Allgemeine Relativitätstheorie: Übungen & Lösungen

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Some further reading

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Oxford Univ. Press (2010)

S. Weinberg, *Gravitation and cosmology*, Wiley (1978)

C.M. Will, *Confrontation between general relativity and experiments*,
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Springer (1986)

Série 2

1. The invariant of Minkowski space reads, with the metric tensor $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$,

$$\mathrm{d}s^2 = \eta_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta = \overline{g}_{\mu\nu}\,\mathrm{d}\overline{x}^\mu\,\mathrm{d}\overline{x}^\nu \tag{1}$$

where the coordinates $\overline{x}^\mu$ come from a coordinate transformation $x^\mu \mapsto \overline{x}^\mu$. Give the metric tensor $\overline{g}_{\mu\nu}$ in the new coordinates.

Solution:
it is enough to write out the respective derivatives (use the expansion $\mathrm{d}x^\alpha = \frac{\partial x^\alpha}{\partial \bar{x}^\mu}\mathrm{d}\bar{x}^\mu$)

$$\begin{aligned}
\mathrm{d}s^2 &= \eta_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta \\
&= \eta_{\alpha\beta}\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\mathrm{d}\bar{x}^\mu \frac{\partial x^\beta}{\partial \bar{x}^\nu}\mathrm{d}\bar{x}^\nu \\
&= \underbrace{\left(\eta_{\alpha\beta}\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\nu}\right)}_{=:\bar{g}_{\mu\nu}} \mathrm{d}\bar{x}^\mu \mathrm{d}\bar{x}^\nu \\
&= \bar{g}_{\mu\nu}\mathrm{d}\bar{x}^\mu \mathrm{d}\bar{x}^\nu
\end{aligned}$$

and one can read off

$$\boxed{\bar{g}_{\mu\nu} = \eta_{\alpha\beta}\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\frac{\partial x^\beta}{\partial \bar{x}^\nu}}$$

2. Consider the determinant of the metric tensors $g := \det g_{\mu\nu}$. Is it Lorentz-invariant ?

Solution:

The metric tensor transforms as follows

$$\bar{g}_{\mu\nu} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu}$$

and this can be viewed as a product of matrices. Taking the determinant

$$\begin{aligned} \bar{g} &:= \det\big(\bar{g}_{\mu\nu}\big) \\ &= \det\big(g_{\alpha\beta}\big) \det\left(\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\right) \det\left(\frac{\partial x^\beta}{\partial \bar{x}^\nu}\right) \\ &= g \left[\det\left(\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\right)\right]^2 \end{aligned}$$

For a general transformation $x \mapsto \bar{x}$, g is invariant if and only if $\det\left(\frac{\partial x^\alpha}{\partial \bar{x}^\mu}\right) = 1$.

For linear transformations, $\Lambda^\alpha_\nu = \frac{\partial x^\alpha}{\partial \bar{x}^\mu}$ is a matrix with constant matrix elements. For a space rotation, $\det \Lambda = 1$ is well-known. For a Lorentz transformation (in x-direction)

$$\Lambda^\alpha_\nu = \begin{pmatrix} \gamma & \gamma v & & \\ \gamma v & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta & & \\ \sinh\theta & \cosh\theta & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \Rightarrow \det \Lambda = \begin{cases} \gamma^2 - \gamma^2 v^2 & = 1 \\ \cosh^2\theta - \sinh^2\theta & = 1 \end{cases}$$

$\Rightarrow$ g *is* Lorentz-invariant, but it is **not** invariant under general transformations.

3. Show that the invariant volume element of a four-dimensional space is given by

$$\mathrm{d}^4 V = \left(-g\right)^{1/2} \mathrm{d}^4\mathrm{x} = \left(-g\right)^{1/2} \mathrm{d}t\mathrm{d}x\mathrm{d}y\mathrm{d}z \qquad (2)$$

where $g := \det g_{\mu\nu}$ is the determinant of the metric tensor $g_{\mu\nu}$.

Solution:
Under the transformation $\mathrm{x} \to \bar{\mathrm{x}}$, the volume element is $\mathrm{d}^4\mathrm{x} = \det\left(\frac{\partial \mathrm{x}}{\partial \bar{\mathrm{x}}}\right)\mathrm{d}^4\bar{\mathrm{x}}$. One of the frames can be assumed to be Minkowski space with the metric tensor $\eta_{\mu\nu} = \mathrm{diag}\,(-1, 1, 1, 1)$. From the previous exercice

$$-\bar{g} = -\det\left(\bar{g}_{\alpha\beta}\right) = -\det\left(\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\frac{\partial x^\nu}{\partial \bar{x}^\beta}\eta_{\mu\nu}\right) = \left[\det\left(\frac{\partial \mathrm{x}}{\partial \bar{\mathrm{x}}}\right)\right]^2 (-\det\eta)$$

Hence $\left(-g\right)^{1/2} = \det\frac{\partial \mathrm{x}}{\partial \bar{\mathrm{x}}}$. In consequence

$$\mathrm{d}^4\bar{V} := \left(-\bar{g}\right)^{1/2}\mathrm{d}^4\bar{\mathrm{x}} = \det\left(\frac{\partial \bar{\mathrm{x}}}{\partial \mathrm{x}}\frac{\partial \mathrm{x}}{\partial \bar{\mathrm{x}}}\right)\left(-g\right)^{1/2}\mathrm{d}^4\mathrm{x} = \mathrm{d}^4 V$$

4. (a) Show that the *invariant* volume element of three-dimensional space, for an observer with the four-velocity $\mathtt{u}$ is given by

$$\mathrm{d}^3 V = (-g)^{1/2}\, u^0\, \mathrm{d}^3 x \tag{3}$$

(b) Write down the *invariant* volume element of the contra-variant momentum $\mathrm{d}^4\mathtt{p}$ in four-dimensional momentum space.
(c) Write down the *invariant* three-dimensional volume element in momentum space *"on the mass shell"*, that is with the constraint $\sqrt{-\mathtt{p}\cdot\mathtt{p}} = m$.

Solution:
(a) at rest, one has clearly $\mathrm{d}^3 V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$. One wants a scalar which to reduces to this at rest.
Start from $\mathrm{d}^4 V$, and introduce the component $u^0 = \frac{\mathrm{d}t}{\mathrm{d}\tau}$ of the four-velocity u, where τ is proper time.

$$\mathrm{d}^4 V = \left(-g\right)^{1/2}\mathrm{d}t\mathrm{d}x\mathrm{d}y\mathrm{d}z = \left(-g\right)^{1/2}\mathrm{d}t\mathrm{d}x\mathrm{d}y\mathrm{d}z\frac{u^0}{u^0} = \left(-g\right)^{1/2}u^0\mathrm{d}\tau\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Since $\mathrm{d}^4 V$ and $\mathrm{d}\tau$ are scalars, $\mathrm{d}^3 V := \left(-g\right)^{1/2}u^0\mathrm{d}x\mathrm{d}y\mathrm{d}z$ must be scalar as well.
(b) The four-momentum $\mathrm{p} = (P^0, \boldsymbol{P})$ transforms as a four-vector. The invariant volume element is

$$\mathrm{d}^4 p = \left(-g\right)^{1/2}\mathrm{d}P^0\mathrm{d}P^x\mathrm{d}P^y\mathrm{d}P^z$$

(c) One has the extra constraint $(-\mathrm{p}\cdot\mathrm{p}) = m$. This gives the invariant $3D$ element

$$\mathrm{d}^3 p = \int (-g)^{1/2}\, \mathrm{d}P^0 \mathrm{d}P^x \mathrm{d}P^y \mathrm{d}P^z\, \delta\left(\left(-g_{\alpha\beta} P^\alpha P^\beta\right)^{1/2} - m \right)$$

From the theory of distributions (see e.g Gelfand & Shilov, *Generalised Functions, Vol. 1*) one recalls the identity $\int \mathrm{d}x\, \delta(f(x)) = \sum_{x_0} \frac{1}{|f'(x_0)|}$, where x_0 runs over all zeros of $f(x)$, that is $f(x_0) = 0$.

With the help of this, one eliminates the integration over P^0 and finds

$$\begin{aligned} \mathrm{d}^3 p &= (-g)^{1/2}\, \mathrm{d}P^x \mathrm{d}P^y \mathrm{d}P^z \left[-\frac{1}{2} \left(-g_{\alpha\beta} P^\alpha P^\beta\right)^{1/2} 2 g_{t\alpha} P^\alpha \right]^{-1} \\ &= (-g)^{1/2}\, \mathrm{d}P^x \mathrm{d}P^y \mathrm{d}P^z \left(\frac{m}{-P_0} \right) \end{aligned} \tag{4}$$

In the rest frame,this reduces indeed to $\mathrm{d}^3 p \to \mathrm{d}P^x \mathrm{d}P^y \mathrm{d}P^z$, as expected.

5. Relativistic electrodynamics is described by the field tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ where $\mathtt{A}$ is the four-vector-potential. On a test body with electric charge q then acts the Lorentz force (frz. *force de Laplace* (sic !)), with four-momentum $\mathtt{p} = m\mathtt{u}$ and proper time τ

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = qF^{\mu\nu} u_\nu \tag{5}$$

(a) Consider first the zeroth component (time component) $\mu = 0$ of the equation (5). Express it via the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ and show that

$$\frac{\mathrm{d}p^0}{\mathrm{d}t} = q\boldsymbol{v} \cdot \boldsymbol{E} \tag{6}$$

(b) Write the equation for $\mathrm{d}\boldsymbol{p}/\mathrm{d}t$, expressed via $\boldsymbol{E}$ and $\boldsymbol{B}$.
Hint: consider the space components of (5).
(c) A particle with electric charge q and mass m moves on a circle with radius R in an uniform magnetic field $\boldsymbol{B} = B\boldsymbol{e}_z$.
(i) Express B in terms of known quantities and the angular frequency ω.
(ii) In the rest system, why the magnetic field $\boldsymbol{B}$ cannot furnish work on the particle ? Was is the finding of an observer, who moves with the relative velocity $\beta\boldsymbol{e}_x$? Which velocity does he find, and in particular, which value of $u^{0'}$?
(iii) Determine $\mathrm{d}u^{0'}/\mathrm{d}\tau$ and hence also $\mathrm{d}p^{0'}/\mathrm{d}\tau$. Why can the energy of the particle change, although the magnetic field $\boldsymbol{B}$ does not furnish work ?

Solution:

(a) set $\mu = 0$ in eq. (5): $\frac{\mathrm{d}p^0}{\mathrm{d}\tau} = qF^{0\nu}u_\nu = qE^i\gamma v_i$, with $i = 1, 2, 3$. Because of $\mathrm{d}\tau = \mathrm{d}t/\gamma$, this gives indeed eq. (6).

almost identical at the non-relativistic form, but p^0 also contains the rest energy.

(b) this is worked out directly

$$\frac{\mathrm{d}p^i}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}p^i}{\mathrm{d}t} = qF^{i\nu}u_\nu = qF^{i0}u_0 + qF^{ij}u_j = q\gamma E^i + q\gamma\varepsilon^{ijk}B_k v_j$$

which is indeed the Lorentz force $\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = q\big(\boldsymbol{E} + \boldsymbol{v}\wedge\boldsymbol{B}\big)$.

(c) (i) from the Lorentz force $\omega|\boldsymbol{p}| = \left|\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}\right| = q|\boldsymbol{v}||\boldsymbol{B}|$. Then $B = |\boldsymbol{B}| = \frac{\omega}{q}\frac{|\boldsymbol{p}|}{|\boldsymbol{v}|} = \frac{m\omega}{q\sqrt{1-v^2}} = \frac{m\omega}{q\sqrt{1-\omega^2R^2}}$ (having set $c = 1$).

(ii) in (6), the change of the energy p^0 does not depend on B, hence $p^0 = \mathrm{cste.}$. No work is neither furnished, nor gained.

In the frame of the laboratory, the components of the four-velocity are

$$u^0 = \left(1-\omega^2R^2\right)^{-1/2}, \ u^x = \frac{\omega y}{\sqrt{1-\omega^2R^2}}, \ u^y = -\frac{\omega x}{\sqrt{1-\omega^2R^2}}$$

On the other hand, for an observer with relative velocity $\beta \boldsymbol{e}_x$, one finds from a Lorentz transformation

$$u'^0 = \gamma\big(u^0 - \beta u^x\big) = \gamma\big(1 - \beta\omega y\big)\big(1 - \omega^2 R^2\big)^{-1/2} \ , \ \gamma = \big(1 - \beta^2\big)^{-1/2} \quad (*)$$

(iii) we have $\frac{\mathrm{d}p'^0}{\mathrm{d}\tau} = m\frac{\mathrm{d}u'^0}{\mathrm{d}\tau} = -\frac{m\omega\gamma\beta u^y}{\sqrt{1-\omega^2 R^2}} \neq 0$.

No contradiction, since the electric/magnetic fields transform as follows

$$\begin{aligned} E'^y &= F'^{02} = \Lambda^0_\mu \Lambda^2_\nu F^{\mu\nu} \\ &= \Lambda^0_0 \Lambda^2_\nu F^{0\nu} + \Lambda^0_1 \Lambda^2_\nu F^{1\nu} \ = \ \Lambda^0_0 \Lambda^2_2 F^{02} + \Lambda^0_1 \Lambda^2_2 F^{12} \\ &= \gamma \cdot 1 \cdot E^y + \big(-\gamma\beta\big) \cdot 1 \cdot B^z \end{aligned}$$

The electric field $\boldsymbol{E}$ does not at all transform as a vector.

