagrange multiplier λ. It follows that $\{\cdots\}$ must be independent of r, or $\partial_r\{\cdots\} = 0$. This gives

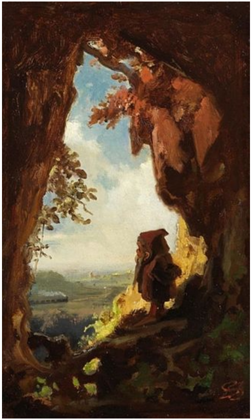
$$\begin{aligned} 0 &= \left(\frac{n'}{p+\rho} - \frac{n(p'+\rho')}{(p+\rho)^2}\right)\left(1 - \frac{2G\mathscr{M}(r)}{r}\right)^{-1/2} \\ &\quad + \frac{Gn}{p+\rho}\left(4\pi r\rho - \frac{\mathscr{M}}{r^2}\right)\left(1 - \frac{2G\mathscr{M}(r)}{r}\right)^{-3/2} - 4\pi Gn\left(1 - \frac{2G\mathscr{M}(r)}{r}\right)^{-3/2} \end{aligned}$$

recall that isentropy lead to $n'(r) = \frac{n(r)\rho'(r)}{p(r)+\rho(r)}$. Insertion into above relation produces

$$-r^2 p' = G\left(1 - \frac{2G\mathscr{M}(r)}{r}\right)^{-1/2} (p+\rho)(\mathscr{M} + 4\pi r^3\rho)$$

which is equivalent to TOV. QED

5.3 Polytropes and white dwarfs



Klarstellung: Ein Astrophysiker denkt bei *'weißen Zwergen'* keinesfalls an Grimm'sche Märchen, oder an alte Wichte . . . Schade !

Bildquellen: `https://de.wikipedia.org/wiki/Schneewittchen`, https://de.wikipedia.org/wiki/Zwerg_(Mythologie)

5.3 Polytropes and white dwarfs

the TOV equation should describe the equilibrium state of a star,
including relativistic corrections

$$r^2\frac{\mathrm{d}p}{\mathrm{d}r} = -G\mathscr{M}(r)\rho(r)\left(1+\frac{p(r)}{\rho(r)}\right)\left(1+\frac{4\pi r^3 p(r)}{\mathscr{M}(r)}\right)\left(1+\frac{2G\mathscr{M}(r)}{r}\right)^{-1}, \quad \mathscr{M}(r) = \int_0^r \mathrm{d}r'\, 4\pi r'^2\rho(r')$$

• for simplicity, begin with non-relativistic stars: that is $u \ll m_N n$ and $p \ll m_N n$ such that $\rho \simeq m_N n$, hence $p \ll \rho \quad \Rightarrow \quad 4\pi r^3 p \ll M$ and $\frac{2G\mathscr{M}}{r} \ll 1$
then the TOV reduces to $-r^2 p'(r) = G\mathscr{M}(r)\rho(r)$ or equivalently

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho(r)}\frac{\mathrm{d}p(r)}{\mathrm{d}r}\right) = -4\pi G r^2 \rho(r)}$$

which is the constitutive newtonian hydrostatic equation for a gas ball.
Should be integrated with initial conditions (here for centre of star)

$$\rho(0) = \rho_0 = \mathrm{cste.} \quad , \quad \rho'(0) = 0$$

N.B.: if $\rho'(0) \neq 0$, it follows from the gas ball equation that $p'(0) = 0$

☞ must still provide the equation of state $p(r) = p\big(\rho(r)\big)$.

Equation of state: *polytrope*

$$\boxed{u = \rho - m_N n \overset{!}{=} \frac{1}{\gamma - 1} p} \quad , \quad \gamma\text{: polytrope exponent}$$

stars considered to be isentropic ($s = \text{cste}$). Therefore

$$0 = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\rho}{n}\right) + p\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{n}\right) = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u}{n}\right) + p\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{n}\right) = \frac{1}{\gamma - 1}\left(\gamma p \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{n}\right) + \frac{1}{n}\frac{\mathrm{d}p}{\mathrm{d}r}\right)$$

solving this differential equation gives $p \sim \left(1/n\right)^{-\gamma}$ and since $\rho \simeq m_N n$

$$\text{polytrope equation} \qquad \boxed{p = K\rho^{\gamma}} \qquad\qquad K = \text{cste.}$$

For a newtonian star, the total mass is dominated by rest mass M, such that $N = \frac{M}{m_N}$.

γ	object	physical model
1	self-gravitating gas sphere	ideal gas
6/5	star	SCHUSTER's exact solution
4/3	sun	EDDINGTON's model
5/3	convective stars globular clusters	convective ideal gas

! integration of hydrostatic equation hard ⇒ turn to equivalent variational problem !

☞ minimise internal energy functional of the star $U = U[\rho] = \big(T + V\big)[\rho]$

$$T = \int_0^R \mathrm{d}r\, 4\pi r^2\, u(r) \quad , \quad \text{thermal energy}$$

$$V = -4\pi G \int_0^R \mathrm{d}r\; r\, \mathscr{M}(r)\rho(r) \quad , \quad \text{gravitational energy}$$

? which polytropic stars are stable ?

for illustration: uniform implosion of stellar matter, with density $\rho = \mathrm{cste.}$

mass $\quad M = \int_0^R \mathrm{d}r\, 4\pi r^2\, \rho = \frac{4\pi}{3}\rho R^3 \simeq N m_N$

thermal $\quad T = \int_0^R \mathrm{d}r\, 4\pi r^2\, u(r) = \frac{4\pi}{\gamma - 1} K \rho^\gamma \int_0^R \mathrm{d}r\; r^2 = \frac{4\pi}{3}\frac{K}{\gamma - 1}\rho^\gamma R^3$

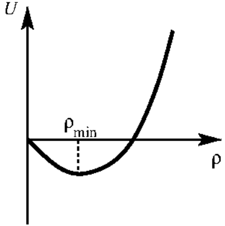
gravit. $\quad V = -4\pi G \int_0^R \mathrm{d}r\; r \left(\int_0^R \mathrm{d}r'\, 4\pi r'^2\, \rho\right)\rho = -\frac{16\pi^2}{15} G \rho^2 R^5$

one scales out the mass M and finds

$$U = T + V = a\rho^{\gamma-1} - b\rho^{1/3} \quad , \quad \text{where } a = \frac{KM}{\gamma-1},\ b = \frac{3}{5}\left(\frac{4\pi}{3}\right)^{1/5} GM^{5/3}$$

for $\gamma > \frac{4}{3}$, $U = U(\rho)$ has minimum at

$$\rho_{\min} = \left(\frac{b}{3a(\gamma-1)}\right)^{1/(\gamma-4/3)}$$



valid within $10 - 20\%$ for $\gamma \simeq \frac{4}{3} - \frac{5}{3}$

leads to **mass-density scaling relation**

$$\boxed{M \simeq \frac{4\pi}{3}\left(\frac{15K}{4\pi G}\right)^{3/2} \rho^{(3\gamma-4)/2}} \quad , \quad \text{if } \gamma > \frac{4}{3}$$

N.B.: coefficient only contains global constants ⇒ universality

☞ *a polytropic and isentropic star is stable for $\gamma > \frac{4}{3}$.*

Application to white dwarfs

preceeding discussion disregarded energy production, and energy radiation

☞ applicable to stars with few or exhausted 'fuel' for nuclear reactions

1. **brown dwarfs**: not heavy enough that nuclear reactions can start
2. **white dwarfs**: one end stadium of stellar evolution when accessible 'nuclear fuel' is used up

stellar evolution: after formation out of a gas cloud, a new star rapidly reaches an equilibrium configuration, characterised by an empirical equation of state (*'main sequence'*) $L_\star = L_\star(T_\star)$ between its luminosity $L_\star$ and its temperature $T_\star$. Energy radiated off is produced by nuclear fusion

(i) first hydrogen fusion $4\mathrm{H} \to {}^4\mathrm{He}$ ☞ main sequence

H most frequent, gives most energy

(ii) then helium fusion $3{}^4\mathrm{He} \to {}^{12}\mathrm{C}$ ☞ red giant

(iii) finally oxygene production ${}^4\mathrm{He} + {}^{12}\mathrm{C} \to {}^{16}\mathrm{O}$

must end at the latest with production of Fe nuclei

fusion of Fe nuclei loses energy

$\Rightarrow$ without energy source, the star will contract when ⋆ contracts, the électrons will fall to the lowest possible energy levels

if temperature low enough, électrons should occupy all energy levels $\varepsilon = \varepsilon(k)$, up to **Fermi momentum** k_{F}

$$\#\ \text{électrons/volume} \qquad n = \frac{4\pi}{(2\pi\hbar)^3} \int_0^{k_{\mathrm{F}}} \mathrm{d}k\, k^2 \cdot 2 = \frac{k_{\mathrm{F}}^3}{3\pi^2\hbar^3}$$

because of **Pauli principle**, have exactly 2 électrons in each quantum state

mass density $\rho = n m_N \mu$, where $\mu = \#$ électrons/per nucleon

Example: $2p + 2e^- \to d + 2\nu + \underbrace{e^+ + e^-}_{\to\, 2\gamma} + e^- \to d + e^- + \text{energy}$

deuteron d has **2** nucleons $\Rightarrow$ $\mu = 2$ for fusion from pure hydrogene

$$\Rightarrow \qquad \boxed{k_{\mathrm{F}} = \hbar \left(\frac{3\pi^2}{m_N \mu}\, \rho \right)^{1/3}}$$

there will be *perfect électron condensation*, if $k_{\mathrm{B}} T \ll \left(k_{\mathrm{F}}^2 + m_e^2\right)^{1/2} - m_e$
if that is so, obtain

$$u = \frac{8\pi}{\left(2\pi\hbar\right)^3} \int_0^{k_{\mathrm{F}}} \mathrm{d}k \; k^2 \left[\left(k_{\mathrm{F}}^2 + m_e^2\right)^{1/2} - m_e\right] \quad , \quad \text{energy}$$

$$p = \frac{8\pi}{3\left(2\pi\hbar\right)^3} \int_0^{k_{\mathrm{F}}} \mathrm{d}k \; k^2 \frac{k \cdot k}{\left(k_{\mathrm{F}}^2 + m_e^2\right)^{1/2}} \quad , \qquad \text{pressure}$$

the Fermi momentum $k_{\mathrm{F}} = k_{\mathrm{F}}(\rho)$ gives the equation of state $p = p\big(k_{\mathrm{F}}(\rho)\big)$.

rappel: from relativistic statistical mechanics $u = \int_0^\infty \mathrm{d}k \; \varepsilon(k) n(k)$, $p = \frac{1}{3} \int_0^\infty \mathrm{d}k \; v(k) \cdot k \; n(k)$
and $v(k) = \frac{\mathrm{d}\varepsilon(k)}{\mathrm{d}k}$; $n(k)$ is the Fermi distribution (limit $T \to 0$).

Definition: *The* **critical density** ρ_c *is given by the condition*

$$\boxed{k_{\mathrm{F,c}} := m_e \overset{!}{=} \hbar \left(\frac{3\pi^2}{m_N \mu} \rho_c\right)^{1/3}} \implies \rho_c \simeq 10^9 \; [\mathrm{kg/m^3}]$$

Definition: *(i) Matter is called* **non-degenerate**, *if* $\rho \ll \rho_c$.
(ii) Matter is called **degenerate**, *if* $\rho \gg \rho_c$.

these two limit cases are examples of the simple polytrope model discussed above

(A) non-degenerate: one has $k_{\mathrm{F}} \ll m_e$, hence

$$u = \frac{3}{2}p \quad , \quad p = \frac{8\pi k_{\mathrm{F}}^5}{15 m_e (2\pi\hbar)^3} = \frac{\hbar^2}{15 m_e \pi^2} \left(\frac{3\pi^2}{m_N \mu}\rho\right)^{5/3}$$

$\Rightarrow$ polytrope, with $\gamma = \frac{5}{3}$ and $K = \frac{\hbar^2}{15 m_e \pi^2} \left(\frac{3\pi^2}{m_N \mu}\right)^{5/3}$.

the isentropic and polytrope model above gives for mass $M_\star$ and radius $R_\star$

$$\boxed{M_\star \simeq 2.79 \mu^{-2} \left(\frac{\rho(0)}{\rho_c}\right)^{1/2} M_\odot \quad , \quad R_\star \simeq 2 \cdot 10^4 \mu^{-1} \left(\frac{\rho(0)}{\rho_c}\right)^{1/6} [\mathrm{km}]}$$

even if $\rho = \rho(r)$ the scaling relation $M \sim \rho^{5/3}$ remains valid, constants shift by 10-20%

(B) degenerate: one has $k_{\mathrm{F}} \gg m_e$, hence

$$u = 3p \quad , \quad p = \frac{8\pi k_{\mathrm{F}}^4}{12 m_e (2\pi\hbar)^3} = \frac{\hbar}{12\pi^2} \left(\frac{3\pi^2}{m_N \mu}\rho\right)^{4/3}$$

$\Rightarrow$ polytrope, with $\gamma = \frac{4}{3}$ and $K = \frac{\hbar}{12\pi^2} \left(\frac{3\pi^2}{m_N \mu}\right)^{4/3}$.

