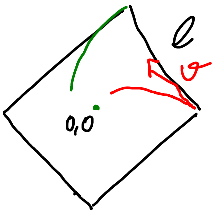
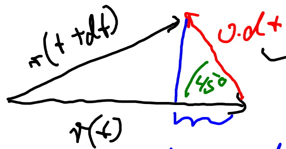
Vorlesung 27.11.20

Schildkröten



a) Wie lange? immer ein Quadrat ist
$\Rightarrow$ Länge der Seiten ist $l - v\cdot t \Rightarrow t_{ende} = \frac{l}{v}$

b) Was ist die Bahn?



$\hookrightarrow$ ist $<< r(t)$

im Maßstab

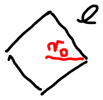
$$dr = v\,dt \cdot \cos 45^0 = v\,dt\,\frac{1}{\sqrt{2}}$$

$$r(t+dt) = r(t) - v\,dt/\sqrt{2}$$

$$\lim_{dt\to 0}\ \frac{r(t+dt)-r(t)}{dt} = -\frac{v}{\sqrt{2}} = \dot r(t) \Rightarrow \text{integrieren}$$

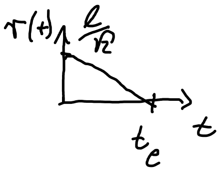
$$\int_{r_0}^{r(t)} dr' = -\int_{t=0}^{t} v\,dt' \Rightarrow r(t) - r_0 = -\frac{v\cdot t}{\sqrt{2}} - 0$$

$$r(t) = r_0 - \frac{v}{\sqrt{2}}t\ ;\quad r_0$$



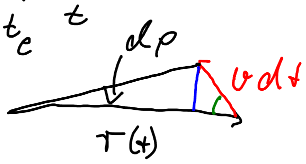
$$2r_0 = \sqrt{2}\,l$$
$$r_0 = \frac{l}{\sqrt{2}}$$

$$r(t) = \frac{1}{\sqrt{2}}(l - vt) \Rightarrow \text{wenn } t_e = \frac{v}{l} \text{ dann } r(t_e) = 0$$



$$\tan\varphi = \frac{v\,dt/\sqrt{2}}{r(t) - dr}$$

um $\varphi(t)$



$$;\quad \tan d\varphi = \frac{\frac{v\,dt}{\sqrt{2}}}{r(t)}$$

$\tan x \approx x$ für $x << 1$

$$d\varphi = \frac{v\,dt}{\sqrt{2}\,r(t)}$$

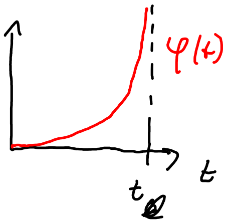
$$d\varphi = \frac{v}{\sqrt{2}\cdot\frac{1}{\sqrt{2}}(l - vt)}\,dt\ ;\quad \text{NR}\quad \int\frac{f'(x)}{f(x)}dx = \int\frac{\frac{df}{dx}}{f(x)}dx = \int\frac{1}{f}df$$
$$= \ln f$$

$$\varphi(t) = \Big[-\ln(l - v\cdot t)\Big]_0^t$$

$$\varphi(t) = -\ln(l - v\cdot t) + \ln l$$

? $\ln l = ?\ \ln(5\,m)$? geht nicht

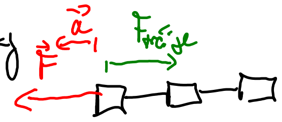
$$\left\|\ \varphi(t) = \ln\frac{l}{l - vt}\ \right\|$$