If $\boldsymbol{E} = \boldsymbol{0}$, one has $E'^y = -\gamma\beta B^z$. From (5), one expects

$$\frac{\mathrm{d}p'^0}{\mathrm{d}\tau} = qE'^y u'^y = -\frac{m\omega\gamma\beta u'^y}{\sqrt{1-\omega^2 R^2}} = -\frac{m\omega\gamma\beta u^y}{\sqrt{1-\omega^2 R^2}}$$

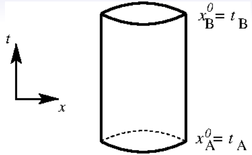
in perfect agreement with (*) above.

The electric field created by Lorentz-transforming the magnetic field $\boldsymbol{B}$ furnishes the work.

6. A vector field $J^\alpha(\mathtt{x})$ satisfies the continuity equation (conservation law) $\partial_\alpha J^\alpha = 0$ and for large distances $r = |\boldsymbol{r}| \to \infty$ it falls off faster than r^{-2}.
(a) Show that $Q := \int \mathrm{d}^3x \, J^0$ is constant in time.
(b) Show that Q is a Lorentz scalar, that is $\int \mathrm{d}^3x \, J^0 = \int \mathrm{d}^3x' \, J^{0'}$.

Therefore, Q is called the **conserved charge** of the conserved four-current J^α.

Solution:



(a) take a domain Ω bounded in time by x_A^0 below and x_B^0 above and with spatial sides far from the origin

using Gauss's theorem

$$
\begin{aligned}
0 &= \int_\Omega \mathrm{d}^4 V \partial_\alpha J^\alpha = \int_\Omega \mathrm{d}t\, \partial_\alpha J^\alpha \mathrm{d}x\mathrm{d}y\mathrm{d}z \\
&= \int_{t_A} J^\alpha \mathrm{d}^3\Sigma^\alpha + \int_{t_B} J^\alpha \mathrm{d}^3\Sigma^\alpha \\
&= \int_{t_B} J^0 \mathrm{d}x\mathrm{d}y\mathrm{d}z - \int_{t_A} J^0 \mathrm{d}x\mathrm{d}y\mathrm{d}z \;=\; Q(t_B) - Q(t_A)
\end{aligned}
$$

$\mathrm{d}\Sigma^\alpha$ is the surface element, oriented *normal* to the surface

- for the only surface here in finite distances, in time-direction

(b) write the charge as $Q = \int \mathrm{d}^4 x\, J^\alpha \partial_\alpha \Theta(n_\beta x^\beta)$, where $n_0 = 1$, $n_1 = n_2 = n_3 = 0$ and $\Theta(u) = \begin{cases} 1 & \text{if } u > 0 \\ 0 & \text{if } u \leq 0 \end{cases}$. To see this, note that Q only contains Lorentz-invariant quantities. It is enough to check it at rest.

$$Q = \int \mathrm{d}^4x \, J^0 \partial_{x^0} \Theta(n_\beta x^\beta) = \int \mathrm{d}^4x \, J^0(x)\delta(x^0) = \int \mathrm{d}^3x J^0(0, \boldsymbol{x}) = Q(0)$$

that is $Q = Q(t) = Q(0)$ is time-independent.
Under a Lorentz transform $Q \mapsto Q' = \int \mathrm{d}^4x \, J^\alpha \partial_\alpha \Theta(n'_\beta x'^\beta)$, with $n'_\beta = \Lambda^\gamma_\beta n_\gamma$.
Hence

$$Q' - Q = \int \mathrm{d}^4x \, \partial_\alpha \Big(J^\alpha(x) \big(\Theta(n'_\beta x'^\beta) - \Theta(n_\beta x^\beta) \big) \Big)$$

Since one knows that (i) $J^\alpha(x) \to 0$ if $|\boldsymbol{x}| \to \infty$ fast enough and (ii) $\Theta(n'_\beta x'^\beta) - \Theta(n_\beta x^\beta) \to 0$ if $|t| \to \infty$, one can again apply Gauss's theorem in $4D$ and express $Q' - Q$ as surface integrals.
This implies $Q' - Q = 0$, hence Q is scalar.

7. Show that the two-dimensional space with the metric

$$\mathrm{d}s^2 = \mathrm{d}v^2 - v^2 \mathrm{d}u^2 \tag{7}$$

is identical to the flat two-dimensional Minkowski-space with the metric $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2$.
Hint: find a coordinate transformation $t = t(v, u)$ and $x = x(v, u)$ which sends the Minkowski metric into the metric (7).

Also show that for a non-accelerated particle the contra-variant component p_u of the 'four-momentum' p is constant. Is this also true for the component p_v ?

Solution:
one might use the analogy with polar coordinates as inspiration

make the ansatz $t = v \sinh u$, $x = v \cosh u$, hence $x^2 - t^2 = v^2$ and $x/t = \coth u$.

$$\begin{aligned} \mathrm{d}t &= \mathrm{d}v \sinh u + \mathrm{d}u \, v \cosh u \\ \mathrm{d}x &= \mathrm{d}v \cosh u + \mathrm{d}u \, v \sinh u \end{aligned}$$

and furthermore $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 = \mathrm{d}v^2 - v^2 \mathrm{d}u^2$. Inverting the above infinitesimal transformation gives
$\mathrm{d}v = \mathrm{d}x \cosh u - \mathrm{d}t \sinh u$ and $\mathrm{d}u = v^{-1}\big(\mathrm{d}t \cosh u - \mathrm{d}x \sinh u\big)$. Next,

$$p_u = g_{uu} p^u = -mv^2 \frac{\mathrm{d}u}{\mathrm{d}\tau} = -mv \cosh u \frac{\mathrm{d}t}{\mathrm{d}\tau} + mv \sinh u \frac{\mathrm{d}x}{\mathrm{d}\tau} = -mx \frac{\mathrm{d}t}{\mathrm{d}\tau} + mt \frac{\mathrm{d}x}{\mathrm{d}\tau}$$

Non-accelerated particle: $x(t) = x_0 + \frac{\mathrm{d}x}{\mathrm{d}t} t$, $\frac{\mathrm{d}t}{\mathrm{d}\tau} = \mathrm{cste.}$, $\frac{\mathrm{d}x}{\mathrm{d}\tau} = \mathrm{cste.}$. Hence
$p_u = -m \frac{\mathrm{d}t}{\mathrm{d}\tau} x_0 = \mathrm{cste.}$, as claimed.
Since $-m^2 = \mathtt{p} \cdot \mathtt{p} = g^{vv} \big(p_v\big)^2 + g^{uu} \big(p_u\big)^2 = \big(p_v\big)^2 - \frac{1}{v^2} \big(p_u\big)^2 \Rightarrow p_v \neq \mathrm{cste.}$.

8. Show that the metric of the surface of the three-dimensional sphere S^3 embedded into $4D$ euclidean space reads:

$$\mathrm{d}s^2 = R^2 \left[\mathrm{d}\alpha^2 + \sin^2\alpha \left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right] \quad (8)$$

$$(R \text{ is the constant radius of the sphere})$$

<u>**Hint:**</u> how would you formulate $4D$ spherical coordinates ?

Solution:
a sphere S^3 with radius R is given by $x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2$. Then introduce the coordinates

$$
\begin{aligned}
x_4 &= R\cos\alpha \\
x_3 &= R\sin\alpha\cos\theta \\
x_2 &= R\sin\alpha\sin\theta\cos\phi \\
x_1 &= R\sin\alpha\sin\theta\sin\phi
\end{aligned}
$$

In cartesian coordinates, the metric is $\mathrm{d}s^2 = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2$ and reproducing (8) is straightforward.

start with $\mathrm{d}x_4 = -R\sin\alpha\mathrm{d}\alpha$ etc.

9. Hyperboloide haben die folgende Parameterdarstellung im dreidimensionalen Raum $\mathbb{R}^3$

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a\sqrt{s^2+d}\ \cos\varphi \\ b\sqrt{s^2+d}\ \sin\varphi \\ c\,s \end{pmatrix} \quad \text{so daß} \quad \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = d$$

wobei a, b, c Konstante sind und $d = \pm 1$. Für $d = +1$ hat man ein **einschaliges Hyperbolid** (**hyperboloïde à une nappe**) H_1 und für $d = -1$ ein **zweischaliges Hyperbolid** (**hyperboloïde à deux nappes**) H_2.
Für ein einschaliges Hyperbolid kann man wählen $s = \sinh\xi$ und für ein zweischaliges Hyperbolid $s = \cosh\xi$.

Geben Sie die Parameterdarstellung in beiden Fällen an und ebenfalls, welche geometrische Bedingung diese beiden Flächen erfüllen. Wie kann man diese geometrisch veranschaulichen ? Wie lautet die Metrik $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2$ (für $a = b = c$) und insbesondere der metrische Tensor in beiden Fällen ?

Solution:

(a) einschaliges Hyperbolid $d = +1$: setzt man $s = \sinh\xi$, so findet man

$$\boldsymbol{r} = \begin{pmatrix} a\,\cosh\xi\cos\varphi \\ b\,\cosh\xi\sin\varphi \\ c\,\sinh\xi \end{pmatrix},$$ was die Oberfläche $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = 1$ parametrisiert.

Es ist äquivalent, $a = b = c$ zu setzen und als Oberfläche zu nehmen $x^2 + y^2 - z^2 = a^2$. Die Parametrisierung verifiziert diese Oberfläche, weil

$$x^2 + y^2 - z^2 = a^2\left(\cosh^2\xi\cos^2\varphi + \cosh^2\xi\sin^2\varphi - \sinh^2\xi\right) = a^2\left(\cosh^2\xi - \sinh^2\xi\right) = a^2$$

Damit wird die Metrik

$$\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2 \\
&= \left(a\sinh\xi\cos\varphi\mathrm{d}\xi - a\cosh\xi\sin\varphi\mathrm{d}\varphi\right)^2 + \left(a\sinh\xi\cos\varphi\mathrm{d}\xi + a\cosh\xi\cos\varphi\mathrm{d}\varphi\right)^2 - \left(a\cosh\xi\mathrm{d}\xi\right)^2 \\
&= \left(a^2\sinh^2\xi\cos^2\varphi\mathrm{d}\xi^2 - 2a^2\cosh\xi\sinh\xi\cos\varphi\sin\varphi\,\mathrm{d}\xi\mathrm{d}\varphi + a^2\cosh^2\xi\sin^2\varphi\mathrm{d}\varphi^2\right) \\
&\quad + \left(a^2\sinh^2\xi\sin^2\varphi\mathrm{d}\xi^2 + 2a^2\cosh\xi\sinh\xi\cos\varphi\sin\varphi\mathrm{d}\xi\mathrm{d}\varphi + a^2\cosh^2\xi\cos^2\varphi\mathrm{d}\varphi^2\right) - a^2\cosh^2\xi\mathrm{d}\xi^2 \\
&= a^2\sinh^2\xi\,\mathrm{d}\xi^2 + a^2\cosh^2\xi\,\mathrm{d}\varphi^2 - a^2\cosh^2\xi\,\mathrm{d}\xi^2 \\
&= a^2\left(-\mathrm{d}\xi^2 + \cosh^2\xi\,\mathrm{d}\varphi^2\right)
\end{aligned}$$

Mit der Notation $(x^1, x^2) = (\xi, \varphi)$ hat man $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ mit dem metrischen Tensor $g_{\mu\nu} = \begin{pmatrix} -a^2 & 0 \\ 0 & a^2\cosh^2\xi \end{pmatrix}$.

(b) zweischaliges Hyperbolid $d=-1$: setzt man $s=\cosh\xi$, so findet man

$$\boldsymbol{r}=\begin{pmatrix} a\ \sinh\xi\cos\varphi \\ b\ \sinh\xi\sin\varphi \\ \pm c\ \cosh\xi \end{pmatrix}, \text{ was die Oberfläche } \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2-\left(\frac{z}{c}\right)^2=-1$$

parametrisiert. Es ist äquivalent, $a=b=c$ zu setzen und als Oberfläche zu nehmen $x^2+y^2-z^2=-a^2$. Die Parametrisierung verifiziert diese Oberfläche, weil

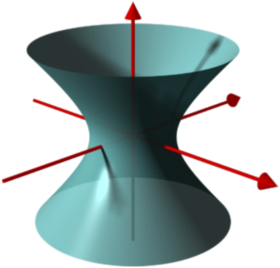
$$x^2+y^2-z^2=a^2\left(\sinh^2\xi\cos^2\varphi+\sinh^2\xi\sin^2\varphi-\cosh^2\xi\right)=a^2\left(-\cosh^2\xi+\sinh^2\xi\right)=-a^2$$

Damit wird die Metrik

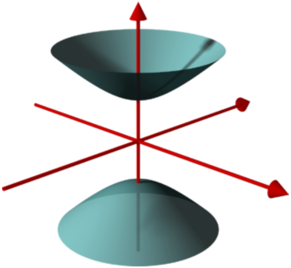
$$\begin{aligned}
\mathrm{d}s^2 &= \mathrm{d}x^2+\mathrm{d}y^2-\mathrm{d}z^2\\
&= \left(a\cosh\xi\cos\varphi\mathrm{d}\xi-a\sinh\xi\sin\varphi\mathrm{d}\varphi\right)^2+\left(a\cosh\xi\cos\varphi\mathrm{d}\xi+a\sinh\xi\cos\varphi\mathrm{d}\varphi\right)^2-\left(a\sinh\xi\mathrm{d}\xi\right)^2\\
&= \left(a^2\cosh^2\xi\cos^2\varphi\mathrm{d}\xi^2-2a^2\cosh\xi\sinh\xi\cos\varphi\sin\varphi\,\mathrm{d}\xi\mathrm{d}\varphi+a^2\sinh^2\xi\sin^2\varphi\mathrm{d}\varphi^2\right)\\
&\quad+\left(a^2\cosh^2\xi\sin^2\varphi\mathrm{d}\xi^2+2a^2\cosh\xi\sinh\xi\cos\varphi\sin\varphi\,\mathrm{d}\xi\mathrm{d}\varphi+a^2\sinh^2\xi\cos^2\varphi\mathrm{d}\varphi^2\right)-a^2\sinh^2\xi\mathrm{d}\xi^2\\
&= a^2\cosh^2\xi\,\mathrm{d}\xi^2+a^2\sinh^2\xi\,\mathrm{d}\varphi^2-a^2\sinh^2\xi\,\mathrm{d}\xi^2\\
&= a^2\left(\mathrm{d}\xi^2+\sinh^2\xi\,\mathrm{d}\varphi^2\right)
\end{aligned}$$

Mit der Notation $(x^1,x^2)=(\xi,\varphi)$ hat man $\mathrm{d}s^2=g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ mit dem metrischen Tensor $g_{\mu\nu}=\begin{pmatrix} a^2 & 0 \\ 0 & a^2\sinh^2\xi \end{pmatrix}$.

eine geometrische Vorstellung ergibt sich aus den Abbildungen:



einschalig/ une nappe



zweischalig/ deux nappes

Physikalische Deutung: falls man die (ausgezeichnete) z-Richtung als Zeitachse in einem Zeit-Raum-Diagramm interpretiert, so ist das zweischalige Hyperboloid eine Illustration des Lichtkegels der Viererimpulses eines massiven Teilchens.