CHANDRASEKHAR

$$\boxed{M_\star \simeq 5.87 \mu^{-2} M_\odot \quad , \quad R_\star \simeq 5.3 \cdot 10^4 \mu^{-1} \left(\frac{\rho(0)}{\rho_c}\right)^{1/3} [\mathrm{km}]}$$

have obtained upper mass limits for stable polytropic stars CHANDRASEKHAR

(A) non-degenerate: $\rho \ll \rho_c$ or $k_{\mathrm{F}} \ll m_e$

$$M_\star \simeq 2.79\mu^{-2}\left(\frac{\rho(0)}{\rho_c}\right)^{1/2} M_\odot \quad , \quad R_\star \simeq 2\cdot 10^4 \mu^{-1}\left(\frac{\rho(0)}{\rho_c}\right)^{1/6} [\mathrm{km}]$$

(B) degenerate: $\rho \gg \rho_c$ or $k_{\mathrm{F}} \gg m_e$

$$M_\star \simeq 5.87\mu^{-2} M_\odot \quad , \quad R_\star \simeq 5.3\cdot 10^4 \mu^{-1}\left(\frac{\rho(0)}{\rho_c}\right)^{1/3} [\mathrm{km}]$$

☞ *entire stars pressed into a ball with merely the double of the Earth's radius*
origin of the name: very compact objects, emitting white light

☞ upper mass limit of stable white dwarf: with $\mu \simeq 2 \Rightarrow$ $M_\star \leq 1.4\, M_\odot$

- importance of relativistic effects: not very large, since

$$\frac{\mathscr{R}}{R_\star} \sim (0.5-1)\mu^{-1}\frac{m_e}{m_N}\left(\frac{\rho(0)}{\rho_c}\right)^{\alpha/3} \lesssim 4\cdot 10^{-4} \quad ; \quad \alpha = \begin{cases} 2 & \text{non-degenerate} \\ 1 & \text{degenerate} \end{cases}$$

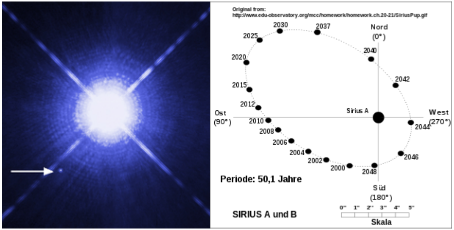
further details: Weinberg, *Gravitation & Cosmology* (1972)

White dwarfs are regularly observed and well-studied

- Sirius A & B

1844 Bessel Sirius suspected double star
1851 Peters orbit determined
1862 Clark first observation

	Sirius A	Sirius B
mass	$2.1 M_\odot$	$0.98 M_\odot$
radius	$1.7 R_\odot$	$0.0087 R_\odot$
luminosity	$25 L_\odot$	$0.03 L_\odot$
temperature	$9900[\mathrm{K}]$	$25000[\mathrm{K}]$
period	$50.1\ \mathrm{y}$	



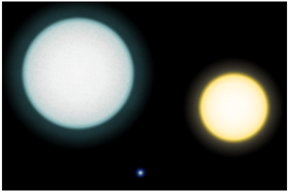
Hubble space telescope image & orbit

Source: `https://de.wikipedia.org/wiki/Sirius`

- IK Peg A & B

1862 Argelander variable star
1927 Harper orbit determined

	IK Peg A	IK Peg B
mass	$1.65 M_\odot$	$1.15 M_\odot$
radius	$1.47 R_\odot$	$0.006 R_\odot$
luminosity	$6.6 L_\odot$	$0.12 L_\odot$
temperature	$7600[\mathrm{K}]$	$35500[\mathrm{K}]$
period	$21.7\ \mathrm{d}$	



IK Peg A/B vs sun (artist's view)

Source: `https://de.wikipedia.org/wiki/IK_Pegasi`

• historically, the evolution of Sirius A/B has been relatively tranquil
stars are remote from each other, semi-major axis $20[\mathrm{AU}] = \mathrm{orbit\ of\ Uranus}$

Formation $240 \cdot 10^6\mathrm{y}$ ago
Sirius B should had initially $5M_\odot$ mass ⇒ rapid evolution to red giant
$140 \cdot 10^6\mathrm{y}$ ago: Sirius B becomes red giant ⇒ He is fusioned to C,O
Sirius B loses 80% of original mass (how much transferred to Sirius A ?)
the burnt-out C- and O-rich nucleus of that red giant we see as Sirius B today
$124 \cdot 10^6\mathrm{y}$ ago: contraction of the nucleus until stabilised by electron degeneracy

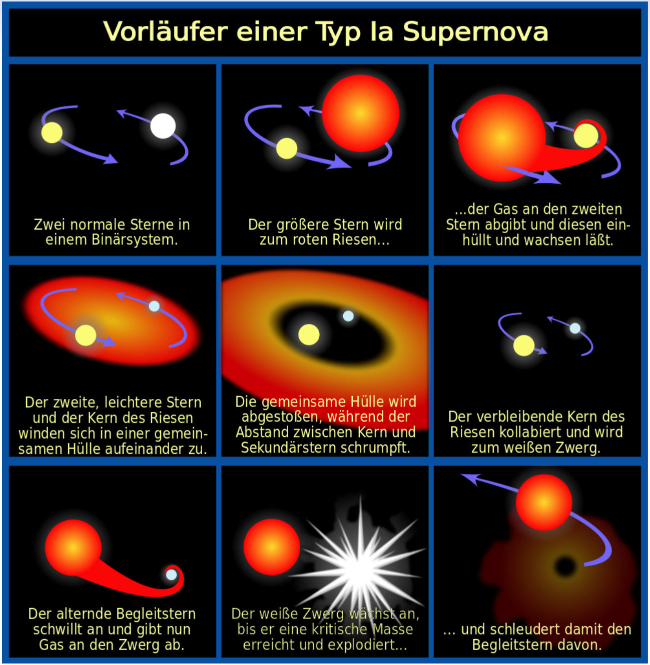
☞ Sirius B was first white dwarf ever observed

• the future evolution of IK Peg A/B has the **potential to become spectacular**
stars are fairly close to each other, semi-major axis $0.3[\mathrm{AU}] = \mathrm{orbit\ of\ Mercury}$

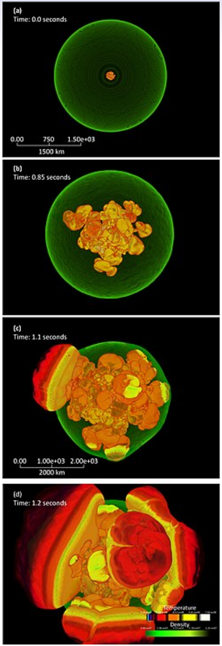
Formation of system $(50 - 600) \cdot 10^6\mathrm{y}$ ago
massive progenitor of IK Peg B, turned into a red giant, lost hydrogen/helium enveloppe
IK Peg B contracts into a white dwarf, consists essentially of C,O
IK Peg A is relatively hot, will turn into a red giant within $(2 - 3) \cdot 10^9\mathrm{y}$
since orbit is close, the mass lost by IK Peg A will mainly fall on IK Peg B
even now, IK Peg B is one of the most heavy white dwarfs known

a white dwarf more massive than the **Chandrasekhar limit** $1.4M_\odot$ will explode

IK Peg is *nearest known candidate* for a future **supernova** explosion (just 150 [ly] away)



Source: `https://fr.wikipedia.org/wiki/Fichier:Progenitor_IA_supernova.svg`



implosion of a white dwarf $\Rightarrow$ **supernova typ IA**

identical physical initial conditions

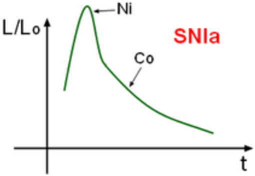
universal light curve and same maximal luminosity

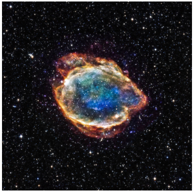
can be used to measure distance of supernovæ

surpernovæ are most bright events in the universe

- peak luminosity comparable to the one of an entire galaxy
- peak luminosities measured up to $570 \cdot 10^9 L_\odot$
- brightest SN seen on Earth in 1006: visible at daytime

☞ use as 'distance candles'





decay of SN light curve from radioactive decays (Ni, Co)

remnants of SNs can have spectacular forms

Source: `https://en.wikipedia.org/wiki/Type_Ia_supernova` white dwarfs are *stabilised by pressure of degenerated électrons*

white dwarfs are **macroscopic quantum objects**

some properties:

masses $(0.17 - 1.35)\, M_\odot$

typically $0.6 M_\odot$

radius $(0.8 - 2) \cdot 10^{-2} R_\odot$

density $10^7 - 10^{10} [\mathrm{kg}/\mathrm{m}^3]$

surface gravity $10^5 g$

temperature $(1 - 4) \cdot 10^4 [\mathrm{K}]$

Fermi energy $\sim 10^9 [\mathrm{K}]$

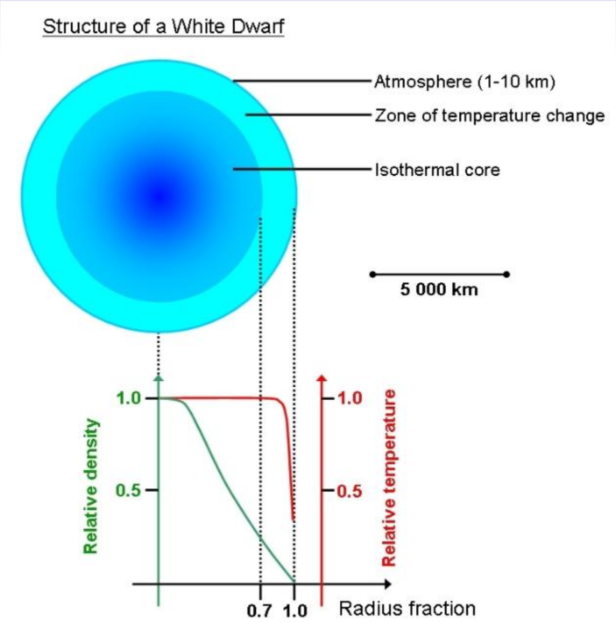
coldest WD: $T = 3900 [\mathrm{K}]$

this WD is $\gtrsim 11 \cdot 10^9 [\mathrm{y}]$ old

typical densities of several materials

material	density $[\mathrm{kg}/\mathrm{m}^3]$
water	1000
sun	1408
osmium	$2.3 \cdot 10^4$
sun (core)	$1.5 \cdot 10^5$
white dwarf	10^9
nucleus	$2.3 \cdot 10^{17}$
neutron star (core)	$10^{17} - 10^{18}$
black hole	$> 2 \cdot 10^{30}$

- white dwarfs are stratified (layers O, C, He, ...)
- interior is opaque (excited électrons do not find free levels deep inside)
- scale height of atmosphere $\sim 10^2 [\mathrm{m}] \Rightarrow$ very thin hot atmosphere
  and crust of few $[\mathrm{km}]$ until one reaches the high-density core
- no internal energy source $\Rightarrow$ slowly cool, on time scales $\gtrsim 10^9 [\mathrm{y}]$



Source: `https://cronodon.com/SpaceTech/WhiteDwarf.html`

5.4 Neutron stars

end state of stellar evolution: described by polytrope, $\star$ stable for $\gamma > \frac{4}{3}$
after exhaustion of nuclear fuel: $\star$ becomes *'white dwarf'*,
stabilised by électron degeneracy pressure

but for $k_\mathrm{F} \gtrsim 5m_e$, électrons captured by protons (**inverse β-decay**)

$$p + e^- \quad \longrightarrow \quad n + \nu$$

also occurs if $M_\star$ larger than Chandrasekhar limit

$\Longrightarrow$ *new star collapse !* $\Longrightarrow$ observed as **supernova** !