$\Rightarrow \infty$ viele Umdrehungen

14 Steinwurf $\Rightarrow$ a) 3,15 s = Steigzeit + Fallzeit,
b) 40,9 m
c) 26,7 m/s
d) 60,9° gegenüber Horizontalen

15 Zug



$$F - 3F_R = 3m \cdot a \Rightarrow a = \frac{F-3F_R}{3m} = 0{,}95\,m/s^2$$

700 N

ein Wagen

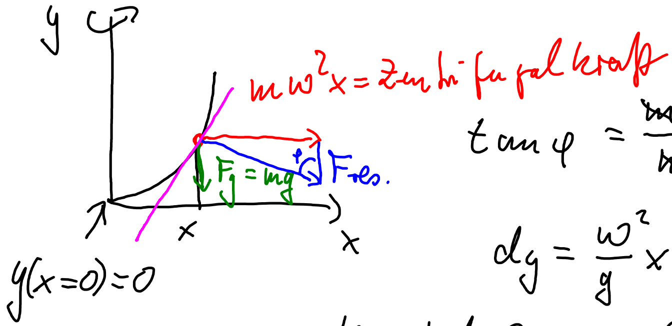
nur die letzten beiden Wagen

$$F_1 - 2F_R = 2ma \Rightarrow F_1 = \frac{2}{3}F = 30\,kN$$

analog letzter Wagen

$$F_2 - F_R = ma \Rightarrow F_2 = \frac{1}{3}F = 15\,kN$$

16 Kaffeebecher



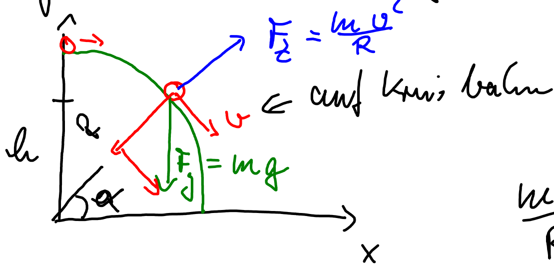
$$\tan\varphi = \frac{\cancel{m}\,\omega^2 x}{\cancel{m}\,g} = \frac{dy(x)}{dx}$$

$$dy = \frac{\omega^2}{g}x\,dx \Rightarrow y = \frac{\omega^2}{g}\int x\,dx = \frac{\omega^2}{2g}x^2 + C$$

Konstante C aus $y(0)=0 \Rightarrow C=0$

$$y = \frac{\omega^2}{2g}x^2 \;\hat{=}\; \text{Parabel}$$

17 gleitender Punkt auf Halbkugel



zum Abheben muss mein Komp. von F_z größer als F_g werden

$$\frac{mv^2}{R} = \sin\alpha \cdot mg\,;$$ v aus Energieerhaltung

$$\frac{mv^2}{2} = mg(R-h(t))$$

$$v^2 = 2g(R-h)$$

$$2g(R-h) = R\cdot\sin\alpha\cdot g\,;\quad \sin\alpha = \frac{h}{R}$$

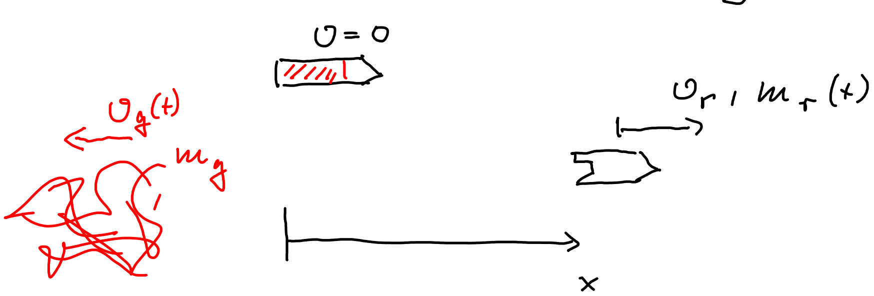
$$2g(R-h) = h\cdot g$$

$$h = \frac{2}{3}R$$ // für φ $\arcsin\frac{2}{3} = 41{,}8^\circ$ //

18 Rakete

t=0

t>0



Immer richtig ist Impulserhaltung $\Rightarrow F_{ext} = \frac{d}{dt}p$; hier $F_{ext}=0$

extern

Ausströmgeschwindigkeit Gase $v_g < 0$

$$0 = \frac{dp}{dt} = \frac{d}{dt}\left(m(t)\cdot v(t)\right) = \dot m_g v_g + m_r(t)\cdot\dot v_r \quad \forall t$$

