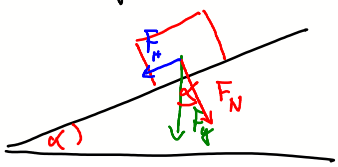
Vorlesung 11.12.2020

24 Betonplatte



geg. $\rho = 2500\ kg/m^3$

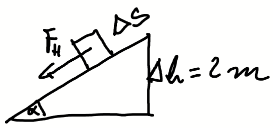
$V = a \cdot b \cdot c = 0{,}5\ m^3$

$m = \rho \cdot V = 0{,}5\ m^3 \cdot \rho = 1250\ kg$

auch 20°

$F_g = m \cdot g$; $F_N = F_g \cos\alpha = 1250\ kg \cdot 9{,}81\ m/s^2 \cdot \cos 30° = \underline{\underline{10{,}6\ kN}}$

b)



$\Delta s = \frac{\Delta h}{\sin\alpha} \overset{\alpha=30}{=} 4m$

gleichmäßig beschleunigte Bewegung $\Delta s = \frac{a}{2} \Delta t^2$; $a = \frac{F_H}{m}$ ← $F_g \cdot \sin\alpha$

$a = \frac{\cancel{m} \cdot g \cdot \sin\alpha}{\cancel{m}}$

$\Delta s = \frac{g}{2} \sin\alpha\ \Delta t^2$

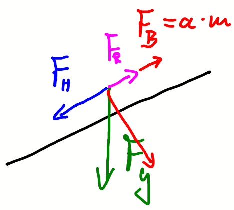
$\Delta t = \sqrt{\frac{2\Delta s}{g \sin\alpha}} = \sqrt{\frac{2\Delta h}{g \underset{0{,}5}{\sin^2\alpha}}} = \sqrt{\frac{8\Delta h}{g}} = \underline{\underline{1{,}28\ s}}$

Geschwindigkeit $\Delta E_{pot} = \Delta E_{kin} = mgh = \frac{m}{2} v^2$

$v = \sqrt{2g\Delta h} = 6{,}26\ m\,s^{-1}$

$v = a \cdot \Delta t = \frac{g}{2} \sin\alpha \cdot \Delta t$

c) mit Reibung, wann geht es los?



$F_R = \mu \cdot F_N$; $\left.\begin{array}{l} F_N = F_g \cdot \cos\alpha \\ F_H = F_g \cdot \sin\alpha \end{array}\right\}$ Bewegung wenn $F_H > F_R$

$F_g \mu \cos\alpha_{max} = F_g \sin\alpha_{max} \Rightarrow \alpha_{max} = \arctan\mu \mathrel{\hat{=}} \underline{\underline{11.3°}}$

d) Beschleunigung mit Reibung

$a = \frac{F_H - F_R}{m} = \frac{mg\sin\alpha - \mu mg\cos\alpha}{m} = g(\sin\alpha - \mu\cos\alpha)$

$\Delta s = \frac{a}{2} \Delta t^2 = \frac{\Delta h}{\sin\alpha} = \frac{1}{2} g(\sin\alpha - \mu\cos\alpha) \cdot \Delta t^2$

$\Delta t = \sqrt{\frac{2\Delta h}{g(\sin\alpha - \mu\cos\alpha)\sin\alpha}} = \underline{\underline{1.58\ s}}$

Geschwindigkeit $v = a \cdot \Delta t = g(\sin\alpha - \mu\cos\alpha) \cdot \Delta t = \underline{\underline{5\ m\,s^{-1}}}$

mit Energie $\Delta E_{pot} - \Delta E_{Reib} = \Delta E_{kin}$

$mg\Delta h - F_R \cdot \Delta s = \frac{m}{2} v^2$

$mg\Delta h - mg\mu\cos\alpha \cdot \frac{\Delta h}{\sin\alpha} = \frac{m}{2} v^2$

$v = \sqrt{2g\Delta h\left(1 - \frac{\mu}{\tan\alpha}\right)} = \underline{\underline{5\ m\,s^{-1}}}$