Bildquelle: `https://de.wikipedia.org/wiki/Hyperboloid`

10. (a) In euclidean spaces the angle θ between two vectors $\boldsymbol{U}$ and $\boldsymbol{V}$ can be found from the scalar product, since $\cos\theta = \frac{\boldsymbol{U}\cdot\boldsymbol{V}}{|\boldsymbol{U}|\,|\boldsymbol{V}|}$. Consider more general spaces, with a metric tensor $g_{\mu\nu}$. How to define the angle between two vectors in such a case ?
(b) Consider **conformal transformations** $x^\mu \mapsto \overline{x}^\mu$, for which the metric tensor transforms as follows, by definition

$$g_{\alpha\beta} \mapsto f(\mathrm{x})g_{\alpha\beta} \tag{9}$$

where $f = f(\mathrm{x}) = f(x^\mu) \neq 0$ is an arbitrary (differentiable) function. Show that conformal transformations keep all angles invariant. How do light-like curves transform ?

Solution:
(a) the cosine θ between two vectors is *defined* as

$$\cos\theta := \frac{\boldsymbol{U}\cdot\boldsymbol{V}}{|\boldsymbol{U}||\boldsymbol{V}|} = \frac{g_{\mu\nu}U^\mu V^\nu}{\left(g_{\mu\nu}U^\mu U^\nu\, g_{\alpha\beta}V^\alpha V^\beta\right)^{1/2}}$$

(b) under a conformal transformation $\mathtt{x} \mapsto \bar{\mathtt{x}}$ one has

$$\cos\theta \mapsto \cos\bar{\theta} = \frac{f(\mathtt{x})g_{\mu\nu}U^\mu V^\nu}{\left(f(\mathtt{x})g_{\mu\nu}U^\mu U^\nu\, f(\mathtt{x})g_{\alpha\beta}V^\alpha V^\beta\right)^{1/2}} = \cos\theta$$

invariance of angles under conformal transformations
* light-like curves maintain this property, since

$$0 = \mathtt{x}\cdot\mathtt{x} = g_{\mu\nu}x^\mu x^\nu \mapsto f(\mathtt{x})g_{\mu\nu}x^\mu x^\nu = 0 = \bar{\mathtt{x}}\cdot\bar{\mathtt{x}}$$

11. Consider the metric

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \left(\frac{3}{13}\mathrm{d}x + \frac{4}{13}\mathrm{d}y + \frac{12}{13}\mathrm{d}z\right)^2 \qquad (10)$$

Is this really a three-dimensional space ? Try to find new coordinates ζ, η such that $\mathrm{d}s^2 = \mathrm{d}\zeta^2 + \mathrm{d}\eta^2$.

Solution:

Criterion: $3D$ space iff $\mathrm{d}^3V = g^{1/2}\mathrm{d}x\mathrm{d}y\mathrm{d}z \neq 0$.

Hence work out the determinant

$$\mathrm{d}^3V = \begin{vmatrix} 1-\left(\frac{3}{13}\right)^2 & -\frac{3}{13}\frac{4}{13} & -\frac{12}{13}\frac{3}{13} \\ -\frac{3}{13}\frac{4}{13} & 1-\left(\frac{4}{13}\right)^2 & -\frac{4}{13}\frac{12}{13} \\ -\frac{12}{13}\frac{3}{13} & -\frac{4}{13}\frac{12}{13} & 1-\left(\frac{12}{13}\right)^2 \end{vmatrix}^{1/2} \mathrm{d}x\mathrm{d}y\mathrm{d}z = 0$$

$\Rightarrow$ the space must be either $1D$ or $2D$.

Since the metric does not depend explicitly on z, one can consider the projection into the xy-plane where

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - \left(\frac{3}{13}\mathrm{d}x + \frac{4}{13}\mathrm{d}y\right)^2$$

$$g = \det\begin{pmatrix} 1-\left(\frac{3}{13}\right)^2 & -\frac{3}{13}\frac{4}{13} \\ -\frac{3}{13}\frac{4}{13} & 1-\left(\frac{4}{13}\right)^2 \end{pmatrix} = \frac{14336}{169^2} \neq 0$$

shows that this projection is indeed $2D$. One can diagonalise $g_{\mu\nu}$ and find $\mathrm{d}s^2 = \mathrm{d}\zeta^2 + \mathrm{d}\eta^2$, where

$$\zeta = \frac{12}{5}\left(\frac{3}{13}x + \frac{4}{13}y\right) \;,\; \eta = \frac{12}{5}\left(-\frac{4}{13}x + \frac{3}{13}y\right)$$

Série 3

1. The covariant derivatives of the metric tensor are defined as follows

$$g_{\mu\nu;\lambda} := g_{\mu\nu,\lambda} - g_{\sigma\nu}\Gamma^{\sigma}_{\mu\lambda} - g_{\mu\sigma}\Gamma^{\sigma}_{\nu\lambda} \qquad (1)$$

$$\text{with} \;\; \Gamma^{\mu}_{\nu\lambda} = g^{\mu\rho}\Gamma_{\rho\nu\lambda} = \frac{1}{2}g^{\mu\rho}\left(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho}\right)$$

where $\Gamma^{\mu}_{\nu\lambda}$ denote the Christoffel symbols. Show that the metric tensor $g_{\mu\nu}$ always has a vanishing covariant derivative, that is $\boxed{g_{\mu\nu;\lambda} = 0}$.

N.B.: *this* **compatibility property** *of the metric is characteristic for Einstein's theory of gravitation. In particular, such metrics are also compatible with flat spaces with a Minkowski metric tensor.*

Solution:
Begin with recalling from (1) the definition of $g_{\mu\nu;\lambda}$. For comparing the Christoffel symbols, it is useful to put all indices down-stairs

$$\Gamma_{\rho\nu\lambda} = \frac{1}{2}\left(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho}\right)$$

We then have

$$\begin{aligned} g_{\mu\nu;\lambda} &= g_{\mu\nu,\lambda} - \Gamma_{\nu\mu\lambda} - \Gamma_{\mu\nu\lambda} \\ &= g_{\mu\nu,\lambda} - \frac{1}{2}\left(g_{\nu\mu,\lambda} + g_{\nu\lambda,\mu} - g_{\mu\lambda,\nu}\right) - \frac{1}{2}\left(g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu} - g_{\nu\lambda,\mu}\right) \\ &= g_{\mu\nu,\lambda} - g_{\mu\nu,\lambda} \;=\; 0 \end{aligned}$$

the symmetry $g_{\mu\nu} = g_{\nu\mu}$ has been frequently used

2. Show that for a *diagonal* metric with metric tensor

$$g_{\mu\nu} = \mathrm{diag}\,(g_{00}, g_{11}, g_{22}, g_{33}) = \begin{pmatrix} g_{00} & & & \\ & g_{11} & & \\ & & g_{22} & \\ & & & g_{33} \end{pmatrix} \qquad (2)$$

the Christoffel symbols have the following values:

$$\Gamma^{\mu}{}_{\nu\lambda} = 0 \quad ; \quad \Gamma^{\mu}{}_{\lambda\lambda} = -\frac{1}{2g_{\mu\mu}}\frac{\partial g_{\lambda\lambda}}{\partial x^{\mu}}$$

$$\Gamma^{\mu}{}_{\mu\lambda} = \frac{\partial}{\partial x^{\lambda}} \ln\sqrt{|g_{\mu\mu}|} \quad ; \quad \Gamma^{\mu}{}_{\mu\mu} = \frac{\partial}{\partial x^{\mu}} \ln\sqrt{|g_{\mu\mu}|} \qquad (3)$$

Herein is always $\mu \neq \nu \neq \lambda \neq \mu$ and there is **no summation** over repeated indices !

Solution:
The Christoffel symbols are given by

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} \left(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho} \right) \qquad (*)$$

(a) since g is diagonal, one must have $\rho = \mu$ in (*). But since $\mu \neq \nu \neq \lambda \neq \mu$, none of the components $\Gamma^{\mu}_{\nu\lambda}$ is non-zero.
(b) if we set $\nu = \lambda$ in (*), we find

$$\Gamma^{\mu}_{\lambda\lambda} = \frac{1}{2} g^{\mu\rho} \left(g_{\rho\lambda,\lambda} + g_{\rho\lambda,\lambda} - g_{\lambda\lambda,\rho} \right) \; = \; -\frac{1}{2} g^{\mu\rho} g_{\lambda\lambda,\rho} \; = \; -\frac{1}{2} \big(g_{\mu\mu} \big)^{-1} g_{\lambda\lambda,\mu}$$

(c) if we set $\nu = \mu$ in (*), we find

$$\Gamma^{\mu}_{\mu\lambda} = \frac{1}{2} g^{\mu\rho} \left(g_{\rho\mu,\lambda} + g_{\rho\lambda,\mu} - g_{\mu\lambda,\rho} \right) \; = \; \frac{1}{2} \big(g_{\mu\mu} \big)^{-1} g_{\mu\mu,\lambda} = \frac{\partial}{\partial x^{\lambda}} \ln\big(|g_{\mu\mu}|^{1/2} \big)$$

(d) simply set $\mu = \lambda$ in (c) and obtain

$$\Gamma^{\mu}_{\mu\mu} = \frac{\partial}{\partial x^{\mu}} \ln\big(|g_{\mu\mu}|^{1/2} \big)$$

3. Die **Pseudosphäre** P^2 hat die Metrik $\mathrm{d}s^2 = a^2\big(\mathrm{d}\xi^2 + \sinh^2\xi\,\mathrm{d}\varphi^2\big)$ eines Hyperboloides. Was ist die Form der geodätischen Kurven ?

The **pseudo-sphere** P^2 has the metric $\mathrm{d}s^2 = a^2\big(\mathrm{d}\xi^2 + \sinh^2\xi\,\mathrm{d}\varphi^2\big)$ of a hyperboloïde. What is the form of the geodetic curves ?

Solution:
This is the metric of a two-sheeted hyperboloïde H_2, as seen in an earlier exercice. Label the coordinates as $(x^1, x^2) = (\xi, \varphi)$. The geodesics are obtained as solutions of the *geodesic equations*

$$\ddot{x}^\mu + \Gamma^\mu_{\kappa\lambda}\dot{x}^\kappa\dot{x}^\lambda = 0 \ , \ \Gamma^\mu_{\kappa\lambda} = \frac{1}{2}g^{\mu\rho}\left(g_{\rho\kappa,\lambda} + g_{\rho\lambda,\kappa} - g_{\kappa\lambda,\rho}\right) = \Gamma^\mu_{\lambda\kappa}$$

First one must find the non-vanishing Christoffel symbols. Since the metric is diagonal, one can use the technique explained in the previous exercice. Also, the non-zero elements of the inverse metric tensor are found, e.g. via $g^{11} = \frac{1}{g_{11}}$. The only non-vanishing Christoffel symbols are

$$\begin{aligned}
\Gamma^1_{22} &= -\frac{1}{2}g^{11}\frac{\partial}{\partial\xi}g_{22} = -\frac{1}{2}\cdot 1\cdot\left(2\sinh\xi\cosh\xi\right) = -\sinh\xi\cosh\xi \\
\Gamma^2_{21} &= \Gamma^2_{12} = \frac{1}{2}g^{22}\frac{\partial}{\partial\xi}g_{22} = \frac{1}{2}\frac{2\sinh\xi\cosh\xi}{\sinh^2\xi} = \coth\xi
\end{aligned}$$

Then the two geodesic equations read

$$\begin{aligned}
\ddot{x}^1 + \Gamma^1_{22}\,\dot{x}^2\dot{x}^2 = 0 \quad &\Rightarrow \quad \ddot{\xi} - \sinh\xi\cosh\xi\,\dot{\varphi}^2 = 0 \\
\ddot{x}^2 + 2\Gamma^2_{12}\,\dot{x}^1\dot{x}^2 = 0 \quad &\Rightarrow \quad \ddot{\varphi} + 2\coth\xi\,\dot{\xi}\dot{\varphi} = 0
\end{aligned}$$