- neutrinos (ν) escape $\Longrightarrow$ unambiguous signal for a supernova
- *stellar matter is transformed into neutrons*

○ matter will be compressed until the *neutrons* become degenerate

both électrons and neutrons are fermions, but neutron mass $m_n \simeq 2000 m_e$
can re-use same model as before, with électrons replaced by neutrons

and $\mu \mapsto 1$

$$\rho \simeq u = \frac{8\pi}{(2\pi\hbar)} \int_0^{k_\mathrm{F}} \mathrm{d}k\, k^2 \left(k^2 + m_n^2\right)^{1/2} = 3\rho_c \int_0^{k_\mathrm{F}/m_n} \mathrm{d}u\, u^2 \sqrt{u^2+1}$$

$$p = \frac{8\pi}{3(2\pi\hbar)} \int_0^{k_\mathrm{F}} \mathrm{d}k\, k^4 \left(k^2 + m_n^2\right)^{-1/2} = \rho_c \int_0^{k_\mathrm{F}/m_n} \mathrm{d}u\, \frac{u^4}{\sqrt{u^2+1}}$$

where the critical density is now

$$\boxed{\rho_c = \frac{8\pi m_n^4 c^3}{3(2\pi\hbar)} \simeq 6 \cdot 10^{18}\ [\mathrm{kg/m^3}]}$$

That is the density of nuclear matter !

consider case of *perfect neutron condensation* $\Rightarrow$ can effectively look at $T \to 0$ limit

(A) non-degenerate: $\rho \ll \rho_c$ or $k_{\rm F} \ll m_n$ newtonian polytrope

$$M_\star \simeq 2.7 \left(\frac{\rho(0)}{\rho_c}\right)^{1/2} M_\odot \quad , \quad R_\star \simeq 11 \left(\frac{\rho(0)}{\rho_c}\right)^{1/6} [\mathrm{km}]$$

N.B.: here $\frac{\mathscr{R}}{R_\star} \approx 0.3$, relativistic effects are becoming important

(B) degenerate: $\rho \gg \rho_c$ or $k_{\rm F} \gg m_n$

$$\rho \simeq \frac{3}{4}\left(\frac{k_{\rm F}}{m_n}\right)\rho_c \quad , \quad p = \frac{1}{4}\left(\frac{k_{\rm F}}{m_n}\right)\rho_c = \frac{1}{3}\rho$$

equation of state of a photon gas $\Rightarrow$ neutrons are extremely relativistic !
The TOV equation becomes in the extreme relativistic case $p = \frac{1}{3}\rho$

$$-r^2 \frac{\mathrm{d}\rho(r)}{\mathrm{d}r} = 4G\mathscr{M}(r)\rho(r)\left(1 + \frac{4\pi r^3 \rho(r)}{\mathscr{M}(r)}\right)\left(1 - \frac{2G\mathscr{M}(r)}{r}\right)^{-1}$$

with the exact solution $\rho(r) = \frac{3}{56\pi}\frac{1}{G}\frac{1}{r^2}$ Misner & Zapolski 1964

but: expect two independent solutions . . .

two qualitative features:

- density $\rho(r)$ diverges for $r \to 0 \Rightarrow$ extreme concentration in the centre
- ◦ slow decay of $\rho(r)$ for $r \to \infty \Rightarrow$ outer layers not fully degenerate

☞ improve TOV equations and solve numerically.

Leads to estimates of upper mass limit (TOV-**limit**) and of radius

$$M_{\star,n} \lesssim (2.2 - 2.9)\, M_{\odot} \quad , \quad R_{\star,n} \approx (10 - 12)\, [\mathrm{km}]$$

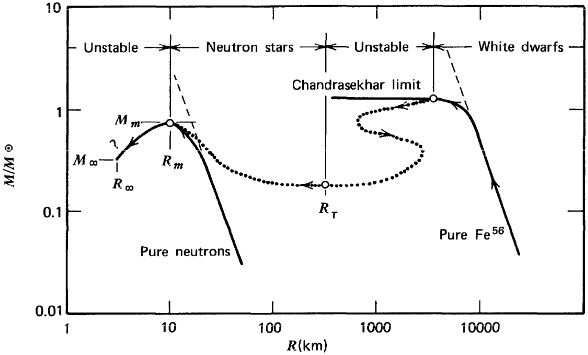
? how is this limit affected if neutron star rotates ?

two branches of stellar equilibrium

(1) pure ^{56}Fe white dwarf

(2) pure neutron star

if the neutron star becomes unstable, no known process can stop collapse into **black hole**



Source: Weinberg, *Gravitation & Cosmology* (1972)

neutron stars have been observed, first with radio waves ('pulsars') then also optically
at present the most heavy known neutron stars include:

PSR J1748-2021B, $M_\star = \left(2.74 \pm 0.21\right) M_\odot$
PSR B1957+20, $M_\star = \left(2.4 \pm 0.12\right) M_\odot$
PSR J2215+5135, $M_\star = \left(2.27 \pm 0.17\right) M_\odot$

the most light known black holes have masses $M_{\mathrm{BH}} \gtrsim \left(3.4\,^{+0.3}_{-0.1}\right) M_\odot$

$\Rightarrow$ the TOV limit should be somewhere in between ...

but: there are candidates for neutron stars beyond the TOV limit !

e.g. GW170817, $M \simeq \left(2.74\,^{+0.04}_{-0.01}\right) M_\odot$, merger of two neutron stars, BH collapse 5-10[s] later ?

Some numerical illustrations:

(a) angular momentum $\ell = MR^2\omega \Rightarrow R^2\omega = \mathrm{cste}$ during collapse

$$\left.\begin{array}{l} \text{sun: } R_\odot = 7\cdot 10^8[\mathrm{m}],\ \omega_\odot = 3\cdot 10^{-6}[\mathrm{s}^{-1}] \\ \text{Neutron star: } R_\star = 5\cdot 10^4[\mathrm{m}] \end{array}\right\} \Rightarrow \omega_\star \sim 10^4[\mathrm{s}^{-1}]$$ very rapid rotation !

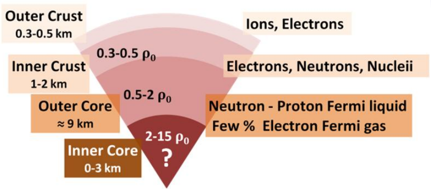
(b) magnetic flux $\Phi = \pi BR^2 \Rightarrow BR^2 = \mathrm{cste}$ during collapse

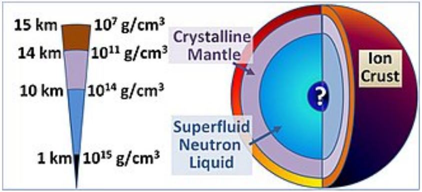
$$\left.\begin{array}{l} \text{sun: } B \simeq 10^{-4}[\mathrm{T}]\ \text{(global)},\ B \simeq 10^{-1}[\mathrm{T}]\ \text{(sun spots)} \\ \text{Neutron star: } R_\star = 5\cdot 10^4[\mathrm{m}] \end{array}\right\} \Rightarrow B_\star \gtrsim 10^4[\mathrm{T}]$$

actual fields $\left(10^4 - 10^{11}\right)[\mathrm{T}] \Rightarrow$ **extremely** strong magnetic field !

N.B.: in labos on Earth: $B_{\max} \lesssim 16[\mathrm{T}]$; dipôle magnets of LHC collider: $8.2[\mathrm{T}]$

schematic section of a neutron star





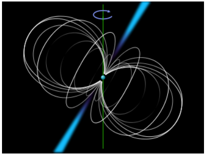
complex inner structure

full neutron degeneracy only deeply inside

atmosphere thickness $\lesssim$ [cm]

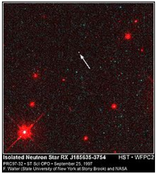
*density increases towards interior (factor 10^8)

*core temperature falls from 10^{11}[K] to 10^4[K] in the first [My] after formation



neutron stars emit intensive periodic radiation in concentrated rays from magnetic pôles

⇒ **pulsar** periods ∼ [ms]



1934 theoretical proposal Baade, Zwicky
1967 observation (radio) Hewitt, Bell
1992 observation (optical) RX J1856.5-3754 – Chandra observatory

Sources: https://de.wikipedia.org/wiki/Neutronenstern, https://en.wikipedia.org/wiki/Neutron_star

Vorlesung XI

Rappel: solve full field equation $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^2} T_{\mu\nu}$,

with $T^{\mu\nu} = \frac{p}{c^2} g^{\mu\nu} + \left(\rho + \frac{p}{c^2}\right) u^\mu u^\nu$ for a perfect fluid, with density ρ pressure p

- interior of spherical star (with $\rho = \mathrm{cste}$) given by inner Schwarzschild metric

$$\mathrm{d}s^2 = -\left[\frac{3}{2}\left(1 - AR^2\right)^{1/2} - \frac{1}{2}\left(1 - Ar^2\right)^{1/2}\right] c^2 \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - Ar^2} + r^2 \mathrm{d}\Omega^2$$

and $A := \frac{8\pi G}{3c^2} \rho$.

explicit solution $\mathrm{d}s^2$ for generic ρ exists

- Pressure profile $p = p(r)$: $\Rightarrow$ Tolman-Oppenheimer-Volkoff (TOV) equation

$$r^2 \frac{\mathrm{d}p}{\mathrm{d}r} = -G\mathscr{M}(r)\rho(r)\left(1 + \frac{p(r)}{\rho(r)}\right)\left(1 + \frac{4\pi r^3 p(r)}{\mathscr{M}(r)}\right)\left(1 + \frac{2G\mathscr{M}(r)}{r}\right)^{-1}$$

where $\mathscr{M}(r) := \int_0^r \mathrm{d}r'\, 4\pi r'^2 \rho(r')$.

$c = 1$

◦ also need equation of state $p(r) = p\big(\rho(r)\big)$.

☞ recast solution of TOV-equation as constrained variational problem

$$M[\rho] = \int_0^\infty \mathrm{d}r\, 4\pi r^2\, \rho(r) \overset{!}{=} \min \quad , \quad N = \int_0^\infty \mathrm{d}r\, 4\pi r^2\, n(r) \left(1 - \frac{2G\mathscr{M}(r)}{r}\right)^{-1/2} \text{ fixed}$$

Alternatively, can minimise energy $U[\rho] = M[\rho] - N m_N = (T+V)[\rho]$

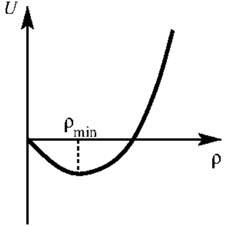
T: thermal energy, V: gravitational energy

Illustration: polytrope equation of state $p = K\rho^\gamma$, isentropic star

for density $\rho = \mathrm{cste}$, find

$$U = U(\rho) = a\rho^{\gamma-1} - b\rho^{1/3}$$

which, for $\gamma > \frac{4}{3}$, has a single minimum at $\rho_{\min}$.