25 Bremsfallschirme

a) $a = -kv^2 = \frac{dv}{dt}$; gew. Dgl. 1. Ord. ⇒ Trennung der Variablen

$$\frac{dv}{v^2} = -k\,dt \quad \Big| \int$$

$$\int_{v_0}^{v_1} \frac{dv}{v^2} = -\int_0^{t_1} k\,dt \Rightarrow \left[-\frac{1}{v}\right]_{v_0}^{v_1} = \left(-\frac{1}{v_1} + \frac{1}{v_0}\right) = -k\cdot t_1$$

$$k\,t = \left(\frac{1}{v(t)} - \frac{1}{v_0}\right) = \frac{v_0 - v(t)}{v(t)\cdot v_0}$$

nach $v(t)$ umstellen

$$t_1 = \frac{1}{k}\left(\frac{1}{v_1} - \frac{1}{v_0}\right) = \frac{v_0 - v_1}{k v_0 \cdot v_1} ; \quad \text{mit } \left.\begin{matrix} v_0 = 50\,m/s \\ v_1 = 1\,m/s \end{matrix}\right\} \; \underline{\underline{t_1 = 24{,}5\,s}}$$

b) Weg ausrechnen aus $v = \frac{ds}{dt}$; allg. $v(t) = \frac{v_0}{1 + k v_0 \cdot t}$

$$s_1 = \int_0^{t_1} v(t)\,dt = \int_0^{t_1} \frac{v_0}{1 + k v_0 t}\,dt ;$$ Substitution mit Grenzen

$z = 1 + k v_0 \cdot t$ $\quad$ $t = 0 \Rightarrow z_0 = 1$

$dz = k v_0\,dt$ $\quad$ $t = t_1 \Rightarrow z_1 = 1 + k v_0 \cdot t_1$

$dt = \frac{1}{k v_0}\,dz$

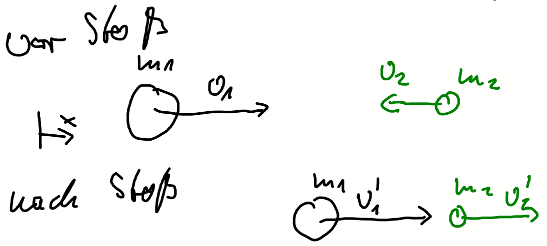
$$s_1 = \frac{1}{k}\int_{z_0}^{z_1} \frac{dz}{z} = \frac{1}{k}\left[\ln z\right]_{z_0}^{z_1} = \frac{1}{k}\left(\ln z_1 - \ln z_0\right) = \frac{1}{k}\ln\frac{z_1}{z_0} = \frac{1}{k}\ln\left(1 + k v_0 \cdot t_1\right)$$

$$s_1 = \frac{1}{k}\ln\frac{v_0}{v_1} ; \quad \text{mit } v_1 = 1\,m/s \Rightarrow \underline{\underline{s_1 = 97{,}8\,m}}$$

Vorlesung allg. Stöße

a) gerade, 100% elastische Stoß

↑ mit Energieerhaltung + Impulserhaltung $\vec{p} = m \cdot \vec{v}$?! immer



$$\frac{m_1}{2} v_1^2 + \frac{m_2}{2} v_2^2 = \frac{m_1}{2} v_1'^2 + \frac{m_2}{2} v_2'^2 \quad \text{Energie}$$

$$m_1 \cdot v_1 + m_2 \cdot v_2 = m_1 v_1' + m_2 v_2' ;$$?! $v_{1,2}$ können auch < 0 sein

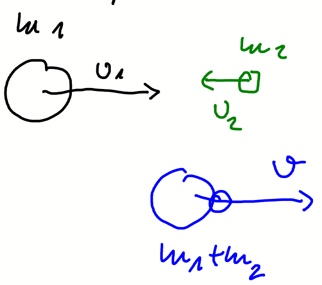
⇒ geg. 2 Gleichungen mit 2 Unbekannten v_1', v_2'

$$v_1' = \frac{(m_1 - m_2)v_1 + 2 m_2 v_2}{m_1 + m_2} ; \quad v_2' = \frac{(m_2 - m_1)v_2 + 2 m_1 v_1}{m_1 + m_2}$$ nachrechnen