If $\dot{\varphi} \neq 0$, the second of these gives

$$\frac{1}{\dot{\varphi}}\frac{\mathrm{d}\dot{\varphi}}{\mathrm{d}\sigma} + 2\coth\xi\,\frac{\mathrm{d}\xi}{\mathrm{d}\sigma} = 0 \;\Rightarrow\; \ln\dot{\varphi} + 2\ln\bigl(\sinh\xi\bigr) = \mathrm{cste.} \;\Rightarrow\; \boxed{\dot{\varphi}\sinh^2\xi = h = \mathrm{cste}}$$

Rather than solving the last remaining geodesic equation, it is more simple to go back to the metric, if one chooses the parameter $\sigma \overset{!}{=} s$ as the arc length. From the metric

$$1 = a^2\left(\frac{\mathrm{d}\xi}{\mathrm{d}s}\right)^2 + a^2\sinh^2\xi\left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)^2 = a^2\left(\frac{\mathrm{d}\xi}{\mathrm{d}s}\right)^2 + \frac{a^2h^2}{\sinh^2\xi}$$

where the conservation law derived above (implicitly taking $\sigma = s$) was inserted. From this, the geodesic equations can be written as

$$\frac{\mathrm{d}\xi}{\mathrm{d}s} = \pm\frac{\sqrt{\sinh^2\xi - a^2h^2}}{a\sinh\xi}\;,\quad \frac{\mathrm{d}\varphi}{\mathrm{d}s} = \frac{h}{\sinh^2\xi}$$

Since we *require the geometric form of the geodesic*, we are really looking for the **orbit**, which we seek in the form $\varphi = \varphi(\xi)$. Hence

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = \frac{\mathrm{d}\varphi}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}\xi} = \pm\frac{h}{\sinh^2\xi}\frac{a\sinh\xi}{\sqrt{\sinh^2\xi - a^2h^2}} = \pm\frac{\mathrm{d}}{\mathrm{d}\xi}\arccos\left(\frac{h}{\sqrt{1/a^2 + h^2}}\coth\xi\right)$$

the details of this integration will be spelled out below

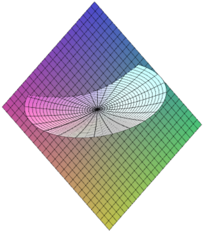
This form of the orbit can be re-expressed as $\cos(\varphi - \varphi_0) - \frac{h}{\sqrt{1/a^2+h^2}} \coth\xi = 0$, which can be re-stated as $A\cos\varphi + B\sin\varphi + C\coth\xi = 0$ with known constantes A, B, C. This can also be rephrased as

$$\boxed{A\sinh\xi\cos\varphi + B\sinh\xi\sin\varphi + C\cosh\xi = 0} \qquad (*)$$

Equations of this kind arise form the intersection of a hyperboloïde given by $x^2 + y^2 - z^2 = -a^2$, and a plane going through the origin which is described by $\alpha x + \beta y + \gamma z = 0$.

recall HESSE's normal form $\boldsymbol{n} \cdot \boldsymbol{r} = d$ for the equation of a plane, with distance d to the origin

Recall that a two-sheeted hyperboloïde has the parametrisation $x = a\sinh\xi\cos\varphi$, $y = a\sinh\xi\cos\varphi$ and $z = \pm a\cosh\xi$. Inserting this into the equation for the plane produces an equation of the form (*).



The figure shows an example of such an intersection.

[Mathematical remarks on the details of the integration:

want to integrate $\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = \pm \frac{ah}{\sinh^2\xi} \frac{\sinh\xi}{\sqrt{\sinh^2\xi - a^2h^2}}$ (it is better **not** to cancel $\sinh\xi$)

$$\varphi - \varphi_0 = \pm ah \int \frac{\mathrm{d}\xi}{\sinh^2\xi} \frac{1}{\sqrt{1 - \frac{a^2h^2}{\sinh^2\xi}}}$$

notice $\frac{1}{\sinh^2\xi} = \coth^2\xi - 1$

$$= \pm ah \int \frac{\mathrm{d}\xi}{\sinh^2\xi} \left[1 - a^2h^2\left(\coth^2\xi - 1\right)\right]^{-1/2}$$

notice $\frac{\mathrm{d}\coth\xi}{\mathrm{d}\xi} = -\frac{1}{\sinh^2\xi}$

$$= \pm ah \int \frac{\mathrm{d}\xi}{\sinh^2\xi} \left[1 + a^2h^2 - a^2h^2\coth^2\xi\right]^{-1/2}$$

set $u = \coth\xi \Rightarrow \mathrm{d}u = -\frac{\mathrm{d}\xi}{\sinh^2\xi}$

$$= \mp ah \int \mathrm{d}u \, \frac{1}{ah} \left[\frac{1 + a^2h^2}{a^2h^2} - u^2\right]^{-1/2}$$

set $u = \sqrt{\frac{1}{a^2h^2} + 1}\,\cos\alpha \Rightarrow \mathrm{d}u = -\sqrt{\frac{1}{a^2h^2} + 1}\,\sin\alpha\,\mathrm{d}\alpha$

$$= \pm \frac{\sqrt{\frac{1}{a^2h^2} + 1}}{\sqrt{\frac{1}{a^2h^2} + 1}} \int \mathrm{d}\alpha \, \frac{\sin\alpha}{\sin\alpha} = \pm\alpha$$

$$= \pm \arccos\left(\frac{1}{\sqrt{\frac{1}{a^2h^2} + 1}} \coth\xi\right) = \pm \arccos\left(\frac{h}{\sqrt{\frac{1}{a^2} + h^2}} \coth\xi\right)$$

as claimed.]

4. In less than 4 dimensions, the Riemann tensor admits simple forms.

Write down simple explicit expressions for the Riemann tensor in $d = 1, 2, 3$ dimensions. How many independent components of the Riemann tensor do you find in each case ?
Hint: for $d < 4$ the Riemann tensor can be expressed uniquely through the Ricci scalar R and the Ricci tensor. Use the known symmetries of the Riemann tensor

$$R_{\mu\nu\lambda\sigma} = R_{\lambda\sigma\mu\nu} \ \ ; \ \ R_{\mu\nu\lambda\sigma} = -R_{\nu\mu\lambda\sigma} = -R_{\mu\nu\sigma\lambda} \ \ ; \ \ R_{\mu[\nu\lambda\sigma]} = 0 \tag{4}$$

☞ In empty space, the field equations of gravitation are $R_{\mu\nu} = 0$. What follows about gravitation in empty space in $d = 2$ or $d = 3$ dimensions ?

Solution:
(a) $d = 1$: the Riemann tensor has a single component $R_{1111} = 0$ because of the symmetries. ☞ *all 1D spaces are flat*.
(b) $d = 2$: the Riemann tensor has a single independent component. One can take into account the symmetries and write the Riemann tensor in the form

$$R_{\alpha\beta\gamma\delta} = \left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right) r$$

with a scalar r.
Verifying the first three symmetries in (4) is obvious. For the last one, the *Bianchi identity*, consider

$$\begin{aligned} R_{\alpha[\beta\gamma\delta]} &:= \frac{1}{3}\left(R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\beta\delta} + R_{\alpha\delta\beta\gamma}\right) \\ &= \frac{r}{3}\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma} + g_{\alpha\delta}g_{\gamma\beta} - g_{\alpha\beta}g_{\gamma\delta} + g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\gamma}g_{\delta\beta}\right) = 0 \end{aligned}$$

We now compute the Ricci scalar

$$R = R^{\alpha\beta}{}_{\alpha\beta} = \left(g^{\alpha}{}_{\alpha}g^{\beta}{}_{\beta} - g^{\alpha}{}_{\beta}g^{\beta}{}_{\alpha}\right) r = \left(2 \cdot 2 - 2\right) r = 2r$$

so that we finally have

$$\boxed{R_{\alpha\beta\gamma\delta} = \frac{1}{2}\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right) R}$$

(c) $d = 3$: The Riemann tensor has 6 independent components. Since in $3D$, the Ricci tensor $R_{\mu\nu} = R_{\nu\mu}$ has 6 independent components as well, one may try to express the Riemann tensor through the $R_{\mu\nu}$. First, the following ansatz takes the symmetries into account

$$\begin{aligned} R_{\mu\nu\lambda\sigma} &= A\big(g_{\mu\lambda}R_{\nu\sigma} - g_{\nu\lambda}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\lambda} + g_{\nu\sigma}R_{\mu\lambda}\big) \\ &\quad + B\big(g_{\mu\lambda}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\lambda}\big)R \end{aligned}$$

where the constants A, B are to be found. (the second line is the same as in $2D$)
By contraction, one obtains

$$\begin{aligned} R^{\mu}{}_{\nu\mu\sigma} &= R_{\nu\sigma} \\ &= A\big(3R_{\nu\sigma} - R_{\nu\sigma} - R_{\nu\sigma} + g_{\nu\sigma}R\big) + B\big(3g_{\nu\sigma} - g_{\nu\sigma}\big)R \\ &= AR_{\nu\sigma} + g_{\nu\sigma}R\big(A + 2B\big) \end{aligned}$$

This gives $A = 1$ and $B = -\frac{1}{2}$.
for a check, contract once more: $R = AR(1+3) + BR2\cdot 3 = \big(4\cdot 1 - \frac{1}{2}\cdot 6\big)R = \big(4-3\big)R$.

* In empty space, the Ricci tensor vanishes $R_{\mu\nu} = 0$ (hence $R = 0$ as well).
Therefore, the $2D/3D$ full Riemann tensor vanishes in empty space
$\Rightarrow$! *no gravitational force in empty space for $d = 2$ or $d = 3$* !
☞ long-range gravitational forces across empty space need $d = 4$ time-space dimensions

5. In the rest frame of a perfect fluid with (proper) mass density ρ and pressure p the energy-momentum tensor has the form

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} . \tag{5}$$

Find the energy-momentum tensor for an element of the liquid with proper mass density ρ and proper pressure p, which moves with the four-velocity $\mathfrak{u}$.

Solution:
In the rest frame, one has

explicitly assumed here: cartesian coordinates

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

In addition, one has $u^0 = 1$, $u^i = 0$, with $i = 1, 2, 3$. This suggests to propose the form

$$T^{\mu\nu} = p g^{\mu\nu} + (\rho + p) u^\mu u^\nu$$

* this reduces to the known expression in the rest frame, where $g^{\mu\nu} = \eta^{\mu\nu}$.
* the proposed form is generally co-variant.

Hence, by the principle of general co-variance, it will hold in general, for all coordinate systems.

N.B.: ermöglicht auf sehr billige Art, $T^{\mu\nu}$ in nichtkartesischen Koordinaten explizit hinzuschreiben, (nehme das Ruhesystem !) sobald man nur den metrischen Tensor $g^{\mu\nu}$ in diesen Koordinaten kennt

Série 4

1. Show that the gravitational force on a test body inside a gravitating hollow sphere vanishes.
Hint: Birkhoff's theorem states, that the Schwarzschild metric is a solution of the field equations $R_{\mu\nu} = 0$ in the case of spherical symmetry. Can such a solution have singularities in the inside of a sphere ?
You may remember an analogous result in Newton's theory of gravity or else in electrostatics.

Solution:
Because of the spherical symmetry, Birkhoff's theorem states that $g_{\mu\nu}$ must be Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1 - \frac{\mathscr{R}}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{\mathscr{R}}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

where $\mathrm{d}\Omega$ is the element of the solid angle.
If $\mathscr{R} \neq 0$, the solution has a singularity at $r = \mathscr{R}$ ($\mathscr{R}$ plays here the rôle of an integration constant). However, singularities are physically inadmissible, since one is in the interior of a sphere,without any masses. Hence $\mathscr{R} = 0$. Then the metric must be

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

which is the flat Minkowski metric. Since the corresponding Riemann tensor vanishes, there is *no gravitational force*.

2. Mathematically a curved ('Riemann') space in d dimensions has a *constant curvature*, if the Riemann tensor $R_{\mu\nu\lambda\kappa}$ can be expressed as follows through the metric tensor $g_{\mu\nu}$ (with $\det g \neq 0$):

$$R_{\mu\nu\lambda\kappa} = K\,(g_{\mu\lambda}g_{\nu\kappa} - g_{\mu\kappa}g_{\nu\lambda}) \tag{1}$$

where the constant K describes the constant curvature. Show that the Ricci tensor has the form

$$R_{\mu\nu} := g^{\sigma\tau}R_{\sigma\mu\tau\nu} = K(d-1)g_{\mu\nu} \tag{2}$$

Can one make a statement about two-dimensional spaces ($d = 2$) ?
Hints: $g^{\sigma\tau}g_{\sigma\tau} = d$. Recall the general form of $R_{\mu\nu\lambda\kappa}$ for $d = 2$.

Solution:
A direct calculation gives

$$\begin{aligned} R_{\mu\nu} &= g^{\sigma\tau} R_{\sigma\mu\tau\nu} \\ &= K g^{\sigma\tau} \big(g_{\sigma\tau} g_{\mu\nu} - g_{\sigma\nu} g_{\mu\tau} \big) \\ &= K d g_{\mu\nu} - K g^{\tau}_{\nu} g_{\mu\tau} \; = \; K(d-1) g_{\mu\nu} \end{aligned}$$

If $d = 2$, we had from a previous exercice that $R_{\alpha\beta\gamma\delta} = \frac{R}{2}\big(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\big)$. Comparison shows that $K = \frac{1}{2}R$. A conformal transformation can be used to make $R = \mathrm{cste.}$.
Hence, for $d = 2$ any space is conformally related to a Riemann space of constant curvature.