☞ universal **mass-density scaling relation**

valid within $10 - 20\%$ for $\gamma \simeq \frac{4}{3} - \frac{5}{3}$

$$\boxed{M \simeq \frac{4\pi}{3} \left(\frac{15K}{4\pi G}\right)^{3/2} \rho^{(3\gamma-4)/2}} \quad , \quad \text{if } \gamma > \frac{4}{3}$$

☞ *a polytropic and isentropic star is stable for $\gamma > \frac{4}{3}$.*

Rappel: Have analysed two possible end stadia of stellar evolution

	white dwarf	neutron star
degeneracy	electrons	neutrons
maximal mass limit	$1.4M_\odot$	$(2.2-2.9)M_\odot$
radius	$\sim 10^4[\mathrm{km}]$	$(10-12)[\mathrm{km}]$
density	$10^9[\mathrm{kg/m^3}]$	$10^{17}[\mathrm{kg/m^3}]$
surface gravity	$10^5 g$	$10^{11} g$

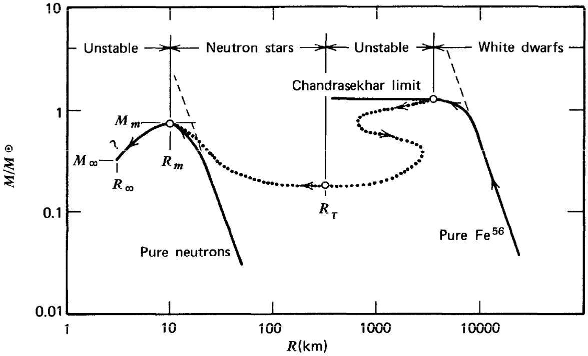
? what about effects of rotation on a white dwarf/neutron star ?

two branches of stellar equilibrium

(1) pure ^{56}Fe white dwarf

(2) pure neutron star

if the neutron star becomes unstable, no known process can stop collapse into **black hole**



Source: Weinberg, *Gravitation & Cosmology* (1972)

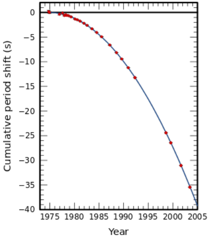
6. Gravitational waves

* new phenomenon in gravitation, quite analogous to electromagnetic waves
* theoretically predicted by EINSTEIN in 1916
* first direct observation announced the 11^{th} of february 2016
* *new 'window' in astrophysics*, complementary to 'optical' observations
* created by **strong** gravitational fields; propagation is a weak-field effect

* 1974 indirect evidence from change of period of binary pulsar

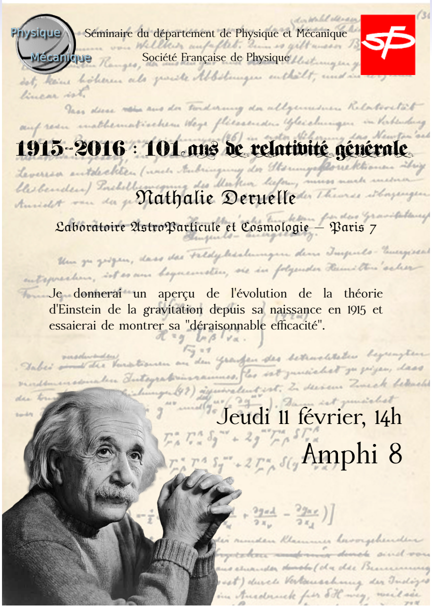
PSR 1913+16 HULSE, TAYLOR

using the world's largest radio telescope (Arecibo)
diameter 305$[\mathrm{m}]$ – collapsed déc. 2020



* since 2015 direct detection (LIGO & VIRGO collab.)

https://www.ligo.caltech.edu/page/what-is-ligo
https://www.sciencesetavenir.fr/espace/univers/peut-on-sauver-le-radiotelescope-d-arecibo_149148
https://de.wikipedia.org/wiki/PSR_1913%2B16



a bit of regional history: a SFP seminar announcement in Nancy the 11th of February 2016 – followed by a certain LIGO press conference . . .

6.1 Linear approximation

a wave equation is derived in the linear approximation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} =: g^{(1)}_{\mu\nu}$$

where $\left|h_{\mu\nu}\right| \ll 1$, merely keep terms of first order
have system with weak curvature
but can still make general coordinate transformations
☞ consider Lorentz transformations on the background metric

$$x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad , \quad x_\mu \mapsto x'_\mu = \bar{\Lambda}_\mu{}^\nu x_\nu$$

$\Lambda^\mu{}_\nu$ is space-independent matrix of Lorentz transformations, $\Lambda^\mu{}_\nu \bar{\Lambda}_\mu{}^\kappa = \delta_\nu{}^\kappa$
The metric tensor transforms as follows

$$g^{\mu\nu} \mapsto g'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma g^{\rho\sigma} \quad , \quad g_{\mu\nu} \mapsto g'_{\mu\nu} = \bar{\Lambda}_\mu{}^\rho \bar{\Lambda}_\nu{}^\sigma g_{\rho\sigma}$$

while the Minkowski metric is invariant $\bar{\Lambda}_\mu{}^\rho \bar{\Lambda}_\nu{}^\sigma \eta_{\rho\sigma} = \eta_{\mu\nu}$

the complete metric $g^{(1)}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ transforms as follows

$$\bar{\Lambda}_\mu{}^\rho \bar{\Lambda}_\nu{}^\sigma g^{(1)}_{\rho\sigma} = \eta_{\mu\nu} + \bar{\Lambda}_\mu{}^\rho \bar{\Lambda}_\nu{}^\sigma h_{\rho\sigma} = g^{(1)}_{\mu\nu} \quad \Longrightarrow \quad \boxed{\bar{\Lambda}_\mu{}^\rho \bar{\Lambda}_\nu{}^\sigma h_{\rho\sigma} = h_{\mu\nu}}$$

a weak gravitational field is described by the tensor $h_{\mu\nu}$, but in flat time-space.

Write down EINSTEIN's field equations: notice first that

$$\begin{aligned} \Gamma^\kappa_{\lambda\mu} &= \frac{1}{2}\eta^{\kappa\rho}\left(h_{\rho\lambda,\mu} + h_{\rho\mu,\lambda} - h_{\lambda\mu,\rho}\right) \\ \Rightarrow \quad R^{(1)}_{\alpha\mu\beta\nu} &= \frac{1}{2}\left(h_{\mu\beta,\alpha\nu} + h_{\alpha\nu,\mu\beta} - h_{\mu\nu,\alpha\beta} - h_{\alpha\beta,\mu\nu}\right) \\ \Rightarrow \quad R^{(1)}_{\mu\nu} = \eta^{\alpha\beta} R^{(1)}_{\alpha\mu\beta\nu} &= \frac{1}{2}\left(h^\alpha_{\mu,\alpha\nu} + h^\alpha_{\nu,\mu\beta} - h_{\mu\nu,}{}^\alpha_\alpha - h_{,\mu\nu}\right) \end{aligned}$$

where $a_{,\lambda}^{\lambda} = \partial_\lambda \partial^\lambda a = \Box a$ (d'Alembert) and $h := h^\mu{}_\mu$.

Next, the Einstein tensor reads $G^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{1}{2} R^{(1)} \eta_{\mu\nu}$.

N.B.: the energy-momentum tensor $T^{(0)}_{\mu\nu}$ does not depend on $h_{\mu\nu}$ – see newtonian limit

$$\boxed{G^{(1)}_{\mu\nu} = \frac{8\pi G}{c^2} T^{(0)}_{\mu\nu}}$$

N.B.: to this order, the conservation law $T^{\mu\nu}{}_{;\nu} = 0$ reduces to $T^{\mu\nu\,(0)}{}_{,\nu} = \partial_\nu T^{\mu\nu\,(0)} = 0$.

field equations $\boxed{G^{(1)}_{\mu\nu} = \frac{8\pi G}{c^2} T^{(0)}_{\mu\nu}}$ for the metric tensor $g^{(1)}_{\mu\nu}$

- are symmetric matrices $G^{(1)}_{\mu\nu} = G^{(1)}_{\nu\mu}$, $T^{(0)}_{\mu\nu} = T^{(0)}_{\nu\mu}$
- obey conservation laws $\partial^\nu G^{(1)}_{\mu\nu} = \partial^\nu T^{(0)}_{\mu\nu} = 0$
  $\Rightarrow$ gives $10 - 4 = 6$ independent field equations

○ metric tensor symmetric $g^{(1)}_{\mu\nu} = g^{(1)}_{\nu\mu}$ and conserved $\partial^\nu g^{(1)}_{\mu\nu} = 0$
  $\Rightarrow$ have $10 - 4 = 6$ independent variables

$\Rightarrow$ *remaining 4 degrees of freedom are used to maintain general covariance !*

Definition: *An infinitesimal* **gauge transformation** *is a change of coordinates*

$$\boxed{x^\mu \mapsto x'^\mu = x^\mu + b^\mu(\mathrm{x}) \quad ; \quad \left|b^\mu(\mathrm{x})\right| \ll 1 \quad , \quad \left|\partial_\nu b^\mu(\mathrm{x})\right| \ll 1}$$

N.B.: terms in b and its derivatives are only kept to first order.

$$\frac{\partial x'^\mu}{\partial x^\alpha} = \delta^\mu{}_\alpha + \partial_\alpha b^\mu \quad \Longrightarrow \quad \frac{\partial x^\mu}{\partial x'^\alpha} = \delta^\mu{}_\alpha - \partial_\alpha b^\mu + \mathrm{O}\left(b^2\right)$$

had seen, for gauge transformation: $\frac{\partial x^\mu}{\partial x'^\alpha} \simeq \delta^\mu{}_\alpha - \partial_\alpha b^\mu$

this implies for the transformation of the metric tensor

$$
\begin{aligned}
{g'_{\alpha\beta}}^{(1)} &= \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} g^{(1)}_{\mu\nu} \\
&\simeq \left(\delta^\mu{}_\alpha - \partial_\alpha b^\mu\right)\left(\delta^\nu{}_\beta - \partial_\beta b^\nu\right)\left(\eta_{\mu\nu} + h_{\mu\nu}\right) \\
&\simeq \delta^\mu{}_\alpha \delta^\nu{}_\beta \left(\eta_{\mu\nu} + h_{\mu\nu}\right) - \partial_\alpha b^\mu \eta_{\mu\nu} - \partial_\beta b^\nu \eta_{\mu\nu}
\end{aligned}
$$

and this gives the gauge transformation of the tensor $h_{\alpha\beta}$: $\quad b_\alpha = b^\mu \eta_{\mu\alpha}$

$$
\boxed{h'_{\alpha\beta} = h_{\alpha\beta} - \partial_\alpha b_\beta - \partial_\beta b_\alpha}
$$

N.B.: maintains the symmetry $h_{\alpha\beta} = h_{\beta\alpha}$.

there are many *analogies* between electromagnetism and linearised gravity:

	electromagnetism	linearised gravity
source	j^μ	$T^{\mu\nu} = T^{\nu\mu}$
conservation law	$\partial_\mu j^\mu = 0$	$\partial_\mu T^{\mu\nu} = 0$
field	A_μ	$h_{\mu\nu} = h_{\nu\mu}$
gauge tranformation	$A_\mu \mapsto A_\mu - \partial_\mu \Lambda$	$h_{\mu\nu} \mapsto h_{\mu\nu} - \partial_\mu b_\nu - \partial_\nu b_\mu$
Lorenz gauge	$\partial^\mu A_\mu = 0$	$\partial^\mu \bar{h}_{\mu\nu} = 0$
field equation (in Lorenz gauge)	$\Box A_\mu = \frac{4\pi}{c} j_\mu$	$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^2} T_{\mu\nu}$

with abbreviation: $\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2} h^\alpha{}_\alpha \eta_{\mu\nu}$

N.B.: the physicists L. Lorenz (Copenhagen) and H.A. Lorentz (Leiden) are distinct people. But there also exists a 'Lorentz-Lorenz equation' in optics.

6.2 Lorenz gauge

Definition: *The* **trace-inverted field** *is given by*

$$\boxed{\bar{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{2} h\, \eta_{\mu\nu} \quad , \quad h := h^{\alpha}{}_{\alpha}}$$

* one has $\bar{h} = \bar{h}^{\mu}_{\mu} = h^{\mu}{}_{\mu} - \frac{1}{2} h \cdot 4 = -h$. explains the name
* gauge transformation $h'_{\alpha\beta} = h_{\alpha\beta} - \partial_\alpha b_\beta - \partial_\beta b_\alpha \;\Rightarrow\; h' = h - 2\partial^\beta b_\beta$.

Consider the gauge transformation

$$\begin{aligned}
\bar{h}'{}_{\alpha\beta} &= h'_{\alpha\beta} - \frac{1}{2} h'\, \eta_{\alpha\beta} \\
&= h_{\alpha\beta} - \partial_\alpha b_\beta - \partial_\beta b_\alpha - \frac{1}{2}\left(h - 2\partial^\gamma b_\gamma\right)\eta_{\alpha\beta} \\
&= h_{\alpha\beta} - \frac{1}{2} h\, \eta_{\alpha\beta} - \partial_\alpha b_\beta - \partial_\beta b_\alpha + \left(\partial^\gamma b_\gamma\right)\eta_{\alpha\beta} \\
&= \bar{h}_{\alpha\beta} - \partial_\alpha b_\beta - \partial_\beta b_\alpha + \left(\partial^\gamma b_\gamma\right)\eta_{\alpha\beta} \qquad \text{(J)}
\end{aligned}$$

take the divergence in eq. (J)

$$\partial^\alpha \bar{h}'_{\alpha\beta} = \partial^\alpha \bar{h}_{\alpha\beta} - \partial^\alpha \partial_\alpha b_\beta \overbrace{- \partial^\alpha \partial_\beta b_\alpha + \partial^\alpha \left(\partial^\gamma b_\gamma\right) \eta_{\alpha\beta}}^{=0} = \partial^\alpha \bar{h}_{\alpha\beta} - \Box b_\beta$$

choosing the gauge transformation such that $\Box b_\beta \stackrel{!}{=} \partial^\alpha \bar{h}_{\alpha\beta}$, one can always achieve that the Lorenz gauge $\partial^\alpha \bar{h}'_{\alpha\beta} = 0$ is satisfied. The function $\mathtt{b(x)}$ is unique up to solutions of $\Box b_\beta = 0$.