Wieviel Gas pro s $\dot m_g > 0 \Rightarrow \dot v_r > 0 \Rightarrow a > 0$

$$\frac{dv_r}{dt} = - v_g \frac{1}{m_r(t)} \cdot \frac{dm_g}{dt} \quad ; \text{ auch } \frac{dm_g}{dt} = - \frac{dm_r(t)}{dt}$$

$$\frac{dv_r}{dt} = v_g \cdot \frac{1}{m_r(t)} \cdot \frac{dm_r}{dt} \quad \Big| \int$$

$$\int_{v_0}^{v_e} dv_r = v_e - v_0 = \Delta v_r = v_g \int_{m_0}^{m_e} \frac{1}{m_r} dm_r = v_g \ln \frac{m_e}{m_0}$$

$$\Rightarrow \Delta v_r = - \underbrace{v_g}_{<0} \underbrace{\ln \frac{m_0}{m_e}}_{>0} \Rightarrow \Delta v_r > 0 \; ; \; \Delta v_r = |v_g| \ln \frac{m_0}{m_e}$$

b) Geschwindigkeit Gase v_s? $F_{Schub} = const$

$$\underset{>0}{F_{Schub}} = \underset{<0}{\frac{dm_r}{dt}} \cdot \underset{<0}{v_g} \quad | \cdot dt \quad | \int$$

$$F_{Schub} \cdot t = v_g \int_{m_0}^{m_e} dm_r \Rightarrow v_g = \frac{F_{Schub} \cdot t}{\underbrace{m_e - m_0}_{<0}} \Rightarrow v_g < 0$$

c) Start von Erde aus $\Rightarrow F_{ext} \neq 0$

$\uparrow x$ $\quad \vec{g} = \begin{pmatrix} -g \\ 0 \\ 0 \end{pmatrix}$ $\quad \downarrow \vec{v}_g$

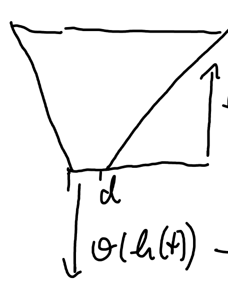
$$\frac{d}{dt}\vec{p} = \vec{F}_{ext} = -mg = \underbrace{\dot{m}_g \cdot v_g}_{F_{Schub}} + m_R(t) \cdot \dot{v}_R(t)$$

an $t=0 \Rightarrow m(t=0) = m_0$

$\dot{v}(t=0) = a_0$

$$-m_0 g - \dot{m}_g v_g = m_0 \cdot a_0 \Rightarrow a_0 = \frac{F_{Schub}}{m_0} - g$$

19 Ausflusszeit; jetzt aber Trichter 45°

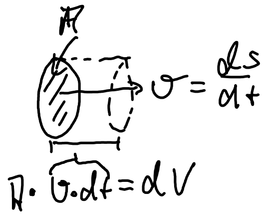


a) reibungsfrei, Energieerhaltung

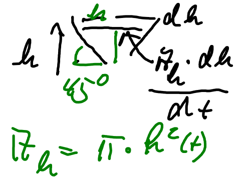
$$\underset{E_{pot}}{\rho g h} = \underset{E_{kin}}{\frac{\rho}{2} v^2} \qquad \text{pro Volumen sonst } \frac{mgh}{V} = \frac{m\rho}{2V}$$

$$v = \sqrt{2gh(t)}$$

Volumenstrom



$$\dot V = \frac{dV}{dt} = A\cdot v(h) = \frac{\pi}{4} d^2 \cdot \sqrt{2g\,h(t)} = -\frac{\overbrace{A_h \cdot dh}^{dV}}{dt}$$