3. A detailed study of the mouvement of galaxies, far from earth, has raised interest in the following variant of Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\kappa T_{\mu\nu} \quad , \quad \kappa := -\frac{8\pi G}{c^2} \tag{3}$$

where Λ is called *cosmological constant*. G is Newton's gravitational constant.

a) What is the dimension of Λ ?

b) Show, via a convenient contraction, that the Ricci scalar $R = \kappa T + 4\Lambda$ (here $T := T^{\mu}_{\mu}$) and derive from (3) the following alternative form of the field equations

$$R_{\mu\nu} = \Lambda g_{\mu\nu} - \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) \tag{4}$$

c) Without external sources, that is for $T_{\mu\nu} = 0$, the flat Minkowski metric a solution of the field equations (4) ? Compare the field equations (4) without sources with the form of a Riemann space with constant curvature, as derived in a previous exercice. Can you interpret geometrically the cosmological constant Λ ?

d) In the non-relativistic limit the component $\mu = \nu = 0$ from eq. (4) reproduces Newton's equations. Show that for $\Lambda \neq 0$ one obtains a generalised Poisson's equation

$$\Delta\phi + \Lambda c^2 = 4\pi G\rho \qquad (5)$$

where $\phi = -\frac{c^2}{2} h_{00}$ is Newton's gravitational potential, $g_{\mu\nu} = \eta_{\mu\nu} - h_{\mu\nu} + \dots$ and Δ denotes the usual Laplace operator.

e) Show that for $\Lambda \neq 0$ one has phenomenologically an additional force $F_\Lambda = \frac{1}{3}\Lambda c^2 r$ in the distance r from the centre of the force. For which class of objects would you expect measurable effects of the constant Λ ?

Hint: in $3D$ spherical coordinates the Laplace operator reads $\Delta f(r) = \frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(rf(r)\right)$, for the case of a spherical symmetry.

Solution:

(a) Since $R_{\mu\nu}$ contains two derivatives $\Rightarrow$ Λ has dimension $[\mathrm{length}^{-2}]$.

(b) From (3) have $R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} - \kappa T_{\mu\nu}$. Contracting, one finds $R = R^{\mu}{}_{\mu} = \frac{1}{2} \cdot 4R - 4\Lambda - \kappa T$, or $R = 4\Lambda + \kappa T$. Insert this into (3) and find

$$R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} \big(4\Lambda + \kappa T \big) - \Lambda g_{\mu\nu} - \kappa T_{\mu\nu} = \Lambda g_{\mu\nu} - \kappa \big(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \big)$$

(c) The Minkowski metric is solution of $R_{\mu\nu} = 0$, which is different from (4) with $\Lambda \neq 0$ and $T_{\mu\nu} = 0$. In a previous exercice we have seen that for spaces with a constant curvature $R_{\mu\nu} = K(d-1) g_{\mu\nu}$. Comparison gives $\Lambda = 3K$. In the absence of sources, *the resulting space has the constant curvature* $K = \frac{\Lambda}{3}$.

(d) In principle, it is enough to repeat the calculations for obtaining the non-relativistic limit from the lecture. For clarity, the main steps are repeated here.

The newtonian limit is a *weak-field limit* where one sets $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with h 'small'. In addition, as $c \to \infty$, one expects $\tau \simeq t$, $\frac{\mathrm{d}x^0}{\mathrm{d}\tau} \simeq c$, $\frac{\mathrm{d}x^i}{\mathrm{d}\tau} \simeq \frac{\mathrm{d}x^i}{\mathrm{d}t} = v^i \ll c$. Furthermore, this is a *static* approximation where the potentials are time-independent. The three spatial geodesic equations become

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} + c^2 \Gamma^i_{00} \big(1 + \mathrm{O}(1/c) \big) = 0 \quad \Rightarrow \quad \frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -c^2 \Gamma^i_{00} = a^i \quad \text{acceleration}$$

which begins to look like a newtonian equation of motion.

One must now work out, in the static approximation and to linear order in h:

$$\Gamma^i_{00} = \frac{1}{2} g^{i\nu}\big(\underbrace{2g_{\nu 0,0}}_{=0} - g_{00,\nu}\big) = -\frac{1}{2} g^{ik} g_{00,k} \simeq -\frac{1}{2}\eta^{ik} h_{00,k} + \mathrm{O}(h^2) = -\frac{1}{2}\nabla^i h_{00}$$

This gives the equation of motion $\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -c^2\Gamma^i_{00} = \frac{c^2}{2}\nabla^i h_{00}$ and should be compared with the newtonian equation $\frac{\mathrm{d}^2 x^i}{\mathrm{d}t^2} = -\nabla^i\phi$. One identifies the newtonian gravitational potential $\boxed{h_{00} = -\frac{2}{c^2}\phi}$, or $g_{00} = -\big(1 + \frac{2}{c^2}\phi\big)$.

In order to find the newtonian limit of the field equation, consider again

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad \Rightarrow \quad g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathrm{O}(h^2)$$

[to see this: $g_{\mu\nu} g^{\nu\kappa} = \big(\eta_{\mu\nu} + h_{\mu\nu}\big)\big(\eta^{\nu\kappa} - h_{\nu\kappa}\big) \simeq \eta_{\mu\nu}\eta^{\nu\kappa} - \eta_{\mu\nu} h_{\nu\kappa} + h_{\mu\nu}\eta^{\nu\kappa} + \mathrm{O}(h^2)$
$= \delta^\kappa_\mu - h^\kappa_\mu + h^\kappa_\mu = \delta^\kappa_\mu$]
Then

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\rho}\big(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho}\big) \simeq \frac{1}{2}\eta^{\mu\rho}\big(h_{\rho\nu,\lambda} + h_{\rho\lambda,\nu} - h_{\nu\lambda,\rho}\big) + \mathrm{O}(h^2)$$

which itself is of first order in h throughout.

Recall the computation of the Ricci tensor

$$R_{\mu\nu} = \Gamma^{\kappa}_{\mu\nu,\kappa} - \Gamma^{\kappa}_{\mu\kappa,\nu} + \underbrace{\Gamma^{\kappa}_{\rho\kappa}\Gamma^{\rho}_{\mu\nu} - \Gamma^{\kappa}_{\rho\nu}\Gamma^{\rho}_{\mu\kappa}}_{=\mathrm{O}(h^2),\ \text{negligible}}$$
$$= \frac{1}{2}\eta^{\kappa\sigma}\left(h_{\sigma\nu,\mu\kappa} + h_{\mu\kappa,\sigma\nu} - h_{\mu\nu,\sigma\kappa} - h_{\sigma\kappa,\mu\nu}\right) + \mathrm{O}(h^2)$$

Concentrate on the component $\mu = \nu = 0$ (use the static approximation !):

$$R_{00} \simeq \frac{1}{2}\eta^{\kappa\sigma}\Big(\underbrace{h_{\sigma 0,0\kappa} + h_{0\kappa,\sigma 0}}_{=0} - h_{00,\sigma\kappa} - \underbrace{h_{\sigma\kappa,00}}_{=0}\Big)$$
$$= -\frac{1}{2}\eta^{\kappa\sigma}h_{00,\sigma\kappa} = -\frac{1}{2}\left(-\frac{1}{c^2}\underbrace{\frac{\partial^2}{\partial t^2}}_{=0} + \boldsymbol{\nabla}^2\right)h_{00} = -\frac{1}{2}\boldsymbol{\nabla}^2 h_{00}$$

Next, in the newtonian limit, the energy-momentum tensor of matter is $T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$

Then $T = T^{\mu}_{\mu} = -\rho$ and $T^{\mu\nu} - \frac{1}{2}g^{\mu\nu}T = \begin{pmatrix} \rho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} + \frac{\rho}{2}\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = \frac{\rho}{2}\delta^{\mu\nu}$.

The field equation (4) now takes the form

$$R_{00} = -\frac{1}{2}\boldsymbol{\nabla}^2 h_{00} = \frac{1}{c^2}\boldsymbol{\nabla}^2\phi \stackrel{!}{=} \Lambda g_{00} - \kappa\frac{\rho}{2}\delta_{00} = -\frac{\kappa}{2}\rho - \Lambda$$

which gives the ‘newtonian’ form

$$\nabla^2\phi + \Lambda c^2 = -c^2\frac{\kappa}{2}\rho$$

For $\Lambda = 0$, this should reproduce the newtonian equation $\boldsymbol{\nabla}\phi = 4\pi G\rho$ from which the κ given in the problem statement (3) follows. The full field equation in the newtonian limit is

$$\boxed{\nabla^2\phi + \Lambda c^2 = 4\pi G\rho} \qquad (*)$$

N.B.: for the sake of comparison with the litterature, we have kept the c ...

(e) On phenomenological consequences: (*) is linear, hence just consider the extra term coming from $\nabla^2\phi_\Lambda = -\Lambda$.

$$\Rightarrow \ \frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\big(r\phi_\Lambda(r)\big) = -\Lambda \ \Rightarrow \ \frac{\mathrm{d}^2}{\mathrm{d}r^2}\big(r\phi_\Lambda(r)\big) = -\Lambda r \ \Rightarrow \ r\phi_\Lambda(r) = -\frac{1}{2\cdot 3}\Lambda r^3$$

such that finally $\phi_\Lambda(r) = -\frac{\Lambda}{6}r^2$. The corresponding *'cosmological force'* is $F_\Lambda(r) = -\frac{\partial\phi_\Lambda(r)}{\partial r} = \frac{1}{3}\Lambda r$.

Since $\Lambda \approx 10^{-52}[\mathrm{m}^{-2}]$, there is a corresponding length scale $\Lambda^{-1/2} \sim 10^{26}[\mathrm{m}]$. In astronomy, an often-used length scale is the light year, which is of order $1[\text{light year}] \sim 10^{16}[\mathrm{m}]$ (which is distance a photon travels in one year). Hence $\Lambda^{-1/2} \sim 10^{10}[\text{light year}]$ which is the order of magnitude of the radius of the visible universe. Effects of the 'cosmological force' should be seen in considerations of the behaviour (e.g. expansion) of the entire universe – and next, one might consider effects on the mouvement of clusters of galaxies which occur at scales at the order of $10^8[\text{light year}]$.

4. At the surface of a pseudo-sphere P^2 one has the metric

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = a^2 \left(\mathrm{d}\chi^2 + \sinh^2\chi \mathrm{d}\phi^2\right) \tag{6}$$

where a is a fixed length scale. Is P^2 a curved space ? Compute the Ricci scalar R as a function of a. Do you see a property in which the pseudo-sphere P^2 is distinct from the usual sphere S^2 ?
Hints: The Riemann tensor R and the Christoffel symbols Γ are given by

$$\begin{aligned} R^\kappa_{\lambda\mu\nu} &= \Gamma^\kappa_{\lambda\nu,\mu} - \Gamma^\kappa_{\lambda\mu,\nu} + \Gamma^\kappa_{\rho\mu}\Gamma^\rho_{\lambda\nu} - \Gamma^\kappa_{\rho\nu}\Gamma^\rho_{\lambda\mu} \\ \Gamma^\mu_{\nu\lambda} &= \frac{1}{2}g^{\mu\rho}\left(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho}\right) \end{aligned} \tag{7}$$

How many independent components the Riemann tensor does have for P^2 ? The Ricci tensor is $R_{\mu\nu} = R^\kappa_{\mu\kappa\nu}$ and the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu} = R^\mu_\mu$.

Solution:

One has the metric, with $(x^1, x^2) = (\chi, \phi)$: $\mathrm{d}s^2 = a^2\big(\mathrm{d}\chi^2 + \sinh^2\chi\,\mathrm{d}\phi^2\big)$. The metric tensor is diagonal, hence using previous exercices one readily has the non-vanishing Christoffel symbols

$$\begin{aligned}
\Gamma^1_{22} &= -\frac{1}{2}\frac{1}{g_{11}}\frac{\partial}{\partial\chi}g_{22} = -\frac{1}{2}\cdot 1\cdot 2\sinh\chi\cosh\chi = -\sinh\chi\cosh\chi \\
\Gamma^2_{21} &= \Gamma^2_{12} = \frac{1}{2}\frac{1}{g_{22}}\frac{\partial}{\partial\chi}g_{22} = \frac{1}{2}\frac{2\sinh\chi\cosh\chi}{\sinh^2\chi} = \coth\chi
\end{aligned}$$

Since P^2 is two-dimensional, there is a single independent component of the Riemann tensor, for example

$$\begin{aligned}
R^1_{212} &= \Gamma^1_{22,1} - \Gamma^1_{21,2} + \Gamma^1_{i1}\Gamma^i_{22} - \Gamma^1_{i2}\Gamma^i_{21} \\
&= \frac{\partial}{\partial\chi}\big(-\sinh\chi\cosh\chi\big) - \Gamma^1_{22}\Gamma^2_{21} \\
&= -\cosh^2\chi - \sinh^2\chi - \big(-\sinh\chi\cosh\chi\big)\coth\chi \\
&= -\cosh^2\chi - \sinh^2\chi + \cosh^2\chi = -\sinh^2\chi \neq 0
\end{aligned}$$

N.B.: all terms not spelled out explicitly here vanish !