☞ *It is always possible to have the Lorenz gauge $\partial^\mu \bar{h}_{\mu\nu} = 0$ satisfied.*

this implies $\partial^\mu h_{\mu\nu} = \frac{1}{2}\partial_\nu h$. Furthermore, for the linearised Ricci scalar

$$R^{(1)} = \partial^\mu \partial^\nu h_{\mu\nu} - \Box h = \partial_\nu \left(\tfrac{1}{2}\partial_\nu h\right) - \Box h = -\frac{1}{2}\Box h$$

and for the linearised Ricci tensor

$$\begin{aligned} R^{(1)}_{\mu\nu} &= \frac{1}{2}\left(\partial^\alpha \partial_\nu h_{\alpha\mu} + \partial_\mu \partial^\alpha h_{\alpha\nu} - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h\right) \\ &= \frac{1}{2}\left(\partial_\nu \frac{1}{2}\partial_\mu h + \partial_\mu \frac{1}{2}\partial_\nu h - \Box h_{\mu\nu} - \partial_\mu \partial_\nu h\right) = -\frac{1}{2}\Box h_{\mu\nu} \end{aligned}$$

The linearised Einstein tensor becomes

$$G^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{1}{2} R^{(1)}\,\eta_{\mu\nu} = -\frac{1}{2}\Box h_{\mu\nu} + \frac{1}{4}\Box h\,\eta_{\mu\nu} = -\frac{1}{2}\Box \bar{h}_{\mu\nu}$$

For linearised gravity, the field equation takes the form of a wave equation

(with external source)

$$\boxed{\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^2} T^{(0)}_{\mu\nu}}$$

Small perturbations of the metric propagate as waves with light velocity

EINSTEIN 1916

retarded formal solution

can add arbitrary solution of $\Box \bar{h}_{\mu\nu} = 0$

$$\bar{h}_{\mu\nu}(t, \boldsymbol{r}) = \frac{4G}{c^2} \int \mathrm{d}\boldsymbol{r}' \, \frac{T^{(0)}_{\mu\nu}\left(t - \frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}, \boldsymbol{r}'\right)}{|\boldsymbol{r} - \boldsymbol{r}'|}$$

retarded potential, quite familiar from electromagnetism

N. Deruelle, J.P. Lesote, *“Les ondes gravitationnelles”*, Paris (2018)

6.3 Plane waves

gravitational wave propagation in the vacuum, without source $T^{(0)}_{\mu\nu} = 0$
$\Rightarrow \Box \bar{h}_{\mu\nu} = 0$, since $\bar{h} = -h$, also have $\Box \bar{h} = 0$

$$\text{wave equation} \qquad \boxed{\Box \bar{h}_{\mu\nu} = 0} \qquad \text{(W)}$$

ansatz: plane waves $\bar{h}_{\mu\nu}(\mathtt{x}) = \varepsilon_{\mu\nu}\, e^{\mathrm{i}\mathtt{k}\cdot\mathtt{x}}$

with $\mathtt{k} = \left(\frac{\omega}{c}, \boldsymbol{k}\right)$: four-momentum of wave, $\varepsilon_{\mu\nu} = \varepsilon_{\nu\mu}$ **polarisation tensor**

* insert into wave equation (W): $\mathtt{k}^2 \varepsilon_{\mu\nu} e^{\mathrm{i}\mathtt{k}\cdot\mathtt{x}} = 0$
  $\Rightarrow \mathtt{k}^2 = k_\alpha k^\alpha = -\frac{\omega^2}{c^2} + \boldsymbol{k}^2 \overset{!}{=} 0$ light-like four-momentum
* must obey Lorenz gauge $\partial^\mu h_{\mu\nu} = 0 \Rightarrow k^\mu \varepsilon_{\mu\nu} = 0$ transverse wave

Gravitational waves are transversally polarised and propagate on the light cone.

Experimental bound: the gravitational wave event GW170817 was also observed as γ-ray burst in the galaxy NGC 4993. The observed time difference of arrival of the gravity and light signals gives

$$\boxed{-3 \cdot 10^{-15} < \frac{c_{\mathrm{grav}} - c_{\mathrm{light}}}{c_{\mathrm{light}}} < 7 \cdot 10^{-16}}$$

6.4 Transverse traceless gauge

(a) can always make further gauge transformations provided $\Box b_\mu = 0$.

one has: $\bar{h}_{\mu\nu} = \varepsilon_{\mu\nu} e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$; and choose: $b_\nu = B_\nu e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$. $\quad B_\nu = \mathrm{cste}$

rappel: eq. (J): $\bar{h}'_{\alpha\beta} = \bar{h}_{\alpha\beta} - \partial_\alpha b_\beta - \partial_\beta b_\alpha + \left(\partial^\gamma b_\gamma\right)\eta_{\alpha\beta}$

gauge-transformed polarisation tensor

$$\varepsilon'_{\alpha\beta} = \varepsilon_{\alpha\beta} - \mathrm{i}k_\alpha B_\beta - \mathrm{i}k_\beta B_\alpha + \mathrm{i}\eta_{\alpha\beta}\left(\mathtt{k}\cdot\mathtt{B}\right) \qquad \text{(J')}$$

take the trace $\varepsilon' := \varepsilon'^{\alpha}{}_{\alpha} = \varepsilon^{\alpha}{}_{\alpha} + 2\mathrm{i}\mathtt{k}\cdot\mathtt{B} = \varepsilon + 2\mathrm{i}\mathtt{k}\cdot\mathtt{B}$

via a convenient choice of B, can always achieve that $\varepsilon' = 0$.

☞ *The polarisation tensor has vanishing trace* $\varepsilon = 0$.

(b) ? can one also obtain that $\varepsilon_{\mu 0} = 0$?

From the gauge transformation (J')

$$\varepsilon'_{\mu 0} = \varepsilon_{\mu 0} - \mathrm{i}k_\mu B_0 - \mathrm{i}k_0 B_\mu + \mathrm{i}\eta_{\mu 0}\left(\mathtt{k}\cdot\mathtt{B}\right) \stackrel{?}{=} 0 \qquad \text{(J'')}$$

in principle, 4 conditions $\varepsilon'_{\mu 0} \stackrel{!}{=} 0$. However, also have $k^\mu \varepsilon_{\mu\nu} = 0$ and $\mathtt{k}^2 = 0$.

Constraint from (J'') $k^\mu \varepsilon'_{\mu 0} = k^\mu \varepsilon_{\mu 0} - \mathtt{k}^2 B_0 - \mathrm{i}k_0\left(\mathtt{k}\cdot\mathtt{B}\right) + \mathrm{i}k_0\left(\mathtt{k}\cdot\mathtt{B}\right) \stackrel{!}{=} 0$.

only three independent conditions in (J'') on the B_ν !

☞ *For the polarisation tensor, can always have* $\varepsilon = 0$ *and* $\varepsilon_{\mu 0} = \varepsilon_{0\mu} = 0$

(b) count number of independent components of the polarisation tensor $\varepsilon_{\mu\nu}$:

a priori:	10 independent components
Lorenz gauge $k^\mu \varepsilon_{\mu\nu} = 0$	4 constraints
trace vanishes $\varepsilon = 0$	1 constraint
transverse gauge $\varepsilon_{\mu 0} = 0$	3 constraints
bilan: $10 - 4 - 1 - 3$	2 independent components

☞ *A gravitational wave has two possible polarisations*

Example: gravitational wave, propagating in z-direction

$\Rightarrow$ wave vector $\mathtt{k} = \frac{1}{c}\big(\omega, 0, 0, \omega\big)$

from transversality: $k^\mu \varepsilon_{\mu 3} = 0$ and $\varepsilon_{\mu 0} = \varepsilon_{0\mu} = 0 \Rightarrow \begin{cases} \omega \varepsilon_{\mu 3} = 0 \\ \varepsilon_{\mu 3} = \varepsilon_{3\mu} = 0 \end{cases}$

gives the metric

$$\bar{h}_{\mu\nu}(\mathtt{x}) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{\mathrm{i}\frac{\omega}{c}\left(z - ct\right)}$$

the amplitudes $h_+, h_\times$ correspond to the 2 possible polarisations

6.5 Effect on test masses

Example 1: a single free particle meets a gravitational wave

initially, particle at rest, with four-velocity $u^\mu(0) = \big(c, 0, 0, 0\big)$.

the effect of the passage of gravitational wave comes from the equation of motion

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\nu\lambda} u^\nu u^\lambda = 0$$

initially, only $u^0 \neq 0$ and $\boldsymbol{u} = \boldsymbol{0}$. Have for $\tau = 0$

$$\left.\frac{\mathrm{d}u^\mu}{\mathrm{d}\tau}\right|_{\tau=0} + \Gamma^\mu_{00} u^0 u^0 = 0 \implies \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} = 0 \qquad \text{for all } \tau$$

since $\Gamma^\mu_{00} = \frac{1}{2}\eta^{\mu\rho}\big(\underbrace{h_{\rho 0,0}}_{=0} + \underbrace{h_{0\rho,0}}_{=0} - \underbrace{h_{00,\rho}}_{=0}\big) = 0$ because all terms vanish.

$\Rightarrow$ particle remains stationary in its rest frame

☞ *Need at least two particles in order to detect gravitational waves*

N.B.: principle of equivalence: all gravitation can be absorbed into changes of coordinates !

Example 2:

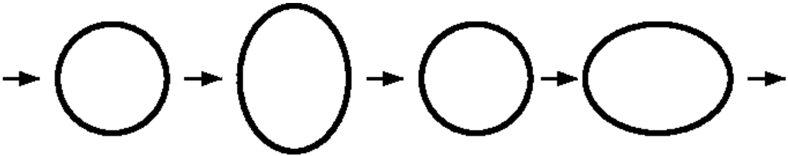
two particles, with distance $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$, meet a gravitational wave

wave propagates in z-direction, take polarisation state $h_+ = h_{11} = h_{11}(ct - z)$

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \left[1 + h_{11}(ct - z)\right]\mathrm{d}x^2 + \left[1 - h_{11}(ct - z)\right]\mathrm{d}y^2 + \mathrm{d}z^2$$

for simplicity, take an instant where $h_{11} > 0$

- two particles with same y-coordinate have spatial distance
  $\mathrm{d}s^2 = \left(1 + h_{11}\right)\mathrm{d}x^2 > \mathrm{d}x^2 \rightarrow$ will separate further
- two particles with same x-coordinate have spatial distance
  $\mathrm{d}s^2 = \left(1 - h_{11}\right)\mathrm{d}y^2 < \mathrm{d}y^2 \rightarrow$ will approach further



Example 3:
two particles, with distance $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$, meet a gravitational wave

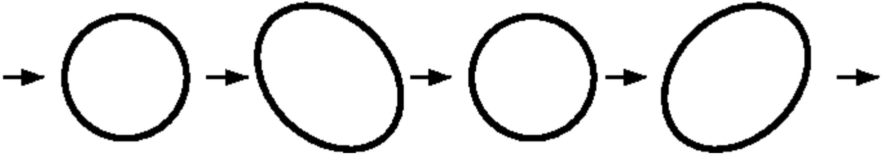
wave propagates in z-direction, take polarisation state $h_\times = h_{12} = h_{12}(ct - z)$

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + 2h_{12}(ct - z)\mathrm{d}x\mathrm{d}y + \mathrm{d}y^2 + \mathrm{d}z^2$$

rotate coordinates by 45^o: $\bar{x} = \frac{1}{\sqrt{2}}\left(x + y\right)$, $\bar{y} = \frac{1}{\sqrt{2}}\left(-x + y\right)$

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \left[1 + h_{12}(ct - z)\right]\mathrm{d}\bar{x}^2 + \left[1 - h_{12}(ct - z)\right]\mathrm{d}\bar{y}^2 + \mathrm{d}z^2$$

same kind of analysis as before



☞ *The passage of a gravitational wave leads to changes in the distance between two particles*