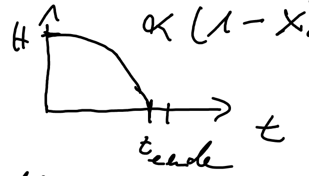
„−" da $\frac{dh}{dt} < 0$, h nimmt ab

$$\frac{\pi}{4} d^2 \sqrt{2gh} = -\pi h^2 \cdot \frac{dh}{dt}$$

$$\frac{d^2}{4}\sqrt{2g}\, dt = -h^{3/2}\, dh \qquad \Big| \int_0^t dt \quad \int_H^{h(t)} dh$$

$$\frac{d^2}{4}\sqrt{2g}\, t = -\frac{2}{5}\left[h^{5/2}\right]_H^{h(t)}$$

Einheiten checken $\quad \left(H^{5/2} - \frac{5}{8} d^2 \sqrt{2g}\cdot t\right)^{2/5} = h(t)$



$\Rightarrow$ Eimer/Trichter wird in endlicher Zeit leer

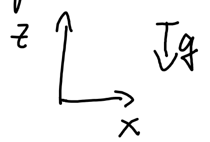
$t_{ende} \Rightarrow h(t_{ende}) = 0$

$$H^{5/2} = \frac{5}{8} d^2 \sqrt{2g}\cdot t_{max}$$

$$t_{max} = \frac{8}{5} H^{5/2} \cdot \frac{1}{d^2\sqrt{2g}}$$

Differentialgl. 1. & 2. Ordnung

2. Bsp. Freier Fall ohne Reibung



$m\ddot z = m\frac{d^2 z(t)}{dt^2} = -mg$ $\quad\hat{=}$ Dgl. 2. Ord.

hier speziell $\frac{dv_z}{dt} = -g$ $\quad$ mit $v_z = \frac{dz}{dt}$

Dgl. 1. Ord., inhomogen

$(y' + f(x)\, y = g(x)$; $g(x) = g)$

$f(x) = 0$

Lösen mit C aus AB

$dv_z = -g\, dt$

$v_z(t) = -g t + C \qquad$ AB: $v_z(0) = v_0$

$v_z(t) = -g t + v_0 \qquad \hat{=}$ konst. Beschl., geht auch so, wenn $a \neq$ const.

z.B. $a = \gamma \cdot t$

$[\gamma] = \frac{m}{s^3}$

nun auch $v_z(t) = \frac{dz(t)}{dt}$

$$\frac{dz}{dt} = -g t + v_0$$

$z(t) = -\frac{g}{2} t^2 + v_0 t + C$; AB2: $z(0) = z_0$

$$\| \; z(t) = -\frac{g}{2} t^2 + v_0 t + z_0 \; \|$$

Bem: • Trennung der Variablen geht immer, wenn

$| y'(x) = f(x)\cdot g(y) \|$ geht nicht wenn

$$y'(x) = f(x) + g(y)$$

$$y' = \sin x + \sin y$$

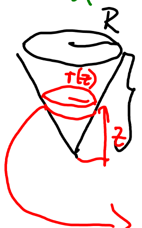
$$\frac{dy}{dx} = \sin x + \sin y$$

• für Dgl. 1. Ordnung brauchen wir 1. AB

für Dgl 2. Ordn. brauchen 2 AB u.s.w.

Bsp. für Integrieren; Formel Volumen Kegel

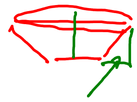
a)



$$r(z) = \frac{R}{H}\cdot z$$

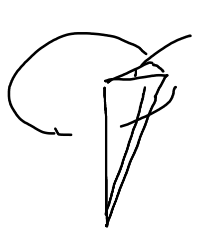
$$A(z) = \pi\, r(z)^2 = \pi \cdot \frac{R^2}{H^2} z^2$$

$$dV = A(z)\cdot dz$$ ≙ Zylinder nicht Kegelstumpf



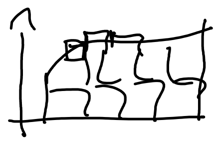
ist klein zu $r(z)$

$$\int_0^V dV = V = \int_0^H \pi \frac{R^2}{H^2}\cdot z^2\, dz = \left[\pi \frac{R^2}{H^2}\cdot\frac{z^3}{3}\right]_0^H = \frac{\pi}{3} R^2 H$$