Since $R^1_{212} \neq 0 \Rightarrow$ *P^2 is curved*

$$R^2_{121} = g^{22} g_{11} R^1_{212} = \frac{1}{a^2 \sinh^2 \chi} a^2 \left(-\sinh^2 \chi\right) = -1 \neq 0$$

N.B.: one might have found R^2_{121} directly as well

$$\begin{aligned} R^2_{121} &= \Gamma^2_{11,2} - \Gamma^2_{12,1} + \Gamma^2_{i2}\Gamma^i_{11} - \Gamma^2_{i1}\Gamma^i_{12} \\ &= -\frac{\partial}{\partial \chi}\left(\coth \chi\right) - \Gamma^2_{21}\Gamma^2_{12} \\ &= -\frac{\sinh^2 \chi - \cosh^2 \chi}{\sinh^2 \chi} - \coth^2 \chi \\ &= -1 + \coth^2 \chi - \coth^2 \chi \; = \; -1 \end{aligned}$$

One computes the Ricci tensor

$$\begin{aligned} R_{11} &= R^i_{1i1} = R^2_{121} = -1 \\ R_{22} &= R^i_{2i2} = R^1_{212} = -\sinh^2 \chi \end{aligned}$$

N.B.: for illustration: $R_{12} = R^i_{1i2} = R^1_{112} + R^2_{122}$ and

$$\begin{aligned} R^1_{112} = \Gamma^1_{12,1} - \Gamma^1_{11,2} + \Gamma^1_{i1}\Gamma^i_{12} - \Gamma^1_{i2}\Gamma^i_{11} \; = \; 0 \\ R^2_{122} = \Gamma^2_{12,2} - \Gamma^2_{12,2} + \Gamma^2_{i2}\Gamma^i_{12} - \Gamma^2_{i2}\Gamma^i_{12} \; = \; 0 \end{aligned}$$

as one might have anticipated from the symmetries of the Riemann tensor. $\Rightarrow R_{12} = 0$.

Finally, the Ricci scalar is

$$R = g^{11}R_{11} + g^{22}R_{22} = \frac{1}{a^2}(-1) + \frac{1}{a^2\sinh^2\chi}\left(-\sinh^2\chi\right) = -\frac{2}{a^2} < 0 \qquad \text{(R)}$$

R is non-vanishing, of dimension $[\mathrm{length}^{-2}]$ and negative. This last property is distinct from the usual sphere S^2, where $R = +\frac{2}{a^2} > 0$ (see lectures).

We also note the matrix forms of the metric and Ricci tensors:

$$g_{\mu\nu} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2\sinh^2\chi \end{pmatrix} \;,\; R_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & -\sinh^2\chi \end{pmatrix}$$

Hence, $\boxed{R_{\mu\nu} = -\frac{1}{a^2}g_{\mu\nu}}$. *The pseudo-sphere P^2 is a space of constant curvature K.*

N.B.: Indeed, it is known from a previous exercice that in $2D$ spaces with constant curvature $K = \frac{1}{2}R$ – in agreement with the explicit results found here, where $K = -\frac{1}{a^2} = \frac{1}{2}R$, see eq. (R).

⚠ the calculations here are for geometric spaces and **not** for Minkowski time-spaces !

Theorem: (Huygens) *The surface area of P^2 is $4\pi a^2$ and its volume $\frac{2}{3}\pi a^3$.*

Theorem: (Hilbert) *It is impossible to embed P^2 into the euclidean $\mathbb{R}^3$.*

Theorem: (Whitney) *For any manifold with $\dim \mathscr{M} = m \leq \frac{n}{2}$ there is an embedding $f : \mathscr{M} \to \mathbb{R}^n$.*

5. (a) Repeat the calculation of the perihelion shift of an elliptical Kepler orbit for the case when $\mathscr{R}/r \ll 1$.
(b) If the central star is flatted through a rapid rotation, its newtonian gravity potential will take phenomenologically the form $V(r) = -Mr^{-1} - AMr^{-3}$, where A describes the flattening. Calculate in the setting of Newton's theory the perihelion shift for an elliptical orbit, which results from the flattening of the star.
(c) What would happen, if the sun were so much flattened, that its flattening alone could explain the perihelion shift of Mercury ? Estimate the perihelion shift for the first four planets in the solar system.
Hint: the ellipticity of the orbits can be neglected.

Solution:
(a) This is a revision of the class. It is helpful to work with $u = 1/r$. Then one arrives at Binet's formula

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{1}{p} + \frac{3}{2}\mathscr{R} u^2$$

For $\mathscr{R} = 0$, one has the well-known elliptic orbit (technically, Binet's formula gives a harmonic oscillator with a constant external force) $u = \frac{1}{p}\big(1 + e\cos\phi\big)$ – with the choice $\phi_0 = 0$. Treating the relativistic correction perturbatively, one finds for the perihelion shift

$$\left.\frac{\Delta\phi}{2\pi}\right|_{\text{per orbit}} \simeq \frac{3}{2}\frac{\mathscr{R}}{r}$$

(b) a newtonian calculation gives readily see next page for details

$$\left.\frac{\Delta\phi}{2\pi}\right|_{\text{per orbit}} \simeq \frac{3A}{r^2} \qquad (*)$$

N.B.:notice the different dependence on r with respect to the relativistic calculation.

We justify the estimate (*). For simplicity of notation, use units such that $M = 1$. Textbooks of classical mechanics (e.g. GOLDSTEIN, or LANDAU & LIFCHITZ) give Binet's formula in the form

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = -\frac{1}{a^2 u^2} f\left(\frac{1}{u}\right) \quad , \quad f(r) = -\frac{\partial V(r)}{\partial r} \quad \text{force}$$

N.B.: for an almost circular orbit, have angular momentum $a^2 \simeq r$ for gravitation.

to see **that**: circular orbit means $\frac{v^2}{r} \sim \frac{1}{r^2}$, angular momentum $a = v \cdot r \sim r^{-1/2} \cdot r \Rightarrow a^2 \sim r$.

$$V(r) = -\frac{1}{r} - \frac{A}{r^3} \quad \Rightarrow \quad f(r) = -\frac{1}{r^2} - \frac{3A}{r^4}$$

hence $u'' + u = \frac{1}{p} + \frac{3A}{a^2} u^2 \Leftrightarrow \boxed{u'' + u\left(1 - \frac{3A}{a^2} u\right) = \frac{1}{p}}$.

If the corrections are small, to leading order the perihelion shift should be of the same order as the corrections:

$$\left.\frac{\Delta\phi}{2\pi}\right|_{\text{per orbit}} \simeq \frac{3A}{a^2}\frac{1}{r} \simeq \frac{3A}{r^2} \qquad (*)$$

(c) for Mercury, one has $\Delta\phi_{\rm Mer} = 43''/[\rm century]$. For another planet, and assuming everything can be explained by the flattening of the star alone, according to (*), one has, where r is the orbit radius and P the period for the planet

$$\Delta\phi = \Delta\phi_{\rm Mer}\left(\frac{r_{\rm Mer}}{r}\right)^2\left(\frac{P_{\rm Mer}}{P}\right) = 43''/[\rm century]\left(\frac{r_{\rm Mer}}{r}\right)^{7/2}$$

where the $3^{\rm rd}$ Kepler's law $P \sim r^{3/2}$ was used. This can be set up in a table, where $\Delta\phi$ is in $''/[\rm century]$.

planet	$r_{\rm Mer}/r$	$\Delta\phi_{\rm GR}$	$\Delta\phi_{\rm flat}$	observed
Mercury	1	43	43	43.11 ± 0.45
Venus	0.536	8.8	4.7	8.4 ± 4.8
Earth	0.386	3.9	1.5	5.0 ± 1.2
Mars	0.245	1.25	0.3	1.5 ± 0.15

In this way, one can clearly distinguish between the two possibilities. The explanation by flatness of the star alone is clearly ruled out.

6. A *photon* moves in a plane orbit in a Schwarzschild metric. Derive the Binet's formula for the orbit

$$\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{3}{2}\frac{\mathscr{R}}{r^2} \tag{8}$$

where $\mathscr{R}$ is the Schwarzschild radius.

Solution:
From the lecture, can take over eqs. (G0,G2,G3). Eq. (G1) is replaced by

$$0 = \left(1 - \frac{\mathscr{R}}{r}\right) \dot{t}^2 - \left(1 - \frac{\mathscr{R}}{r}\right)^{-1} \frac{\dot{r}^2}{c^2} - \frac{r^2}{c^2}\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) \qquad \text{(G1')}$$

As before, fix the plane $\theta = \frac{\pi}{2}$, get from (G3) angular momentum conservation $r^2\dot{\phi} = a = \mathrm{cste.}$ and recall from (G0) that $\left(1 - \frac{\mathscr{R}}{r}\right)\dot{t} = b = \mathrm{cste.}$. Using also that $\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\tau} = \frac{\mathrm{d}r}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{\mathrm{d}r}{\mathrm{d}\phi}\dot{\phi}$ and the relation $\frac{1}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\right)^2 = \left(\frac{\mathrm{d}}{\mathrm{d}\phi}\frac{1}{r}\right)^2$, the eq. (G1') is replaced by

$$\frac{b^2}{a^2} - \left(\frac{\mathrm{d}}{\mathrm{d}\phi}\frac{1}{r}\right)^2 - \frac{1}{r^2}\left(1 - \frac{\mathscr{R}}{r}\right) = 0$$

Derive this with respect to ϕ. For a non-circular orbit, $\frac{\mathrm{d}}{\mathrm{d}\phi}\frac{1}{r} \neq 0$, so that one arrives at Binet's formula (8)

$$\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{3}{2}\frac{\mathscr{R}}{r^2}$$

7. Use the result (8) from the preceding exercice in order to derive the bending of light rays in the gravitational field of the sun.

Solution:
If $\mathscr{R}/r \ll 1$, one can solve (8) perturbatively:

$$\text{if } \frac{\mathscr{R}}{r} = 0 \text{ then } \quad \frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\frac{1}{r} + \frac{1}{r} = 0 \quad \Rightarrow \quad \frac{1}{r} = \frac{1}{r_0}\cos\phi$$

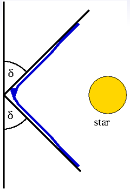
with the implicit choice $\phi_0 = 0$. In first order, insert this solution into the right-hand-side of (8)

$$\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}\frac{1}{r} + \frac{1}{r} \simeq \frac{3}{2}\frac{\mathscr{R}}{r_0^2}\cos^2\phi \qquad (8')$$

A particular solution of this is $\frac{1}{r} = \frac{\mathscr{R}}{2r_0^2}\left(1 + \sin^2\phi\right)$, as is readily verified. Hence the general solution of (8') is

$$\frac{1}{r} = \frac{1}{r_0}\cos\phi + \frac{\mathscr{R}}{2r_0^2}\left(1 + \sin^2\phi\right)$$

N.B.: This is a solution of (8) only to leading order in $\mathscr{R}/r_0$.



Asymptotically, for $r \to \infty$, one has
(i) if $\mathscr{R} = 0$, one expects $\phi \to \pm\frac{\pi}{2}$
(ii) if $\mathscr{R} \neq 0$, one expects a deviation $\phi \to \pm\left(\frac{\pi}{2} + \delta\right)$.
Herein, δ follows from the condition, to leading order

$$0 \stackrel{!}{=} -\frac{1}{r_0}\sin\delta + \frac{\mathscr{R}}{2r_0^2}\left(1+\cos^2\delta\right) \quad \Rightarrow \quad \delta \simeq \frac{\mathscr{R}}{r_0}$$

The final scattering angle is $\Delta = 2\delta = 2\frac{\mathscr{R}}{r_0} = \frac{4MG}{r_0 c^2}$.

For a light deviation just at the border of the sun, one finds $\Delta = 1.75''$.

Remarkably, this is a parameter-free prediction. In 1915/16, it was considered a major innovation to predict *curved orbits for light*, as it went against long-held views going back to NEWTON ! (and beyond)

EDDINGTON organised in 1919 two expeditions to measure Δ at a solar eclipse. The results were

Príncipe: $\Delta = 1.60 \pm 0.31''$, Sobral (Brazil): $\Delta = 1.98 \pm 0.12''$

None of these really agree with the theory, but the average is not too far off

☞ very important event which turned EINSTEIN into a pop star of science

8. A particle is in a circular orbit, with radius r, around a fixed star with mass M and stellar radius $R > \mathscr{R}$, where $\mathscr{R} = \frac{2GM}{c^2}$ is the Schwarzschild radius. Derive Kepler's third law in the form, as it would appear to an observer far away from the system

$$\Omega^2 = \frac{1}{2}\frac{\mathscr{R}}{r^3} \tag{9}$$

where Ω is the angular velocity of the particle on its orbit.

Solution:
Begin with the Schwarzschild metric

$$\mathrm{d}s^2 = -\left(1-\frac{\mathscr{R}}{r}\right)\mathrm{d}t^2 + \left(1-\frac{\mathscr{R}}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2$$

and the geodesic equation $\ddot{x}^\mu + \Gamma^\mu_{\kappa\lambda}\dot{x}^\kappa\dot{x}^\lambda = 0$.
For a circular orbit, only $\dot{t} = \frac{\mathrm{d}x^0}{\mathrm{d}\tau} = u^0$ and $\dot{\phi} = \frac{\mathrm{d}x^\phi}{\mathrm{d}\tau}$ are non-vanishing. Write down the r-component of the geodesic equation

$$0 = \Gamma^r_{\nu\lambda}u^\nu u^\lambda = \left(\Gamma^r_{tt}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + 2\Gamma^r_{t\phi}\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \Gamma^r_{\phi\phi}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2\right) \qquad (*)$$

Choose coordinates such that the orbit is in the plane of the equator, hence $\theta = \frac{\theta}{2} = \mathrm{cste.}$. We need three Christoffel symbols in (*)

$$\begin{aligned}
\Gamma^r_{tt} &= -\tfrac{1}{2}g^{rr}g_{tt,r} = \tfrac{1}{2}\left(1-\frac{\mathscr{R}}{r}\right)\frac{\mathscr{R}}{r^2} \\
\Gamma^r_{t\phi} &= -g^{rr}g_{t\phi,r} = 0 \\
\Gamma^r_{\phi\phi} &= -\tfrac{1}{2}g^{rr}g_{\phi\phi,r} = -\left(1-\frac{\mathscr{R}}{r}\right)r
\end{aligned}$$

Since $\Omega = \frac{\mathrm{d}\phi}{\mathrm{d}t}$ (which is indeed the angular velocity as seen from far away),
(*) $\Rightarrow 0 = \frac{1}{2}\frac{\mathscr{R}}{r^2} - r\left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2$. That gives (9).