Quantitative Estimates

* since fields are very small indeed, newtonian description essentially enough
* two particles at positions $\boldsymbol{r}$ and $\boldsymbol{r}+\boldsymbol{s}$. The accelerations are

$$\frac{\mathrm{d}^2 r^i}{\mathrm{d}t^2} = -\nabla^i \phi(\boldsymbol{r}) \ , \quad \frac{\mathrm{d}^2 (r^i + s^i)}{\mathrm{d}t^2} = -\nabla^i \phi(\boldsymbol{r}+\boldsymbol{s})$$

with the expansion $\phi\big(\boldsymbol{r}+\boldsymbol{s}\big) \simeq \phi(\boldsymbol{r}) + \boldsymbol{s}\cdot\boldsymbol{\nabla}\phi(\boldsymbol{r}) + \ldots$, find for separation

$$\frac{\mathrm{d}^2 s^i}{\mathrm{d}t^2} = -\frac{\partial^2\phi}{\partial r^i \partial r^j} s^j \quad \Longrightarrow \quad \boxed{\left|\frac{\text{relative acceleration}}{\text{distance}}\right| = \frac{\partial^2\phi}{\partial r^i \partial r^j}}$$

Illustration: newtonian potential $\phi = -\frac{GM}{r}$ $\qquad r = |\boldsymbol{r}|$

$$\frac{\partial^2\phi}{\partial r^i \partial r^j} = -\frac{GM}{|\boldsymbol{r}|^3}\left(\delta_{ij} - 3\frac{r^i r^j}{|\boldsymbol{r}|^2}\right)$$

if 1^{st} particle on z-axis, $\boldsymbol{r} = (0,0,r)$, one has

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{pmatrix} s^x \\ s^y \\ s^z \end{pmatrix} = -\frac{GM}{|\boldsymbol{r}|^3}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}\begin{pmatrix} s^x \\ s^y \\ s^z \end{pmatrix}$$

☞ **tidal forces**: expansion in z-direction (longit.), contraction in xy-directions (transv.)

a gravitational wave propagating in z-direction generates the perturbation

$$\delta\phi = -f\,\frac{GM}{|\boldsymbol{r}|}\sin\big(kz-\omega t\big) \quad , \quad k=\frac{\omega}{c}$$

with f: relativistic correction factor $\Rightarrow \frac{\partial^2}{\partial z^2}\phi \simeq f\,\frac{GM}{|\boldsymbol{r}|c^2}\omega^2\sin\big(kz-\omega t\big)$
the relative amplitude of the position change

$$\frac{\delta s}{s} \sim f\,\frac{GM}{|\boldsymbol{r}|c^2} \qquad \text{detailed analysis: } f \sim \left(\frac{v}{c}\right)$$

possible sources: neutron stars, in ‘cosmic neighbourhood’
– until Virgo super-cluster of galaxies

$$|\boldsymbol{r}| \simeq 15[\mathrm{Mpc}] \simeq 50\cdot 10^6[\text{light years}] \quad , \quad f\sim 0.1 \quad , \quad M \simeq M_\odot$$

leads to $\boxed{\frac{\delta s}{s} \sim 10^{-21}}$ ☞ **extremely weak amplitudes** !

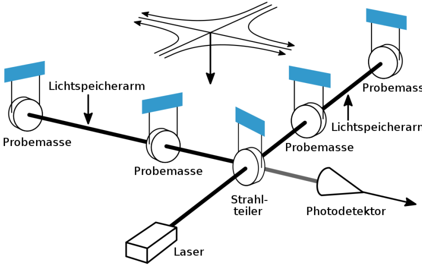
even for a distance $s \simeq 10[\mathrm{km}]$, have $\delta s \sim 10^{-17}[\mathrm{m}] \sim 0.01$ [nuclear radius]

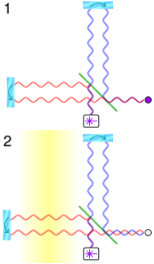
☞ *Direct detection of gravitational waves requires extraordinary efforts*

Detector of gravitational waves

main principle: interferometry

Source: https://en.wikipedia.org/wiki/Gravitational_wave





two suspended mirrors as test masses, arranged as Pérot-Fabry interferometer,

length of arm L light n times reflected, storage time $\Delta t_n = n \cdot \frac{L}{c}$

passage of a gravitational wave changes interference pattern

main task: eliminate all real and imaginable background noise !

☞ use at least two instruments in coïncidence

Instruments: LIGO (Hanford & Livingston (U.S.A.)) $L = 4[\mathrm{km}]$ > 1200 members

VIRGO (Cascina (Italy)), $L = 3[\mathrm{km}]$ > 550 members

☞ on 6^{th} of decembre 2020: **20** confirmed gravitational wave events, 52 candidates

for details, see https://en.wikipedia.org/wiki/List_of_gravitational_wave_observations

6.6 Energy flux of gravitational waves

- restrict to linear approximation
- gravitational waves move on minkowskian background
- gravitational waves also transport energy and momentum

$$g_{\mu\nu} = g^{(f)}_{\mu\nu} + h_{\mu\nu} \ , \ \ g^{(f)}_{\mu\nu} = \eta_{\mu\nu} + \mathrm{O}(h^2) \quad \text{`flat' background metric}$$

gives analogous decomposition of the Ricci tensor

$$R_{\mu\nu} = R^{(f)}_{\mu\nu} + \underbrace{R^{(1)}_{\mu\nu}}_{1^{\mathrm{st}}\,order} + \underbrace{R^{(2)}_{\mu\nu}}_{2^{\mathrm{nd}}\,order} + \dots$$

Vacuum field equation: $R_{\mu\nu} = 0$. At first order, have seen that $R^{(1)}_{\mu\nu} = 0$

$$R^{(f)}_{\mu\nu} + R^{(2)}_{\mu\nu} = 0$$

Both terms are of second order $\mathrm{O}\left(h^2\right)$. small curvature of background by grav. wave

The energy-momentum tensor $t_{\mu\nu}$ of the gravitational wave is given by

$$R^{(f)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(f)} = -\frac{8\pi G}{c^2}t_{\mu\nu} \implies \boxed{t_{\mu\nu} = \frac{c^2}{8\pi G}\left(R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(2)}\right)}$$

to be averaged over a volume large enough: locally, any time-space is flat in certain coordinates !

$$\boxed{t_{\mu\nu} = \frac{c^2}{8\pi G}\left(\left\langle R^{(2)}_{\mu\nu}\right\rangle - \frac{1}{2}\eta_{\mu\nu}\left\langle R^{(2)}\right\rangle\right)}$$

Example: linearly polarised wave (here h_+-state) propagating in z-direction

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1+\bar{h}_+ & 0 & \\ & 0 & 1-\bar{h}_+ & \\ & & & 1 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1-\bar{h}_+ & 0 & \\ & 0 & 1+\bar{h}_+ & \\ & & & 1 \end{pmatrix}$$

with $\bar{h}_+ = h_{0+}\cos\big(\omega(t - z/c)\big)$. The averaged Ricci tensor is

$$\left\langle R^{(2)}_{\mu\nu}\right\rangle = \left\langle \Gamma^{\alpha}_{\alpha\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\nu\alpha}\right\rangle$$

The non-vanishing Christoffel symbols: $\begin{cases} \Gamma^1_{10} = \Gamma^1_{01} = \quad \Gamma^0_{11} = \frac{1}{2}\left(\bar{h}_{+,0} - \bar{h}_+\bar{h}_{+,0}\right) \\ \Gamma^1_{13} = \Gamma^1_{31} = -\Gamma^3_{11} = -\frac{1}{2}\left(\bar{h}_{+,0} - \bar{h}_+\bar{h}_{+,0}\right) \end{cases}$.

$\Rightarrow R^{(2)}_{00} = R^{(2)}_{33} = \frac{1}{2}\left(\partial_0\bar{h}_+\right)$, $R^{(2)}_{11} = R^{(2)}_{22} = 0$ and $R^{(2)} = \eta^{\mu\nu}R^{(2)}_{\mu\nu} = 0$.
energy density of gravitational plane wave (contribution of $\bar{h}_+$ polarisation only)

$$t_{00} = \frac{c^2}{16\pi G}\left\langle\left(\partial_0\bar{h}_+\right)^2\right\rangle$$

to be averaged over a volume large enough: locally, any time-space is flat in certain coordinates !

$$\boxed{t_{\mu\nu} = \frac{c^2}{8\pi G}\left(\left\langle R^{(2)}_{\mu\nu}\right\rangle - \frac{1}{2}\eta_{\mu\nu}\left\langle R^{(2)}\right\rangle\right)}$$

Example: linearly polarised wave (here h_+-state) propagating in z-direction

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1+\bar{h}_+ & 0 & \\ & 0 & 1-\bar{h}_+ & \\ & & & 1 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1-\bar{h}_+ & 0 & \\ & 0 & 1+\bar{h}_+ & \\ & & & 1 \end{pmatrix}$$

with $\bar{h}_+ = h_{+0}\cos\big(\omega(t - z/c)\big)$. The averaged Ricci tensor is

$$\left\langle R^{(2)}_{\mu\nu}\right\rangle = \left\langle \Gamma^{\alpha}_{\alpha\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\nu\alpha}\right\rangle$$

The non-vanishing Christoffel symbols: $\begin{cases} \Gamma^1_{10} = \Gamma^1_{01} = \quad \Gamma^0_{11} = \frac{1}{2}\left(\bar{h}_{+,0} - \bar{h}_+\bar{h}_{+,0}\right) \\ \Gamma^1_{13} = \Gamma^1_{31} = -\Gamma^3_{11} = -\frac{1}{2}\left(\bar{h}_{+,0} - \bar{h}_+\bar{h}_{+,0}\right) \end{cases}$.

$\Rightarrow R^{(2)}_{00} = R^{(2)}_{33} = \frac{1}{2}\left(\partial_0\bar{h}_+\right)$, $R^{(2)}_{11} = R^{(2)}_{22} = 0$ and $R^{(2)} = \eta^{\mu\nu}R^{(2)}_{\mu\nu} = 0$.

energy density of gravitational plane wave (contributions of both $\bar{h}_+$ and $\bar{h}_\times$ polarisations)

$$t_{00} = \frac{c^2}{16\pi G}\left\langle \left(\partial_0\bar{h}_+\right)^2 + \left(\partial_0\bar{h}_\times\right)^2\right\rangle$$

to be averaged over a volume large enough: locally, any time-space is flat in certain coordinates !

$$\boxed{t_{\mu\nu} = \frac{c^2}{8\pi G}\left(\left\langle R^{(2)}_{\mu\nu}\right\rangle - \frac{1}{2}\eta_{\mu\nu}\left\langle R^{(2)}\right\rangle\right)}$$

Example: linearly polarised wave (here h_+-state) propagating in z-direction

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1+\bar{h}_+ & 0 & \\ & 0 & 1-\bar{h}_+ & \\ & & & 1 \end{pmatrix} \quad , \quad g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1-\bar{h}_+ & 0 & \\ & 0 & 1+\bar{h}_+ & \\ & & & 1 \end{pmatrix}$$

with $\bar{h}_+ = h_{+0}\cos\big(\omega(t - z/c)\big)$. The averaged Ricci tensor is

$$\left\langle R^{(2)}_{\mu\nu}\right\rangle = \left\langle \Gamma^{\alpha}_{\alpha\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\nu\alpha}\right\rangle$$

The non-vanishing Christoffel symbols: $\begin{cases} \Gamma^1_{10} = \Gamma^1_{01} = \Gamma^0_{11} = \frac{1}{2}\left(\bar{h}_{+,0} - \bar{h}_+\bar{h}_{+,0}\right) \\ \Gamma^1_{13} = \Gamma^1_{31} = -\Gamma^3_{11} = -\frac{1}{2}\left(\bar{h}_{+,0} - \bar{h}_+\bar{h}_{+,0}\right) \end{cases}$.