$$V = \int dV \qquad dV = V_K \cdot \frac{d\varphi}{2\pi}$$

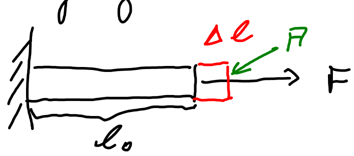
$$V = \int_0^{2\pi} V_K \cdot \frac{d\varphi}{2\pi} = V_K$$



b) Dehnung durch Eigengewicht

Spannung $\sigma = \frac{F}{A} = E \cdot \frac{\Delta l}{l_0}$

Hookes-Gesetz



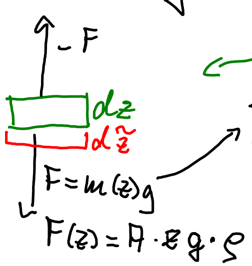
Elastizitätsmodul
Materialkonst.

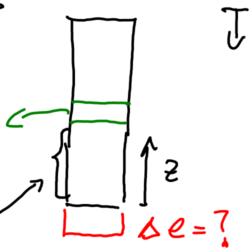
z.B. $E = 5\cdot 10^6 \frac{N}{m^2} = 5\cdot 10^6\, Pa$

Gummi

Bsp. $\Delta l = l_0 \cdot \frac{F}{A}\cdot\frac{1}{E}$; $l_0 = 1m$, $r = 2{,}5\, mm^2$, $F = 1N$ $\Rightarrow \Delta l = 1\,cm$

nun im $\vec{g}$





$\downarrow \vec{g}$ wenn $\rho = 1{,}25\, g/cm^3 \Rightarrow m = 25 g$

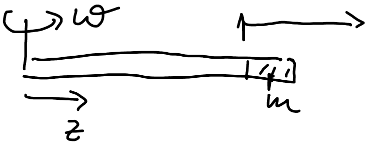
$$d\tilde{z} = dz \cdot \frac{F(z)}{A}\cdot\frac{1}{E} \Rightarrow \int_0^{\Delta\tilde{z}} d\tilde{z} = \int_0^{l} \frac{A z \rho g}{A}\cdot\frac{1}{E}\, dz$$

← nur bis l nicht $l+\Delta l$

$$\Delta \tilde{z} = \frac{\varrho g \ell^2}{2} \cdot \frac{1}{E} \Rightarrow \Delta \tilde{z} = \Delta \ell = 0{,}12\,\text{cm}$$

$$[\Delta \ell] = \frac{m}{s^2} \cdot \frac{kg}{m^3} \cdot \frac{m^2}{\frac{kg\, m}{s^2 m^2}} = m$$

Praktische Anwendung



$F(z) = m(z) \cdot z\,\omega^2$

bis Spannung $m(z)$

$\Rightarrow$ wird länger durch Drehen

$\Rightarrow$ Windräder

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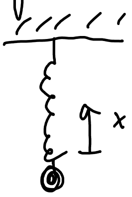
lin. Dgl. 2. Ord. mit konst. Koeff.

$$y''(x) + a_1 y'(x) + a_2 y(x) = g(x)$$

$a_1, a_2 = \text{const.}$ kein $f(x)$

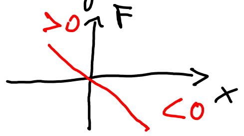
$g(x) \hat{=}$ inhomogenität

Bsp. Federschwinger ohne Reibung



$$m\ddot{x} = -k \cdot x$$

„$-$" weil Kraft immer Auslenkung entgegen zur Ruhelage zurück, auch für $x<0$



$$\ddot{x} + \frac{k}{m} x = 0 \quad \left(\begin{array}{l} a_1 = 0 \\ a_2 = \frac{k}{m} \end{array} \quad g(x) = 0 \right)$$