Bem: speziell wenn $m_1 = m_2 \;\hat{=}\;$ Billiard

b) gerade 100% inelastische Stoß

Vor dem Stoß

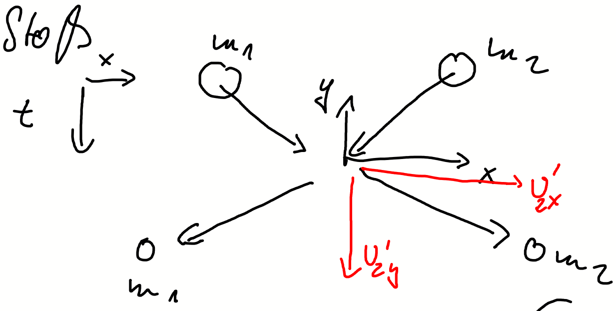


keine Energieerhaltung aber immer Impulserhaltung

$$m_1 v_1 + m_2 v_2 = (m_1+m_2)\, v$$

$$v = \frac{m_1 v_1 + m_2 v_2}{m_1+m_2}$$

c) Bem. schiefer Stoß

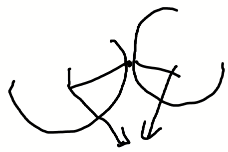


$\Rightarrow$ mit Energieerhaltung $\Rightarrow$ 1 Gl.

ges. v_{2x}'; v_{2y}'; v_{1x}'; v_{2y}'

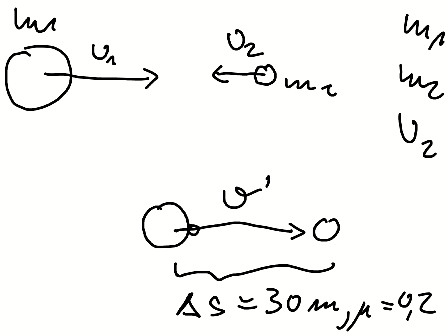
Impulserhaltung in x & y

weitere 2 Gl. durch Winkel



$\Rightarrow$ viel rechnen :-/

Zurück zu 2.6 - Auto-Crash $\hat{=}$ 100% inelastische



$m_1 = 2000\,kg$
$m_2 = 800\,kg$
$v_2 = 42\,km/h$

$$E_{kin}' = W_{Reibung}$$

$$\frac{m_1+m_2}{2} v' = \mu (m_1+m_2) g \cdot \Delta s$$

$$v' = \sqrt{2\mu g s} = 39\,km/h$$

Impulserhaltung

wenn $v>0$ gerechnet

$$m_1 v_1 - m_2 v_2 = (m_1+m_2)\cdot v' \Rightarrow v_1 = 74\,km/h$$

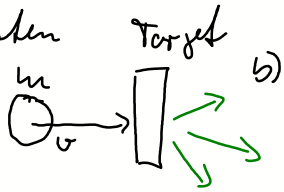
b) $$\Delta E = E_{kin,vor} - E_{kin,nach} = \frac{m_1}{2} v_1^2 + \frac{m_2}{2} v_2^2 - \frac{m_1+m_2}{2} v'^2$$

$$\frac{\Delta E}{E_{kin,vor}} = 1 - \frac{(m_1+m_2)\, v'^2}{m_1 v_1^2 + m_2 v_2^2} = 0{,}63 \hat{=} 63\%$$ der kin. durch Stoß in Deformationsarbeit, andere 37% für Reibung

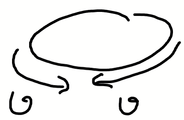
Wichtig für Beschleuniger (Teilchen)

2 Arten

a)



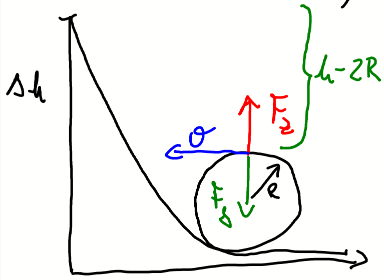
b)