Supplément: On peut ajouter le suivant: pour un observateur qui se déplace en orbite avec la particule, la vitesse angulaire serait $\omega = \frac{\mathrm{d}\phi}{\mathrm{d}\tau} \neq \Omega$. L'équation (*) donne

$$0 = \frac{1}{2}\frac{\mathscr{R}}{r^2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2$$

De plus, en cours on a démontré la loi de conservation $\left(1 - \frac{\mathscr{R}}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\tau} = b = \frac{E}{m} = \mathrm{cste}.$ Ainsi

$$\omega^2 = \left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 = \frac{1}{2}\frac{\mathscr{R}}{r^3}\left(1 - \frac{\mathscr{R}}{r}\right)^{-2}\left(\frac{E}{m}\right)^2$$

Pour des rayons d'orbite $r \gg \mathscr{R}$, on retrouve la 3^e loi de Kepler, puis des corrections relativistes.

Série 5

1. Consider the following purely hypothetical variant of the Einstein field equations

$$R_{\mu\nu} - \alpha g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{1}$$

where $g_{\mu\nu}$ denotes the metric tensor, $T_{\mu\nu}$ the energy-momentum tensor, $R_{\mu\nu}$ and R are the Ricci tensor and scalar, respectively, G is the gravitational constant and α is a free constant parameter.

Show that for $\alpha \neq \frac{1}{2}$ one does not recover Newton's theory in the non-relativistic limit.

Solution:
Recall the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$. Its covariant divergence $G^{\mu\nu}{}_{;\mu} = 0$ vanishes.
Take the co-variant divergence from eq. (1): $\left(\frac{1}{2} - \alpha\right) R_{;\mu} = 8\pi G \, T_\mu{}^\nu{}_{;\nu}$.
On the other hand, one can contract (1) first and take the divergence afterwards, such that

$$\left(1 - 4\alpha\right) R_{;\mu} = 8\pi G T_{;\mu} \quad , \quad T = T^\mu{}_\mu$$

such that

$$T_\mu{}^\nu{}_{;\nu} = \frac{\frac{1}{2} - \alpha}{1 - 4\alpha} T_{;\mu} \qquad (*)$$

In the newtonian limit, have the energy-momentum tensor

$$T_{\mu\nu} = \begin{pmatrix} \rho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

Taking the component $\mu = 0$ from (*) shows $\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \left(\rho \boldsymbol{v}\right) = \frac{1}{2} \frac{1-2\alpha}{1-4\alpha} \frac{\partial \rho}{\partial t}$.
The newtonian mass conservation law is not reproduced for $\alpha \neq \frac{1}{2}$.

2. Ein Teilchen fällt radial unter dem Einfluß einer Schwarzschildmetrik. In Bezug auf die Eigenzeit im Unendlichen, wie groß ist die einwärts gerichtete Geschwindigkeit $\frac{\mathrm{d}r}{\mathrm{d}t}$ für einen Radius r ? Wie groß ist die *lokal gemessene* Geschwindigkeit relativ zu einem stationären Beobachter am gleichen Radius r ?

Hinweis: für radialen Fall kann man mit Hilfe der sogenannten Killingvektoren zeigen, daß u_0 konstant ist, wobei $\mathtt{u}$ die Vierergeschwindigkeit ist. Erinnern Sie sich auch an die Eigenschaft $\mathtt{u} \cdot \mathtt{u} = -1$.

Lösung:
Der metrische Tensor lautet

$$g_{\mu\nu} = \mathrm{diag}\left(-\left(1-\frac{\mathscr{R}}{r}\right), \left(1-\frac{\mathscr{R}}{r}\right)^{-1}, r^2, r^2\sin^2\theta\right)$$

Die Vierergeschwindigkeit sei $\mathtt{u} = \frac{\mathrm{dx}}{\mathrm{d}\tau}$, und man hat $\mathtt{u}\cdot\mathtt{u} = -1$. Dann, für radialen Einfall

$$\begin{aligned}
\mathtt{u}\cdot\mathtt{u} = -1 &= -\left(1-\frac{\mathscr{R}}{r}\right)\left(u^0\right)^2 + \left(1-\frac{\mathscr{R}}{r}\right)^{-1}\left(u^1\right)^2 + 0 \\
&= \left[-\left(1-\frac{\mathscr{R}}{r}\right) + \left(1-\frac{\mathscr{R}}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\right]\left(u^0\right)^2 \\
&= \left[-\left(1-\frac{\mathscr{R}}{r}\right) + \left(1-\frac{\mathscr{R}}{r}\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2\right]\left(u_0\right)^2\left(1-\frac{\mathscr{R}}{r}\right)^{-2}
\end{aligned}$$

die gesuchte Geschwindigkeit ist $$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \left(1-\frac{\mathscr{R}}{r}\right)^2\left[1-\left(1-\frac{\mathscr{R}}{r}\right)\left(u_0\right)^{-2}\right]$$

Ein stationärer Beobachter mißt die Zeitintervalle $\mathrm{d}\hat{t} = \sqrt{g_{00}}\,\mathrm{d}t = \left(1-\frac{\mathscr{R}}{r}\right)^{1/2}\mathrm{d}t$ und die radialen Abstände $\mathrm{d}\hat{r} = \sqrt{g_{11}}\,\mathrm{d}t = \left(1-\frac{\mathscr{R}}{r}\right)^{-1/2}\mathrm{d}r$ und mißt daher die Geschwindigkeit

$$\frac{\mathrm{d}\hat{r}}{\mathrm{d}\hat{t}} = \left(1-\frac{\mathscr{R}}{r}\right)^{-1}\frac{\mathrm{d}r}{\mathrm{d}t}$$

3. Consider a spherical mass distribution with total mass M and Schwarzschild radius $\mathscr{R} = \frac{2GM}{c^2}$. Its metric is therefore the Schwarzschild metric. A test particle (more exciting: a space craft !) crosses at some moment the Schwarzschild radius $\mathscr{R}$. Show that the particle (the space craft !) arrives at the centre at the latest in its proper time $\tau_{\max} = \frac{\pi}{2}\mathscr{R}$.

Can you imagine any forces which could change this result ? Or Mr Spock in all logic ?

Hints: in the interior of a the Schwarzschild radius, particles always fall towards the centre, hence $\mathrm{d}r/\mathrm{d}\tau < 0$. Notice the identity $\int \mathrm{d}r \ \left(\frac{R}{r} - 1\right)^{-1/2} = -\frac{R}{2} \arctan\left(\frac{1}{2}\ \frac{R-2\,r}{\sqrt{r}\sqrt{R-r}}\right) - \sqrt{r}\sqrt{R-r}$.

Solution:
The particle moves along a time-like orbit with $\mathtt{u}\cdot\mathtt{u} = -1$, where $\mathtt{u}$ is its four-velocity. With the Schwarzschild metric one has

$$1 = -\mathtt{u}\cdot\mathtt{u} = \left(1 - \frac{\mathscr{R}}{r}\right)\dot{t}^2 - \left(1 - \frac{\mathscr{R}}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\,\dot{\phi}^2$$

Since $r < \mathscr{R}$, all terms with the exception of one are negative. One therefore requires

$$\left(\frac{\mathscr{R}}{r} - 1\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 \overset{!}{>} 1$$

Since $\dot{r} < 0$, one has $\mathrm{d}r < -\left(\frac{\mathscr{R}}{r} - 1\right)^{-1/2}\mathrm{d}\tau$. The *maximal* proper time before the particle arrives at the centre is therefore

$$\tau_{\max} = -\int_{\mathscr{R}}^{0}\mathrm{d}r\left(\frac{\mathscr{R}}{r} - 1\right)^{-1/2} = \left(-\frac{\mathscr{R}}{2}\arctan\left(1/2\,\frac{\mathscr{R} - 2\,r}{\sqrt{r}\sqrt{\mathscr{R} - r}}\right) - \sqrt{r}\sqrt{\mathscr{R} - r}\right)\Bigg|_{\mathscr{R}}^{0} = \frac{\pi\mathscr{R}}{2}$$

This argument is so general and basic that it is very difficult indeed (well, impossible) to invent any way around.

3a. What would happen in the previous exercice, if the Schwarzschild metric would be replaced by a Schwarzschild-de Sitter metric ?

Solution:
not much, since the extra ‘cosmological’ force vanishes near the centre.

4. Consider the metric

$$\mathrm{d}s^2 = -\mathrm{d}w^2 + \frac{4}{9}\left[\frac{9\mathscr{R}}{4(z-w)}\right]^{2/3}\mathrm{d}z^2 + \left[\frac{9\mathscr{R}}{4}(z-w)^2\right]^{2/3}\mathrm{d}\Omega^2 \quad (2)$$

(a) Since the elements of the metric tensor depend explicit on the 'time coordinate' w, one might think that this were a dynamic metric. Show via a convenient change of coordinates that $\mathrm{d}s^2$ reduces to the outer Schwarzschild metric.
Hint: define first a new radial coordinate $r = \left[\frac{9}{4}\mathscr{R}(z-w)^2\right]^{1/3}$. Then diagonalise the resulting form of $\mathrm{d}s^2$, via the ansatz $z = t + F(r)$, an choose the function F conveniently.
(b) What is the motion in the coordinates w, z ? ($\mathrm{d}\Omega$ is the usual element of the solid angle)
(c) Show that, in the coordinates w, z, stationary observers are in free fall.

Solution:

(a) one identifies the coefficient of $\mathrm{d}\Omega^2$ with r, hence $r = \left[\frac{9\mathscr{R}}{4}\left(z-w\right)^2\right]^{1/3} \Rightarrow$
$w = z - \sqrt{\frac{4}{9\mathscr{R}} r^3}$. It follows $\mathrm{d}w = \mathrm{d}z - \frac{2}{3}\frac{3}{2}\sqrt{\frac{r}{\mathscr{R}}}\,\mathrm{d}r = \mathrm{d}z - \sqrt{\frac{r}{\mathscr{R}}}\,\mathrm{d}r$.

$$\begin{aligned}\Rightarrow \mathrm{d}s^2 &= -\mathrm{d}z^2 + 2\sqrt{\frac{r}{\mathscr{R}}}\,\mathrm{d}z\mathrm{d}r - \frac{r}{\mathscr{R}}\mathrm{d}r^2 + \frac{4}{9}\left(\frac{9\mathscr{R}}{4}\right)^2\left(\frac{4}{4\mathscr{R}(z-w)^2}\frac{9\mathscr{R}}{4}\right)^{1/3}\mathrm{d}z^2 + r^2\mathrm{d}\Omega^2 \\ &= -\left(1-\frac{\mathscr{R}}{r}\right)\mathrm{d}z^2 + 2\sqrt{\frac{r}{\mathscr{R}}}\,\mathrm{d}z\mathrm{d}r - \frac{r}{\mathscr{R}}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\end{aligned}$$

This quadratic form is diagonalised by setting $z = t + F(r)$. In order to find $F = F(r)$

$$\mathrm{d}s^2 = -\left(1-\frac{\mathscr{R}}{r}\right)\left(\mathrm{d}t^2 + 2F'\mathrm{d}t\mathrm{d}r + F'^2\mathrm{d}r^2\right) + 2\sqrt{\frac{r}{\mathscr{R}}}\,\mathrm{d}r\left(\mathrm{d}t + F'\mathrm{d}r\right) - \frac{r}{\mathscr{R}}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2$$

and choose F such that the mixed terms vanish: $-\left(1-\frac{\mathscr{R}}{r}\right)2F' + 2\sqrt{\frac{r}{\mathscr{R}}} \stackrel{!}{=} 0$.

This gives $F'(r) = \sqrt{\frac{r}{\mathscr{R}}}\left(1-\frac{\mathscr{R}}{r}\right)^{-1}$. Then, as asserted

$$\begin{aligned}\mathrm{d}s^2 &= -\left(1-\frac{\mathscr{R}}{r}\right)\mathrm{d}t^2 - \left(1-\frac{\mathscr{R}}{r}\right)\frac{\mathscr{R}}{r}\left(1-\frac{\mathscr{R}}{r}\right)^{-2}\mathrm{d}r^2 + 2\sqrt{\frac{r}{\mathscr{R}}}\sqrt{\frac{r}{\mathscr{R}}}\left(1-\frac{\mathscr{R}}{r}\right)^{-1}\mathrm{d}r^2 \\ &= -\left(1-\frac{\mathscr{R}}{r}\right)\mathrm{d}t^2 + \left(1-\frac{\mathscr{R}}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\end{aligned}$$

N.B.: in the **Lemaître coordinates** w, z, the singularity at $\mathscr{R}$ has disappeared, see (2).

(b) Using (2), $\mathrm{d}w = \mathrm{d}\tau$ is the proper time interval for an observer in free fall. A freely falling observer is moving along a line $z = \mathrm{cste}$.

(c) on the lines $z = \mathrm{cste}$ that is $\mathrm{d}z = 0$, one has

$$\mathrm{d}z = 0 = \mathrm{d}t + F'(r)\mathrm{d}r = \mathrm{d}t + \sqrt{\frac{r}{\mathscr{R}}}\left(1 - \frac{\mathscr{R}}{r}\right)^{-1}\mathrm{d}r$$

Division by $\mathrm{d}t$ leads to

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \frac{r}{\mathscr{R}}\left(1 - \frac{\mathscr{R}}{r}\right)^{-2}$$

which we recognise as the geodesic equation of the free fall (with $u_0 = 1$) of a previous exercice.