Methode Exponentialansatz

$$\left.\begin{array}{l} x(t) = x_0 e^{\lambda t} \\ \dot{x}(t) = x_0 \lambda e^{\lambda t} \\ \ddot{x}(t) = x_0 \lambda^2 e^{\lambda t} \end{array}\right\} \text{einsetzen in Dgl.}$$

$$x_0 \lambda^2 e^{\lambda t} + \frac{k}{m} x e^{\lambda t} = 0 \quad ; e^{\lambda t} \neq 0$$

$$\lambda^2 + \frac{k}{m} = 0 \quad \hat{=} \text{ „charakteristische Gl. der Dgl."}$$

$$\lambda_{1/2} = \pm\sqrt{-\frac{k}{m}} = \pm i \sqrt{\frac{k}{m}} \rightsquigarrow \text{Lsg } x(t) = x_0 e^{i\sqrt{\frac{k}{m}}t} \text{ oder } x_0 e^{-i\sqrt{\frac{k}{m}}t}$$

$$x(t) = x_0 \left[\cos\sqrt{\frac{k}{m}}t + i \sin\sqrt{\frac{k}{m}}t\right]$$

durch Einsetzen können wir zeigen,

$$\left.\begin{array}{l} \text{Re}(x(t)) = x_0 \cos\sqrt{\frac{k}{m}}t \\ \text{Im}(x(t)) = x_0 \sin\sqrt{\frac{k}{m}}t \end{array}\right\} \text{beides Lsg.}$$

$$\Rightarrow \text{allg. Lsg. } x(t) = A \cos\sqrt{\frac{k}{m}}t + B \sin\sqrt{\frac{k}{m}}t$$

Bew. Einsetzen

$$\left.\begin{array}{l} \dot{x} = -A\sqrt{\ }\sin\sqrt{\ }t + B\sqrt{\ }\cos\sqrt{\ }t \\ \ddot{x} = -A\sqrt{\ }^2\cos\sqrt{\ }t - B\sqrt{\ }^2\sin\sqrt{\ }t \end{array}\right\} \text{Einsetzen} \rightsquigarrow 0 = 0 \text{ okay}$$

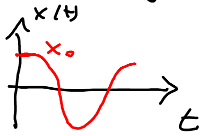
Bestimmung der A & B aus AB ← Anfangsbed.

↑ ↑ Integrationskonst.

1.) $t=0$ Auslenkung um x_0; aber $v(0) = \dot{x}(0) = 0$

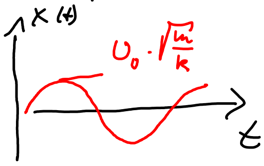
$$x(0) = x_0 \, ; \; \dot{x} = 0$$

$$\left.\begin{aligned} x_0 &= A \cos\sqrt{\ } t + B \sin\sqrt{\ } t \Big|_{t=0} = A \cos 0 = A = x_0 \\ \dot{x}_0 &= 0 = -A\sqrt{\ } \sin 0 + B\sqrt{\ } \cos\sqrt{\ } t \Rightarrow B = 0 \end{aligned}\right\} \; x(t) = x_0 \cos\sqrt{\frac{k}{m}}\, t$$



2.) $t=0$; $x(0)=0$; aber $v(0) = v_0 \hat{=}$ Anstoß aus Ruhelage

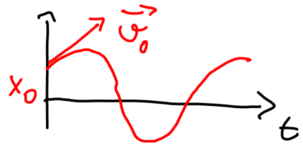
$$\left.\begin{aligned} x(0) &= A \cos 0 = 0 \Rightarrow A = 0 \\ \dot{x}(0) &= B\sqrt{\frac{k}{m}} \cos 0 \Rightarrow B = v_0 \Big/ \sqrt{\frac{k}{m}} = v_0 \sqrt{\frac{m}{k}} \end{aligned}\right\} \; x(t) = \sqrt{\frac{m}{k}}\, v_0 \sin\sqrt{\frac{k}{m}}\, t$$