$\Rightarrow R^{(2)}_{00} = R^{(2)}_{33} = \frac{1}{2}\left(\partial_0\bar{h}_+\right)$, $R^{(2)}_{11} = R^{(2)}_{22} = 0$ and $R^{(2)} = \eta^{\mu\nu}R^{(2)}_{\mu\nu} = 0$.

energy density of gravitational plane wave (with $\bar{h}_+ = h_{11} = -h_{22}$ and $\bar{h}_\times = h_{12} = h_{21}$)

$$t_{00} = \frac{c^2}{16\pi G}\left\langle \left(\partial_0\bar{h}_+\right)^2 + \left(\partial_0\bar{h}_\times\right)^2\right\rangle = \frac{c^2}{32\pi G}\left\langle \frac{\partial h_{ij}}{\partial t}\frac{\partial h_{ij}}{\partial t}\right\rangle$$

transverse traceless gauge

to be averaged over a volume large enough: locally, any time-space is flat in certain coordinates !

$$t_{\mu\nu} = \frac{c^2}{8\pi G}\left(\left\langle R^{(2)}_{\mu\nu}\right\rangle - \frac{1}{2}\eta_{\mu\nu}\left\langle R^{(2)}\right\rangle\right)$$

Example: linearly polarised wave (here h_+-state) propagating in z-direction

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1+\bar{h}_+ & 0 & \\ & 0 & 1-\bar{h}_+ & \\ & & & 1 \end{pmatrix} \ , \ g^{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1-\bar{h}_+ & 0 & \\ & 0 & 1+\bar{h}_+ & \\ & & & 1 \end{pmatrix}$$

with $\bar{h}_+ = h_{+0}\cos\big(\omega(t - z/c)\big)$. The averaged Ricci tensor is

$$\left\langle R^{(2)}_{\mu\nu}\right\rangle = \left\langle \Gamma^{\alpha}_{\alpha\lambda}\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\nu\alpha}\right\rangle$$

The non-vanishing Christoffel symbols: $\begin{cases} \Gamma^1_{10} = \Gamma^1_{01} = \quad \Gamma^0_{11} = \frac{1}{2}\left(\bar{h}_{+,0} - \bar{h}_+\bar{h}_{+,0}\right) \\ \Gamma^1_{13} = \Gamma^1_{31} = -\Gamma^3_{11} = -\frac{1}{2}\left(\bar{h}_{+,0} - \bar{h}_+\bar{h}_{+,0}\right) \end{cases}$.

$\Rightarrow R^{(2)}_{00} = R^{(2)}_{33} = \frac{1}{2}\left(\partial_0\bar{h}_+\right)$, $R^{(2)}_{11} = R^{(2)}_{22} = 0$ and $R^{(2)} = \eta^{\mu\nu}R^{(2)}_{\mu\nu} = 0$.

energy density of gravitational plane wave (with $\bar{h}_+ = h_{11} = -h_{22}$ and $\bar{h}_\times = h_{12} = h_{21}$)

$$t_{00} = \frac{c^2}{32\pi G}\left\langle \frac{\partial h_{ij}}{\partial t}\frac{\partial h_{ij}}{\partial t}\right\rangle \Rightarrow \text{energy flux} \ \ f := c\, t_{00} = \frac{c^3}{32\pi G}\left\langle \frac{\partial h_{ij}}{\partial t}\frac{\partial h_{ij}}{\partial t}\right\rangle$$

analogue of electromagnetic Poynting vector

transverse traceless gauge

6.7 Radiation of a rotating binary source

rappel: recall the retarded wave, for $|\boldsymbol{r}| \gg |\boldsymbol{r}'|$

$$h_{\mu\nu}(t,\boldsymbol{r}) = \frac{4G}{c^2}\int \mathrm{d}\boldsymbol{r}'\,\frac{T_{\mu\nu}\left(t-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c},\boldsymbol{r}'\right)}{|\boldsymbol{r}-\boldsymbol{r}'|} \simeq \frac{4G}{c^2}\frac{1}{|\boldsymbol{r}|}\int \mathrm{d}\boldsymbol{r}'\,T_{\mu\nu}(\overbrace{t-\frac{|\boldsymbol{r}|}{c}}^{=:t_r},\boldsymbol{r}')$$

for weak fields, conservation law ${T^{\mu\nu}}_{,\nu} = 0$:

(i) set $\mu = 0$ and derive by x^0: ${T^{00}}_{,00} = -\frac{\partial}{\partial x^0}\left(\frac{\partial T^{0i}}{\partial x^i}\right) = -\frac{\partial}{\partial x^i}\left(\frac{\partial T^{0i}}{\partial x^0}\right) = -{T^{0i}}_{,0i}$

(ii) set $\mu = k$: ${T^{k0}}_{,0} + {T^{kj}}_{,j} = 0$. $\Rightarrow$ Both together give

$$ {T^{00}}_{,00} = {T^{jk}}_{,jk} \qquad (*)$$

carry out the following transformation, $\Omega \subset \mathbb{R}^3$ + bound. cond. $\quad (\boldsymbol{r} = (x^1, x^2, x^3))$

$$\int_\Omega \mathrm{d}\boldsymbol{r}\, x^m x^n \frac{\partial^2 T^{jk}}{\partial x^j \partial x^k} = \underbrace{x^m x^n \frac{\partial T^{jk}}{\partial x^j}\bigg|_{\partial\Omega}}_{=0} - \int_\Omega \mathrm{d}\boldsymbol{r}\,\left({\delta^m}_k x^n + {\delta^n}_k x^m\right)\frac{\partial T^{jk}}{\partial x^j}$$

$$= -\int_\Omega \mathrm{d}\boldsymbol{r}\,\left(x^n \frac{\partial T^{jm}}{\partial x^j} + x^m \frac{\partial T^{jn}}{\partial x^j}\right)$$

$$= -\underbrace{\left(x^n T^{jm} + x^m T^{jn}\right)\Big|_{\partial\Omega}}_{=0} + \int_\Omega \mathrm{d}\boldsymbol{r}\,\left({\delta^j}_n T^{jm} + {\delta^j}_m T^{jn}\right) = 2\int_\Omega \mathrm{d}\boldsymbol{r}\, T^{mn}$$

together with (*) this gives

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int_\Omega \mathrm{d}\boldsymbol{r}\, x^m x^n T^{00} = 2\int_\Omega \mathrm{d}\boldsymbol{r}\ T^{mn}$$

for the retarded wave, at large distances

$$h^{mn}(t,\boldsymbol{r}) \simeq \frac{2G}{c^2}\frac{1}{r}2\int \mathrm{d}\boldsymbol{r}'\ T^{mn} = \frac{2G}{c^4}\frac{1}{r}\frac{\partial^2}{\partial t^2}\int \mathrm{d}\boldsymbol{r}'\ T^{00}x^m x^n$$

For the planned applications here, the NR limit is enough: T^{00} given by the mass density ρ.

$$\boxed{h^{mn}(t,\boldsymbol{r}) = \frac{2G}{c^4}\frac{1}{r}\ddot{I}^{mn}(t-\frac{r}{c}) \ ,\ \ I^{mn}(t) = \int \mathrm{d}\boldsymbol{r}\ \rho(t,\boldsymbol{r})x^m x^n}$$

Quadrupole formula for gravitational radiation, I^{mn}: **quadrupole moment**

☞ *Gravitational radiation is quadrupole radiation*

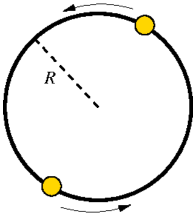
in contrast to the dipole electromagnetic radiation

Case study: two stars of equal mass M, in a circular orbit of radius R

angular frequency of orbit: $\omega = \frac{2\pi}{P} = \left(\frac{GM}{4R^3}\right)^{1/2}$ 3rd KEPLER's law

at time t the stars have the positions

$$(x, y, z) = \begin{cases} (R\cos\omega t, R\sin\omega t, 0) \\ (-R\cos\omega t, -R\sin\omega t, 0) \end{cases}$$



such that

$$I^{mn} = 2M\, x^m(t) x^n(t)$$

Example: $I^{11}(t) = 2MR^2\cos^2\omega t$

effective angular frequency 2ω after half a period have identical ⋆ configuration

$$I(t) = MR^2 \begin{pmatrix} 1+\cos 2\omega t & \sin 2\omega t & \\ \sin 2\omega t & 1-\cos 2\omega t & \\ & & 0 \end{pmatrix}, \quad \ddot{I}(t) = -4\omega^2 MR^2 \begin{pmatrix} \cos 2\omega t & \sin 2\omega t & \\ \sin 2\omega t & -\cos 2\omega t & \\ & & 0 \end{pmatrix}$$

and the metric reads already in transverse traceless gauge

$$h^{\mu\nu}(t, \boldsymbol{r}) = \frac{8MG}{c^4}\frac{R^2\omega^2}{r} \begin{pmatrix} 0 & & & \\ & \cos 2\omega t_r & \sin 2\omega t_r & \\ & \sin 2\omega t_r & -\cos 2\omega t_r & \\ & & & 0 \end{pmatrix} \quad ; \quad t_r = t - r/c$$

N.B.: only describes emission in z-direction next, describe emission in x-direction:

require: $h_{11} = h_{12} = 0 \Rightarrow$ only $h_{22} \neq 0$ remains $\Rightarrow h_{\mu\nu}$ not traceless

formal trick to solve this: if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\mathrm{tr}\, M = a + d$. Now

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \frac{a+d}{2}\mathbf{1} + \frac{a+d}{2}\mathbf{1} = \underbrace{\begin{pmatrix} \frac{a-d}{2} & b \\ c & -\frac{a-d}{2} \end{pmatrix}} + \underbrace{\begin{pmatrix} \frac{a+d}{2} & 0 \\ 0 & \frac{a+d}{2} \end{pmatrix}}$$

- blue term is in transverse traceless gauge;
- red term does not contribute to gravitational field

$\Rightarrow$ for emission in x-direction, have

$$h^{\mu\nu}(t, \boldsymbol{r}) = \frac{8MG}{c^4} \frac{R^2\omega^2}{r} \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \cos 2\omega t_r & 0 \\ & & 0 & -\cos 2\omega t_r \end{pmatrix} \quad ; \qquad t_r = t - r/c$$

* with these metrics, find the energy density t_{00} of a gravitational wave

if emission in z-direction $\qquad t_{00}^{(z)} = \frac{c^2}{16\pi G}\left[(h_{+,0})^2 + (h_{\times,0})^2\right]$

can now put everything together:

(a) flux in z-direction

have $h_+ \sim \cos 2\omega t_r$, $h_\times \sim \sin 2\omega t_r$ and $h^2_{+,0} + h^2_{\times,0} \sim 1$

$$f_z = ct^{(z)}_{00} = \frac{c^3}{16\pi G}\left(\frac{8MG}{c^4}\right)^4\left(\frac{R^2\omega^2}{r}\right)^2 4\omega^4 = \frac{16G}{\pi c^5}M^2R^4\frac{\omega^6}{r}$$

(b) flux in x-direction

have only terms $\cos 2\omega t_r$

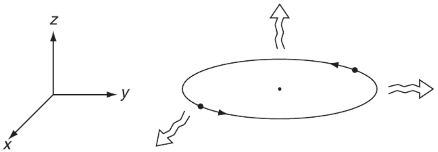
$\Rightarrow$ gives oscillating signal, best perform average over at least one period

$$\bar{f}_x = c\overline{t^{(x)}_{00}} = \frac{2G}{\pi c^5}M^2R^4\frac{\omega^6}{r} = \frac{1}{8}f_z$$

☞ *highly anisotropic emission*

Source: L. Ryder, *General Relativity* (2009)

* flux decreases as $1/r$ and **not** as $1/r^2$!



Orbiting stars in the xy plane. Radiation is emitted in all directions, but not with equal strength.

6.8 Radiated energy

energy density of a gravitational wave, via the quadrupole moment

$$t_{00} = \frac{G}{8\pi c^6} \frac{1}{r^2} \langle \dddot{I}_{mn} \dddot{I}^{mn} \rangle$$

where the average is over at least one period

⚠ It is assumed implicitly here that the tensor I^{mn} is transverse traceless !
otherwise $I^{mn} \mapsto I^{mn} - \frac{1}{3}\delta^{mn} I^k{}_k$

Then the total energy emitted is

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = \int_{S^d} \mathrm{d}\Omega\, r^2\, c t_{00} = \frac{G}{8\pi c^5} \int_{S^d} \mathrm{d}\Omega\, \langle \dddot{I}_{mn} \dddot{I}^{mn} \rangle = \frac{G}{5c^5} \langle \dddot{I}_{mn} \dddot{I}^{mn} \rangle}$$

* only third derivative $\dddot{I}^{mn}$ enters into the energy dissipation
* numerically very small pre-factor $G\, c^{-5}$

☞ *even in strong fields the gravitational radiation will be very weak*

much more weak than electromagnetic radiation

Case study: two stars of equal mass M, in a circular orbit of radius R

had already found the quadrupole tensor $\Rightarrow \langle \dddot{I}_{mn} \dddot{I}^{mn} \rangle = 128 M^2 R^4 \omega^6$

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{128G}{5c^5} M^2 R^4 \omega^6 = \frac{2G^4}{5c^5} \frac{M^5}{R^5} = \frac{1}{80} \frac{c^5}{G} \left(\frac{\mathscr{R}}{R} \right)^5$$

☞ **!** *a binary star system radiates gravitational waves* **!**

effect notable only if orbit radius is very close to Schwarzschild radius of binary system

total energy of binary system in circular orbit & angular frequency (3rd Kepler)

$$E = 2Mc^2 - \frac{GM^2}{4R} \ , \ \ \omega = \left(\frac{GM}{4R^3} \right)^{1/2}$$

energy loss $\Rightarrow$ reduction of orbital radius R and reduction of period P

$$\frac{\mathrm{d}P}{P} = \frac{3}{2} \frac{\mathrm{d}R}{R} = -\frac{3}{2} \frac{\mathrm{d}E}{E} = -\frac{48\pi}{5\sqrt{32}} \left(\frac{\mathscr{R}}{R} \right)^{5/2}$$

This is a prediction for a **strong** gravitational field !