Ringbeschleuniger wenn $m_1 = m_2$
$v_1 = -v_2$

$\Rightarrow$ 100% in neue Energieformen vorhanden

28 - Looping, Wagen ohne Rotationsenergie



$mg = F_Z$ $\qquad$ v aus Energieerhaltung

$\frac{mv^2}{R} = mg$ $\qquad$ $mg(h-2R) = \frac{m}{2}v^2$

$v = \sqrt{2g(h-2R)}$

$v^2 = 2g(h-2R)$

$R = \frac{v^2}{g} = 2(h-2R)$; ges. $R(h)$

$R = 2h - 4R$

$R = \frac{2}{5}h = 0.4m$

b) Geschwindigkeit am Ende $v = \sqrt{2gh} = 4{,}4\,m\,s^{-1}$

c) Kugel wäre langsamer, da auch ΔE_{pot} in E_{rot} umgewandelt wird

Ergebnis; a) nicht vom Kugelradius abhängig

b) $R = \frac{10}{27}h = \underline{\underline{0{,}37\,m < 0{,}4\,m}}$

Vorlesungszeug, Potenzielle Energie aus $\oint \vec{F}\cdot d\vec{s}$ ← entlang Kurve

? Vektorfeld

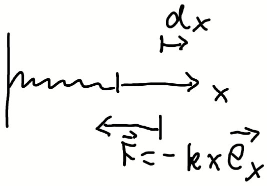
wenn Feld $\vec{F}(\vec{r})$ konservativ, dann

Weg unabhängig $\Leftrightarrow$ $\text{rot}\,\vec{F}(\vec{r}) = \vec{0}$; $\text{rot}\,\vec{F} = \vec{\nabla}\times\vec{F}$; $\vec{\nabla} = \begin{pmatrix}\partial_x\\ \partial_y\\ \partial_z\end{pmatrix}$

Def. $E_{pot} = V = -\oint \vec{F}\cdot d\vec{s} = U$

Umkehrung: $\boxed{\vec{F} = -\text{grad}\,U = -\vec{\nabla}U = -\begin{pmatrix}\partial_x U\\ \partial_y U\\ \partial_z U\end{pmatrix}}$ ← $\vec{\nabla}$ in kartesischen Koor.

Bsp. Feder in 1D



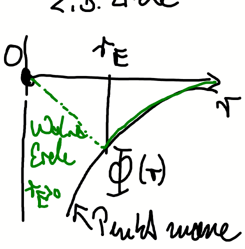
$\Rightarrow U = -\int \vec{F}\cdot d\vec{s} = \frac{k}{2}x^2$

$\vec{F} = -\vec{\nabla}U = -\frac{k}{2}2x\vec{e}_x = -kx\vec{e}_x$

zurück zu Gravitationspot. $\Phi(\vec{r}) = -\frac{GM}{r}$

Wdh: $f(x)\Big|_{x=x_0} = f(x_0) + \frac{1}{1!}\frac{df(x)}{dx}\Big|_{x_0}\cdot(x-x_0)^1 + \frac{1}{2!}\frac{d^2f(x)}{dx^2}\Big|_{x_0}\cdot(x-x_0)^2 + \dots$

M z.B. Erde



Taylorreihe um $r = r_E$

$\Phi(r)\Big|_{r=r_E} = -\frac{GM}{r_E} + \frac{GM}{r^2}\Big|_{r_E}\cdot\underbrace{(r-r_E)^1}_{h} + \dots$

$\frac{GM}{r_E^2} = g = 9{,}81\,m/s^2$

$\Rightarrow \Delta E_{pot}(h \ll r_E) = m\cdot\Phi = mgh$

Umrechnung $\vec F = -\nabla E_{pot}$ $\qquad \vec\nabla = \begin{pmatrix}\partial_x\\ \partial_y\\ \partial_z\end{pmatrix}$

einfach $E_{pot} = mgz \Rightarrow \vec F = \begin{pmatrix}0\\0\\-mg\end{pmatrix}$

besser $E_{pot} = -m\frac{GM}{r} \Rightarrow F = +GmM\begin{pmatrix}\partial_x \frac{1}{\sqrt{x^2+y^2+z^2}}\\ \partial_y \frac{1}{\sqrt{}}\\ \partial_z \frac{1}{\sqrt{}}\end{pmatrix} = -GmM\begin{pmatrix}\frac{x}{\sqrt{}^{3/2}}\\ \frac{y}{\sqrt{}^{3/2}}\\ \frac{z}{\sqrt{}^{3/2}}\end{pmatrix}$