3) $t=0$ $x(0) = x_0$; $v_0 = v_0 \hat{=}$ allgemeiner Fall

$$\left.\begin{aligned} x(0) &= x_0 = A \\ v(0) &= \dot{x}(0) = v_0 = B \cdot \sqrt{\frac{k}{m}} \end{aligned}\right\} \; A = x_0 \, ; \; B = v_0 \sqrt{\frac{m}{k}}$$

$$x(t) = x_0 \cos\sqrt{\frac{k}{m}}\, t + v_0 \cdot \sqrt{\frac{m}{k}} \sin\sqrt{\frac{k}{m}}\, t$$



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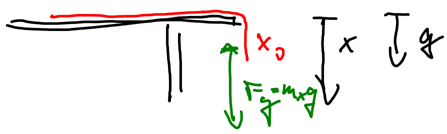
anderes Bsp.: rutschende Kette/Seil



Wie rutscht das Seil vom Tisch?

$$m \cdot \ddot{x} = m_x \cdot g \, ; \quad \frac{m}{l} = \frac{m_x}{x}$$

m_x - Anteil der Masse zur Beschleunigung der Kette

l - Gesamtlänge der Kette

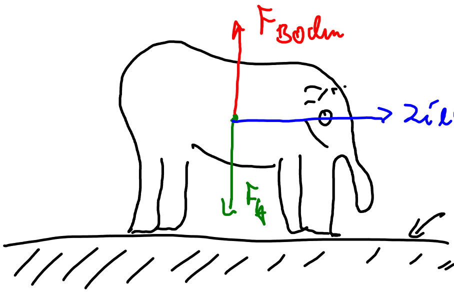
$\ddot{x}$: Übungsaufgabe

5. Arbeit und Energie (Mechanik)

verrichten — Speichern

W = Werk = Kraft • Weg $\Rightarrow [W] = N \cdot m = J =$ Joule

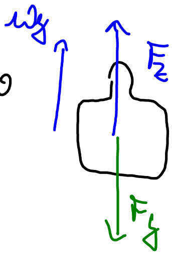
genauer[1] — Vektor — Skalarprodukt — Richtung da



$\Rightarrow F \perp \text{Weg} \Rightarrow W = 0$

es kostet keine Arbeit einen Elefanten über den Tisch zu ziehen

aber

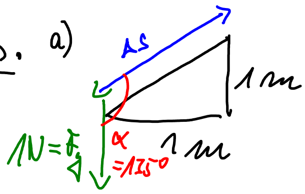


hochziehen (sehr langsam, kein $\ddot{z}$)

$\Rightarrow W > 0$

$$W = \vec{F} \cdot \Delta \vec{s} = |\vec{F}|\,|\Delta\vec{s}| \cdot \cos\alpha$$

Bsp. a)

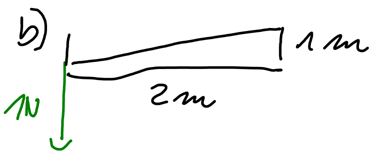


$\alpha = 135^\circ$; $\Delta s = \sqrt{2}\,m$ zu verrichten

$$W = 1N \cdot \sqrt{2}\,m \cdot \underbrace{\cos 135^\circ}_{-\frac{1}{\sqrt{2}}} = \underline{\underline{-1\,Nm}} = -F_g \cdot \Delta h$$

$\Delta h = h_{ende} - h_{anfang}$

b)



$\alpha = \frac{\pi}{2} + \arctan\frac{1}{2}$

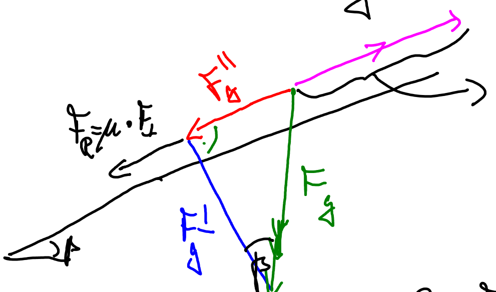
$\Delta s = \sqrt{5}\,m$

$$W = 1N \cdot \sqrt{5}\,m \cdot \underbrace{\cos(\ )}_{= -\frac{1}{\sqrt{5}}} = -1\,Nm = -1\,J$$