5. Zwei identische Uhren A und B werden am Äquator der Erde synchronisiert. Danach wird die Uhr A zum Nordpol transportiert und verbleibt dort ein Jahr, bevor sie wieder an den gleichen Ort am Äquator zurückkehrt. Während dieser Zeit ist die Uhr B an ihrem Ort auf der Erdoberfläche am Äquator verblieben. Beobachtet man einen Gangunterschied zwischen den beiden Uhren ?

Verwenden Sie die (äußere) Schwarzschildmetrik, um Ihre Antwort zu begründen. Die Reisezeit der Uhr A zum Nordpol und zum Äquator zurück kann vernachlässigt werden (warum ?).
(a) Betrachten Sie zunächst eine perfekt kugelförmige Erde, mit Radius $R = 6,38 \cdot 10^6[\mathrm{m}]$ und der Masse $M = 5,97 \cdot 10^{24}[\mathrm{kg}]$. Man hat $G = 6.67 \cdot 10^{-11}[\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}^{-2}]$.
(b) Bekanntlich ist die Erde an den Polen abgeplattet (aplâtie), der Unterschied der Radien R_e und R_p, am Äquator und an den Polen, beträgt $R_e - R_p \approx 21[\mathrm{km}]$. Kann der Einfluß der Abplattung auf den eventuellen Gangunterschied der Uhren vernachlässigt werden ? Ist es praktisch wichtig, ob am Nordpol oder am Südpol gemessen wird ?

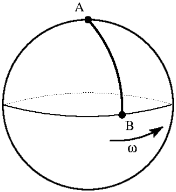
Lösung:
Wir betrachten das Feld der Erde (als perfekte Kugel angenommen). Damit Schwarzschildmetrik

$$\mathrm{d}s^2 = -\left(1-\frac{\mathscr{R}}{r}\right)c^2\mathrm{d}t^2 + \left(1-\frac{\mathscr{R}}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2$$

In erster Näherung kann die Reisezeit zum und vom Nordpol vernachlässigt werden. Aus der Vorlesung weiß man, daß die Reise zu Gangunterschieden der Größenordnung $\sim 10^{-12}$ führt. Die Reisezeit beträgt etwa $10^{-3} - 10^{-4}$ der kompletten Meßzeit (ein Jahr). Daher sollte der Beitrag der Reisezeit zum Gesamteffekt nur klein sein.

(a) für eine perfekte Kugel haben wir:

1. am Nordpol: $\mathrm{d}r = \mathrm{d}\theta = 0$, $r = R$ (Erdradius), $\theta = 0$.
$\mathrm{d}\phi = \omega\mathrm{d}t$.
$\Rightarrow \mathrm{d}s^2 = -c^2\mathrm{d}\tau_A^2 = -\left(1-\frac{\mathscr{R}}{r}\right)c^2\mathrm{d}t^2$. Also
Eigenzeit der Uhr A: $\mathrm{d}\tau_A = \left(1-\frac{\mathscr{R}}{r}\right)^{1/2}\mathrm{d}t$.



2. am Äquator: $\mathrm{d}r = \mathrm{d}\theta = 0$, $r = R$ (Erdradius), $\theta = \frac{\pi}{2}$.
$\mathrm{d}\phi = \omega\mathrm{d}t$.
$\Rightarrow \mathrm{d}s^2 = -c^2\mathrm{d}\tau_B^2 = -\left(1-\frac{\mathscr{R}}{r}\right)c^2\mathrm{d}t^2 + R^2\omega^2\mathrm{d}t^2$. Also
Eigenzeit der Uhr B: $\mathrm{d}\tau_B = \left[\left(1-\frac{\mathscr{R}}{r}\right) - \left(\frac{R\omega}{c}\right)^2\right]^{1/2}\mathrm{d}t$.

Damit relative Zeitverschiebung

$$\Delta = \frac{\mathrm{d}\tau_B - \mathrm{d}\tau_A}{\mathrm{d}\tau_A} = \frac{\left[\left(1-\frac{\mathscr{R}}{R}\right) - \left(\frac{R\omega}{c}\right)^2\right]^{1/2} - \left[1-\frac{\mathscr{R}}{R}\right]^{1/2}}{\left[1-\frac{\mathscr{R}}{R}\right]^{1/2}}$$

Zahlenwerte: für die Erde $\mathscr{R} = \frac{2GM}{c^2} \simeq 9 \cdot 10^{-3}[\mathrm{m}] \Rightarrow \frac{\mathscr{R}}{R} \sim 10^{-9}$
Rotationsterm: $\omega = \frac{2\pi}{T} \simeq \frac{2\pi}{24\cdot 3600}[\mathrm{s}^{-1}] \simeq 7.3 \cdot 10^{-5}[\mathrm{s}^{-1}]$
$\Rightarrow \frac{R\omega}{c} \simeq \frac{76.38\cdot 7.3}{2.99} \cdot 10^{6-5-8} \simeq 1.56 \cdot 10^{-6}$
also Abschätzung: $\Delta \simeq 1 - \frac{1}{2}\frac{\mathscr{R}}{R} - \frac{1}{2}\left(\frac{R\omega}{c}\right)^2 - 1 + \frac{1}{2}\frac{\mathscr{R}}{R} + \mathrm{O}\left(\left(\frac{\mathscr{R}}{R}\right)^2\right) \simeq -1.2 \cdot 10^{-12}$ das ist negatif ! Die Erdrotation erzeugt diesen Effekt (die Uhr am Pol dreht sich nicht mit). Grundsätzlich sind solche Zeitunterschiede meßbar.

(b) unter Berücksichtigung der Erdabplattung hat man am Pol und am Äquator verschiedene Radien. (wir unterstellen, daß die Schwarzschildmetrik noch nicht wesentlich geändert wird).

1. am Nordpol: $r = R_p$, $\theta = 0$, $\mathrm{d}\phi = \omega \mathrm{d}t \Rightarrow \mathrm{d}\tau_A = \left(1 - \frac{\mathscr{R}}{R_p}\right)^{1/2} \mathrm{d}t.$
2. am Äquator: $r = R_e$, $\theta = \frac{\pi}{2}$, $\mathrm{d}\phi = \omega \mathrm{d}t \Rightarrow \mathrm{d}\tau_B = \left[\left(1 - \frac{\mathscr{R}}{R_e}\right) - \left(\frac{R_e\omega}{c}\right)^2\right]^{1/2} \mathrm{d}t$

Damit relative Zeitverschiebung

$$\Delta = \frac{\mathrm{d}\tau_B - \mathrm{d}\tau_A}{\mathrm{d}\tau_A} = \frac{\left[\left(1-\frac{\mathscr{R}}{R_e}\right)-\left(\frac{R_e\omega}{c}\right)^2\right]^{1/2}-\left[1-\frac{\mathscr{R}}{R_p}\right]^{1/2}}{\left[1-\frac{\mathscr{R}}{R_p}\right]^{1/2}}$$

$$\simeq 1-\tfrac{1}{2}\frac{\mathscr{R}}{R_e}-\tfrac{1}{2}\left(\frac{R_e\omega}{c}\right)^2-1+\tfrac{1}{2}\frac{\mathscr{R}}{R_p}+\mathrm{O}\left(\left(\frac{\mathscr{R}}{R_{e,p}}\right)^2\right)$$

$$\simeq -\frac{\mathscr{R}}{2}\left(\frac{1}{R_e}-\frac{1}{R_p}\right)-\frac{1}{2}\left(\frac{R\omega}{c}\right)^2 \qquad \text{mit } R\text{: mittlerer Erdradius}$$

$$= \frac{1}{2}\frac{\mathscr{R}}{R}\frac{R_e-R_p}{R}-\frac{1}{2}\left(\frac{R\omega}{c}\right)^2$$

der zusätzliche Term ist etwa $\frac{1}{2}\frac{9\cdot10^{-3}}{6\cdot10^{6}}\frac{2.1\cdot10^{4}}{6.38\cdot10^{6}} \simeq 2.3\cdot10^{-12}$. Damit wird $\Delta \simeq (2.3-1.2)\cdot10^{-12} \simeq +1\cdot10^{-12}$ positif ! $\Rightarrow$ Die Abplattung ist wesentlich und überkompensiert den Rotationsterm.

* Nordpol und Südpol der Erde unterscheiden sich durch eine geographische Höhe von etwa 3[km]. Der Effekt auf Δ ist eine Größenordnung kleiner als die Effekte der Abplattung (in modernen Präzisionsexperimenten *a priori* sichtbar).

6. Wir betrachten die Erweiterung der Einsteinschen Feldgleichungen durch einen Zusatzterm, parametrisiert durch eine 'kosmologische Konstante' Λ. Außerhalb einer Masse M lautet im Falle sphärischer Symmetrie die Lösung der erweiterten Feldgleichungen $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0$ dann

$$\mathrm{d}s^2 = -Ac^2\,\mathrm{d}t^2 + A^{-1}\,\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2 \;\;;\;\; A := 1 - \frac{\mathcal{R}}{r} - \frac{\Lambda}{3} r^2 \qquad (4)$$

wobei $\mathcal{R} = 2GM/c^2$ der Schwarzschildradius ist. Zeigen Sie, daß für einen weit entfernten Beobachter die Kreisfrequenz $\Omega = \mathrm{d}\phi/\mathrm{d}t$ der Umlaufbahn gegeben ist durch:

$$\Omega^2 = \frac{c^2}{2}\frac{\mathcal{R}}{r^3} - \frac{\Lambda c^2}{3} \qquad (5)$$

Welches bekannte physikalische Gesetz erhält man im Fall $\Lambda = 0$? Gibt es eine drastische qualitative Konsequenz, wenn $\Lambda > 0$? Was würde für $\Lambda < 0$ passieren ?

Lösung:
Man beginnt mit der Metrik ($c = 1$)

$$\mathrm{d}s^2 = -A\mathrm{d}t^2 + A^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2 \;\;,\;\; A = 1 - \frac{\mathscr{R}}{r} - \frac{\Lambda}{3}r^2$$

und man hat die Bewegungsgleichungen $\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu{}_{\nu\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau} = 0$, wobei $\mu = (t, r, \theta, \phi)$. Für eine Kreisbahn sind nur t, ϕ variabel. Damit insbesondere

$$0 = \Gamma^r_{\nu\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\lambda}{\mathrm{d}\tau} = \Gamma^r_{tt}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + 2\Gamma^r_{t\phi}\frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \Gamma^r_{\phi\phi}\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 \qquad (*)$$

mit den Christoffelsymbolen $\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\rho}\left(g_{\rho\nu,\lambda} + g_{\rho\lambda,\nu} - g_{\nu\lambda,\rho}\right)$. Explizite Rechnung

$$\begin{aligned}
\Gamma^r_{tt} &= \frac{1}{2}g^{rr}\left(-g_{tt,r}\right) = \frac{1}{2}g^{rr}\frac{\partial}{\partial r}\left(1 - \frac{\mathscr{R}}{r} - \frac{\Lambda}{3}r^2\right) = g^{rr}\left(\frac{\mathscr{R}}{2r^2} - \frac{\Lambda}{3}r\right) \\
\Gamma^r_{t\phi} &= 0 \\
\Gamma^r_{\phi\phi} &= \frac{1}{2}g^{rr}\left(-g_{\phi\phi,r}\right) = -\frac{1}{2}g^{rr}2r = -g^{rr}r
\end{aligned}$$

Einsetzen in (*) ergibt

$$\left(\frac{1}{2}\frac{\mathscr{R}}{r^2} - \frac{\Lambda}{3}r\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 - r\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 = 0$$

Und für einen weit entferntern Beobachter findet man (5)

$$\Omega^2 = \left(\frac{\mathrm{d}\phi}{\mathrm{d}t}\right)^2 = \frac{1}{2}\frac{\mathscr{R}}{r^3} - \frac{\Lambda}{3}$$

* für $\Lambda = 0$ ist das das bekannte 3. Keplersche Gesetz, wie erwartet.

* für $\Lambda > 0$ gibt es einen maximalen Radius für eine Kreisbahn $r_{\max} = \left(\frac{3}{2}\frac{\mathscr{R}}{\Lambda}\right)^{1/3}$. Die kosmologische Konstante gibt damit eine maximale Größe für gravitativ gebundene Objekte an.

Zahlenbeispiel: die Masse von typischen Galaxien ist $M_g \simeq 4 \cdot 10^{11} M_\odot$. Die Kosmologen behaupten, daß $\Lambda \simeq 10^{-52}[\mathrm{m}^{-2}] > 0$. Dann $r_{\max} \simeq \left(1.5 \cdot 4 \cdot 10^{11} \cdot 3000 \cdot 10^{52}\right)^{1/3}[\mathrm{m}] \simeq 2.5 \cdot 10^{22}[\mathrm{m}] \sim 10^6[\mathrm{pc}]$ distance typique entre grandes galaxies $\Rightarrow$ Stabilität von Galaxienhaufen könnte durch Λ beschränkt sein.

N.B.: $r_{\max}$ hängt nicht sehr empfindlich von $\mathscr{R}$ oder M ab.

N.B.': solche 'transversalen Effekte' treten auf wesentlich kleineren Skalen als $\Lambda^{-1/2}$ auf.

* für $\Lambda < 0$ hätte man $\Omega^2 \to \frac{|\Lambda|}{3}$ für sehr große Kreisbahnen mit $r \to \infty$ (starre Rotation).