Abkürzung $\sqrt{x^2+y^2+z^2}$ $\qquad F = -\nabla E_{pot}$

$$\vec F = -G\frac{mM}{r^2}\frac{\vec r}{|\vec r|}$$

geht auch in Kugelkoor aber ?! $\vec\nabla_{Kugelkoor} = \begin{pmatrix}\partial_r\\ \frac{1}{r}\partial_\vartheta\\ \frac{1}{r\sin\varphi}\partial_\varphi\end{pmatrix} \begin{matrix}\leftarrow \vec e_r\\ \leftarrow \vec e_\vartheta\\ \leftarrow \vec e_\varphi\end{matrix}$

oft $U(r,\vartheta,\varphi)$ nur $U(r,\cancel{\vartheta},\cancel{\varphi})$

z.B. $U = -G\frac{M}{r}$

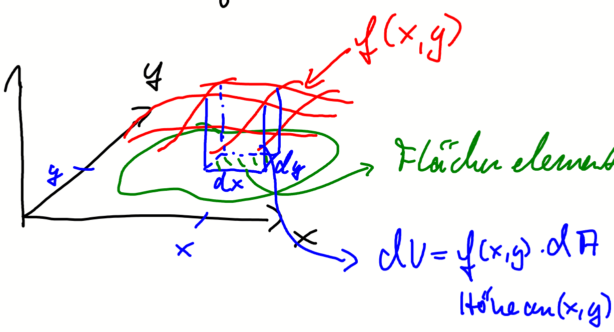
$$\vec F = \begin{pmatrix} mGM\,\partial_r\left(\frac{1}{r}\right)\\ mGM\,\frac{1}{r}\partial_\vartheta\left(\frac{1}{r}\right)\\ \dots\ \partial_\varphi\left(\frac{1}{r}\right)\end{pmatrix}\begin{matrix}\leftarrow \vec e_r\\ \leftarrow \vec e_\vartheta\\ \leftarrow \vec e_\varphi\end{matrix} = \begin{pmatrix}-G\frac{mM}{r^2}\\ 0\\ 0\end{pmatrix};\quad \vec F = -G\frac{mM}{r^2}\vec e_r$$

Bisher Linienintegrale $\int \vec F\cdot d\vec s$

Was sind Flächen / Volumen integrale (nicht Oberflächenintegrale)

Integral über Fläche

Volumen von $f(x,y)$ über Fläche



$V = \iint f(x,y)\,dA = \int dV$

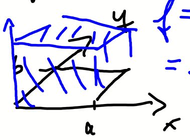
$[dA] = m^2$

V = Volumen, wenn $[f] = m$

V = Masse, wenn $[f] = \frac{kg}{m^2}$

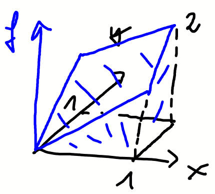
Beispiele

a)



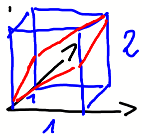
$f = 1 = f(x,y)$ $\qquad V = \int_0^a dx \underbrace{\int_0^b dy\cdot 1}_{b} = \int_0^a dx\, b = a\cdot b$ ✓ weil $\left[\iint f\,dA\right] = [f]\cdot[dA]$

b) $f = x+y$



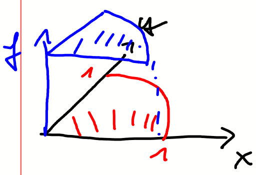
$$V = \int_0^1 dx\int_0^1 dy\,(x+y) = \int_0^1 dy\left(\frac{x^2}{2}+xy\right) = \left[\frac{x^2y}{2}+\frac{xy^2}{2}\right]_{0,0}^{1,1} = \underline{\underline{1}}$$

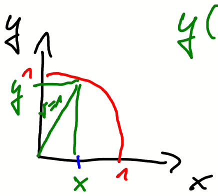
$\hat{=}\ \frac{\text{Volumen}}{2}$ von diesem Quader