Binary pulsar PSR B1913+16

neutron stars emit very regular pulses (radio waves, $59[\mathrm{ms}]$) ☞ **pulsar**

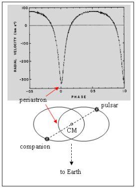
– one of the best clocks available

Source: `https://physicsfromplanetearth.wordpress.com/2016/04/19/gravitational-radiation-1/`

1974 HULSE & TAYLOR find yet another pulsar

but also observe unexplained shift in pulse frequency

explained as Doppler shift caused by companion ⋆



☞ very close binary system: $a = 1.950100 \cdot 10^6 [\mathrm{km}] \simeq 2.8 R_\odot$

two neutron stars on a highly elliptic orbit $e = 0.6171334$

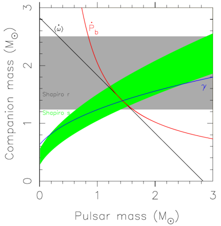
$M_1 = 1.438 \pm 0.001 M_\odot$, $M_2 = 1.390 \pm 0.001 M_\odot$

☞ very precise astronomical characterisation of this binary pulsar companion invisible

☞ can repeat all classical tests of general relativity
for **strong** fields, including

- shift of periastron
- Shapiro time delay

lead to consistent constraints on masses $M_{1,2}$ of binary



1978 change of period $\dot{P}_{\mathrm{obs}} = -\left(2.40263 \pm 0.00005\right) \cdot 10^{-12} < 0$

J.M. Weisberg, Y. Huang, ApJ **829**, 55 (2016)

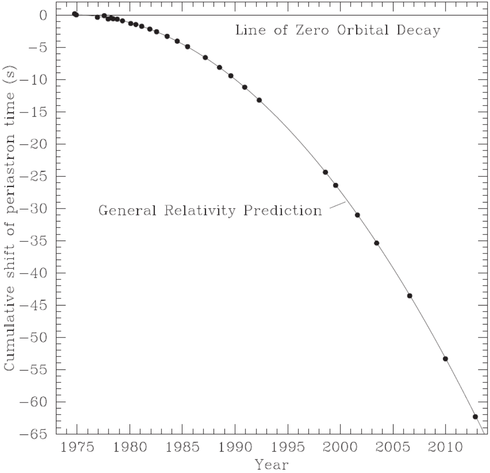


Figure 3. Orbital decay of PSR B1913+16 as a function of time. The curve represents the orbital phase shift expected from gravitational wave emission according to General Relativity. The points, with error bars too small to show, represent our measurements.

Comparison of the change $\dot{P}$ in the orbital period observed in the binary pulsar PSR B1913+16 with the prediction of general relativity, over > 35 years.

Quantitatively, $\boxed{\frac{\dot{P}_{\mathrm{obs}}}{\dot{P}_{GR}} = 0.9983 \pm 0.0016}$. parameter-free prediction

PSR	$\dot{P}_b^{\rm intr}/\dot{P}_b^{\rm GR}$	References
J0348+0432	1.05 ± 0.18	Antoniadis et al. (2013)
J0737-3039	1.003 ± 0.014	Kramer et al. (2006)
J1141-6545	1.04 ± 0.06	Bhat et al. (2008)
B1534+12	0.91 ± 0.06	Stairs et al. (2002)
J1738+0333	0.94 ± 0.13	Freire et al. (2012)
J1756-2251	1.08 ± 0.03	Ferdman et al. (2014)
J1906+0746	1.01 ± 0.05[a]	van Leeuwen et al. (2015)
B1913+16	0.9983 ± 0.0016	This work
B2127+11C	1.00 ± 0.03	Jacoby et al. (2006)

Source: J.M. Weisberg, Y. Huang, ApJ **829**, 55 (2016)

main source of uncertainty: lack of precise knowledge of mouvement of matter in the galaxy

Similar observations have been carried out for several binary pulsars. This permits observational tests of general relativity for **strong fields**. Listed here are the measurements for $\dot{P}_{\rm obs}/\dot{P}_{\rm GR}$.

N.B.: for PSR J0737-3039, both neutrons stars are seen as pulsars $\rightarrow$ **double pulsar**

$\Rightarrow$ **firm conclusion**:

gravitational waves do exist, as predicted by General Relativity.

 frequency of emitted gravitational waves

here for $M_1 = M_2 = M$

$$f = \frac{2\omega}{2\pi} = \frac{1}{\pi}\left(\frac{GM}{4R^3}\right)^{1/2} = \frac{1}{\sqrt{8}\,\pi}\frac{c}{R}\left(\frac{\mathscr{R}}{R}\right)^{1/2}$$

as ⋆ ⋆ loose energy, orbital radius R decreases

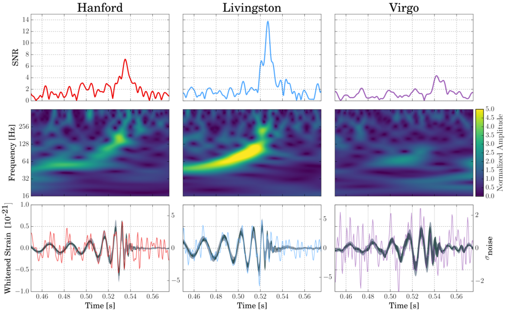
☞ ⋆ ⋆ will finally collide PSR B1913+16: time to collapse $\sim 300[\mathrm{My}]$

right before collision, frequency of gravitational wave will increase ☞ '*chirp*'

measured gravitational radiation from such a collision

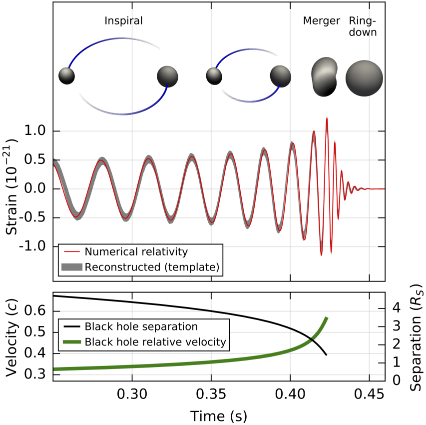
(usually two black holes)

both LIGO and VIRGO



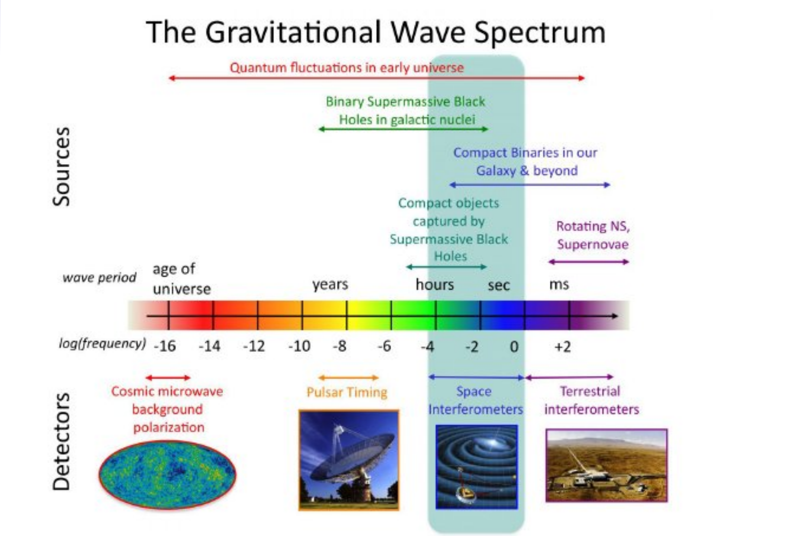
GW event 14th of August 2017

Source: `http://ligo.org/science/Publication-GW170814/index.php`



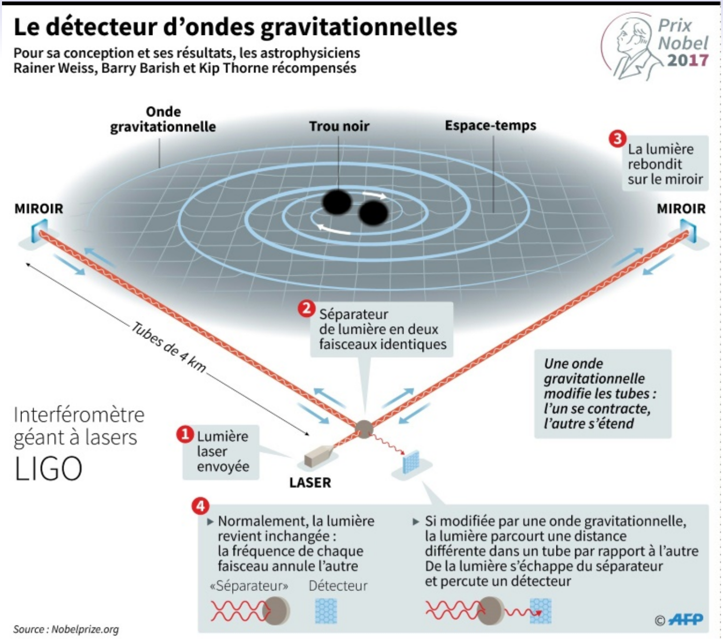
here is an illustrative reconstruction of the latest stages of the fusion of two black holes

Source: `https://ima.org.uk/7430/gravitational-waves-new-window-universe/`



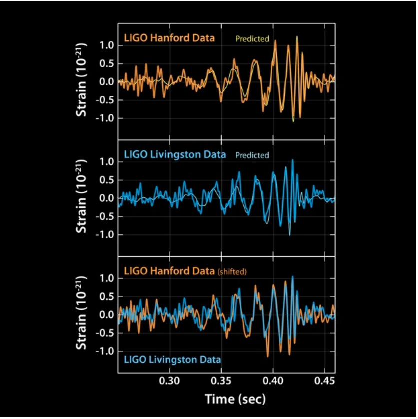
a new window into the universe: **gravitational waves**
(predicted 1916, indirect evidence 1974/75, first direct detection 2015/16)

Source: `https://en.wikipedia.org/wiki/Gravitational_wave`



https://www.sciencesetavenir.fr/sciences/les-detecteurs-d-ondes-gravitationnelles-qui-revolutionnent-la-physique_117440 the first observed event presented the 11^{th} of February 2016

simultaneously observed by LIGO Hanford & Livingston

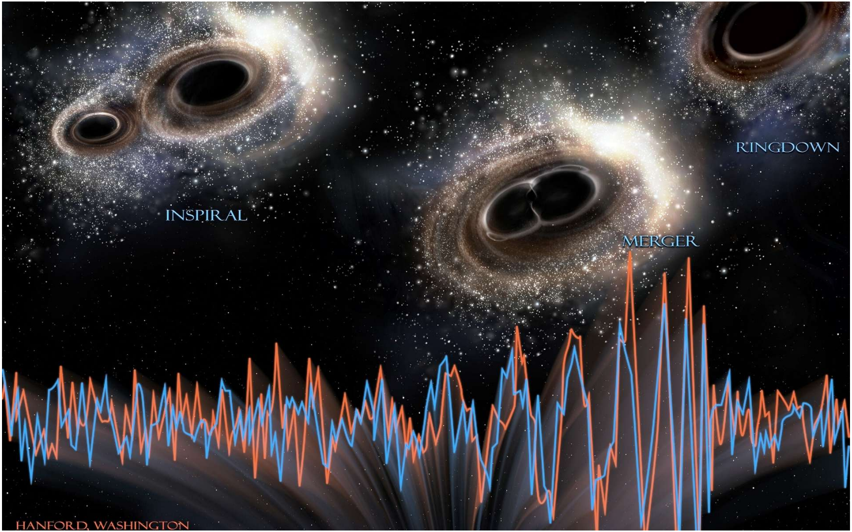


essential: equal & simultaneous signal form at two different places

notice the characteristic 'chirp'

https://www.ligo.org/detections/GW150914.php can be produced through fusion of compact objects

here: fusion of 2 black holes



Source: https://www.pigeonroost.net/gravitational-waves-a-new-window-to-the-universe/