c) $\uparrow \Delta s$; $\alpha = 180^\circ \Rightarrow W = -1\,Nm$

nun mit Reibung

Extras



$$F_g \sin\beta + \mu \cdot \cos\beta \cdot F_g = F_{Work} = |F| \cdot \cos\beta$$

$$\Delta s = \frac{\Delta h}{\sin\beta}$$

$$W = F_{Work} \cdot \Delta s = \Delta h \cdot F_g + \frac{F_g \cdot \mu}{\tan\beta} = w(\beta)$$

dann ist Work minimal, wenn $\beta = 90^\circ \hat{=} \uparrow \Delta s$

(2) genauer

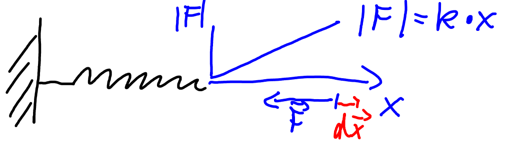
veränderliche Kraft,

$$W = \sum w_i$$

$$= \sum \vec{F}_i \cdot \Delta\vec{s}_i$$

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{s} \;\hat{=}\; \text{Wegintegral}$$

Bsp. zuerst $d\vec{s}$ in selbe Richtung, aber $F(x)$ also Ortsabhängig

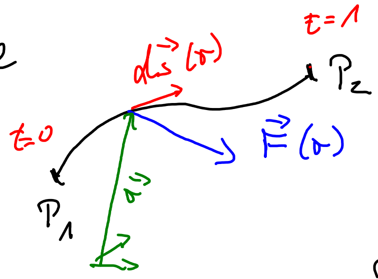


$$W = \int_{\uparrow} -k\cdot x\, dx = -\underset{\uparrow}{\frac{k}{2}} x^2$$

$\alpha = 180°$ zu verrichten < 0

$\hat{=}$ gespeicherte Energie in Feder $= E_{pot} = \frac{k}{2}x^2$

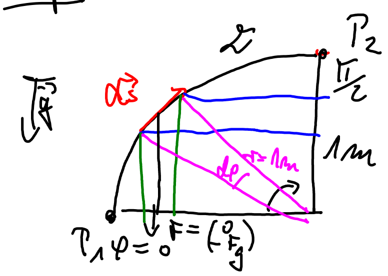
genauer(3): krumme Wege



$$W = \int_{P_1}^{P_2} \vec{F}\cdot d\vec{s}$$

Kurvenintegral, Lösen durch Parameterdarstellung $d\vec{s}(t)$

Bsp. Parameterdarstellung Kurve



Kurve C durch Parameter $\varphi = 0 .. \pi/2$

$|ds| = r\,d\varphi$; $d\vec{s} = \begin{pmatrix} |d\vec{s}|\cdot\sin\varphi \\ |d\vec{s}|\cdot\cos\varphi \end{pmatrix}$ ← x-Komp. / y-Komp. von $d\vec{s}$

$$\vec{F} = \begin{pmatrix} 0 \\ -F_g \end{pmatrix}$$

$$W = \int_0^{\pi/2} \begin{pmatrix} 0 \\ -F_g \end{pmatrix} \bullet \begin{pmatrix} r\,d\varphi\cdot\sin\varphi \\ r\,d\varphi\cdot\cos\varphi \end{pmatrix} = \int_0^{\pi/2} \left(0 - F_g\cdot r\cos\varphi\, d\varphi\right)$$

$$= \left[-F_g\, r\sin\varphi\right]_0^{\pi/2} \qquad \int\cos\varphi\, d\varphi = \sin\varphi$$

$$= -\underset{\substack{\uparrow\\1N}}{F_g}\cdot\underset{\substack{\uparrow\\1m}}{r}\cdot 1 = -1\ Nm$$ //

Hurra :o)