$V = 1\cdot 1\cdot 2/2 = 1$

c) $f(x,y)=1$, Fläche $\frac{1}{2}$ Kreis



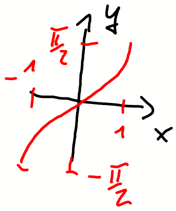


$y(x)=\sqrt{1-x^2}$

$$V=\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \cdot \overset{f(x,y)}{1} = \int_0^1 dx \,\Big[y\Big]_0^{\sqrt{1-x^2}} = \int_0^1 dx \sqrt{1-x^2}$$

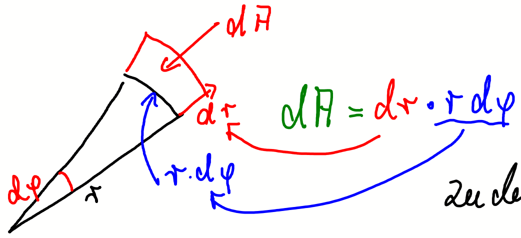
Tafelwerk

$$V=\frac{1}{2}\Big[x\sqrt{1-x^2}+\arcsin x\Big]_0^1 = \frac{\pi}{4}$$



Logo $\hat{=}\ \frac{V}{4}$ Zylinder mit $r=1$, $h=1$ $\quad V_{Zylind}=\pi r^2 h$

nun Polarkoor. $(x,y) \rightsquigarrow (r,\varphi)$

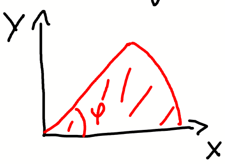


$dA = dr \cdot r\,d\varphi$

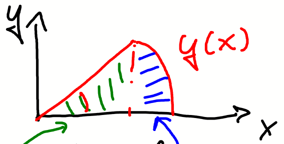
$dx \cdot dx \rightarrow r\,dr\,d\varphi$

zu dem $\frac{1}{4}$ Zylinder $\quad V=\int_0^1 dr\, r \int_0^{\frac{\pi}{2}} d\varphi \cdot \overset{f(r,\varphi)}{1} = \int_0^1 r\,dr \cdot \frac{\pi}{2} = \frac{\pi}{4}$

so auch einfach wenn



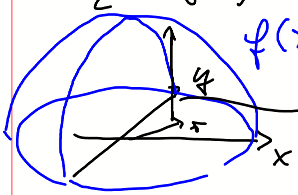
in kartesischen



$$V=\int \ldots dx\,dy + \int \ldots dx\,dy$$

in Polarkoor. $\quad V=\int_0^1 r\,dr \int_0^{\varphi} d\hat{\varphi} \cdot 1 = \frac{1}{2}\varphi$

d) $\frac{1}{2}$ Kugel; $R=1$

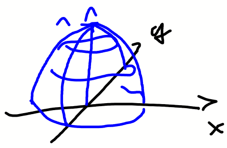


$f(x,y)$ besser $\rightarrow f(r,\cancel{\varphi}) = \sqrt{R^2-r^2}=\sqrt{1-r^2}$;

$$V_{K/2}=\int_0^{2\pi} d\varphi \int_0^{R=1} dr \cdot \underset{\text{Polarkk.}}{r} \cdot \sqrt{1-r^2} = 2\pi \int_0^1 dr\, r\sqrt{1-r^2} = 2\pi\Big[-\frac{1}{3}\sqrt{1-r^3}\Big]_0^1$$

$$=\frac{2}{3}\pi \ \hat{=}\ \frac{\frac{4}{3}\pi R^3}{2} \ \leftarrow R=1$$

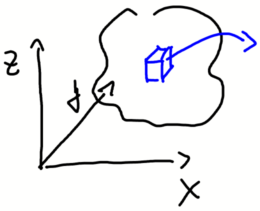
e) Paraboloid



$f(r,\cancel{\varphi}) = 1-r^2$

$$V=\int_0^{2\pi} d\varphi \int_0^1 dr\, r \cdot (1-r^2)$$

f) Volumenintegral

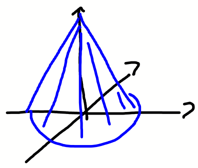


$dV = dx\,dy\,dz$

das $f(x,y,z) \hat{=}$ z.B Dichte ϱ in $\frac{kg}{m^3}$

$$M = \varrho \cdot V = \int dm$$
$$= \int dV \cdot \varrho$$

Bsp. Kegel mit $\varrho = 1 \frac{kg}{m^3}$; $R = H = 1$

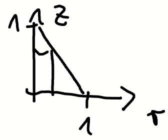


$$V = \int dx \int dy \int dz \cdot \varrho(x,y,z)$$, Grenzen

$x \in [-1, 1]$

$y \in [-\sqrt{R^2-x^2}, +\sqrt{R^2-x^2}]$

$z \in [0, 1-r]$, $r = \sqrt{x^2+y^2}$



$$V = \int_{-1}^{1} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int_{0}^{1-\sqrt{x^2+y^2}} dz \cdot \underbrace{1}_{\varrho}$$

$$V = \int_{-1}^{1} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \cdot \left(1 - \sqrt{x^2+y^2}\right) = \int_{-1}^{1} dx \left[x - \frac{1}{2} x \sqrt{x^2+y^2} - \frac{1}{2} y^2 \ln\left[x + \sqrt{x^2+y^2}\right] \right]_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}$$

⋮ Computeralgebra

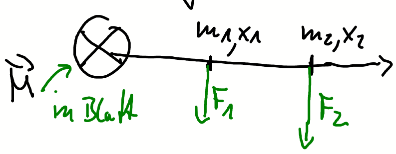
$$V = \frac{\pi}{3}$$ //

Lerneffekt: Grenzen angeben

Massenschwerpunkt

in 1D. Wdhl. Drehmoment $\vec{M} = \vec{r} \times \vec{F}$

2 Massen auf x-Achse haben

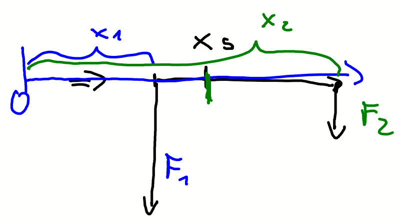


$$|\vec{M}| = m_1 \cdot x_1 + m_2 x_2$$

$\Downarrow$ Def x_S

$$= (m_1 + m_2) \cdot x_S$$ ← Schwerpunkt

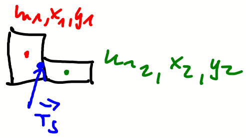
$$x_S = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$



Viele Massen im $\mathbb{R}^3$, jede Koordinate unabhängig

$$\vec r_S = \frac{\sum \vec r_i \cdot m_i}{\underbrace{\sum m_i}_{M}} \; ; \quad \vec r_S = \begin{pmatrix} x_S \\ y_S \\ z_S \end{pmatrix}$$

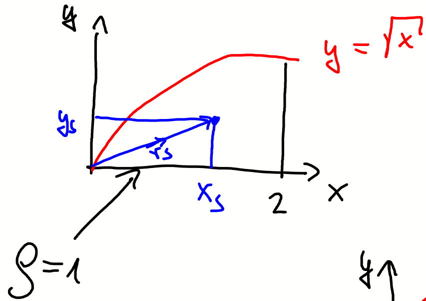
z.B. auch



$$\vec r_S = \frac{\int \vec r \cdot \rho(\vec r)\, dV}{\underbrace{\int \underbrace{\rho(\vec r)\, dV}_{dm}}_{M}} = \begin{pmatrix} \int x \,\rho(x,y,z)\, dV \\ \int y \,\rho(x,y,z)\, dV \\ \int z \,\rho(x,y,z)\, dV \end{pmatrix}$$

oft $\rho(x,y,z) = \text{const.}$ aber es bleibt x, y, z

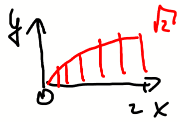
Bsp.: in 2D



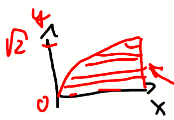
Varianten

a) $$x_S = \frac{\int_0^2 dx \int_0^{\sqrt{x}} dy \cdot x}{M} = \frac{\int_0^2 dx \; x\sqrt{x}}{\frac{4}{3}\sqrt{2}} = \frac{6}{5}$$

für x_S



$$M = \int_0^2 dx \int_0^{\sqrt{x}} dy \longrightarrow \frac{4}{3}\sqrt{2}$$



$$y_S = \frac{\int_0^{\sqrt{2}} y(2-y^2)\, dy}{M} = \frac{3}{4\sqrt{2}}$$