2. Bifurkationen

2.1 Eigenwert-Null-Bifurkation

2.2 Hopf-Bifurkation

2.3 Lokale Bifurkationen von Grenzzyklen

2.4 Globale Bifurkationen von Grenzzyklen

2.5 Bifurkation von räumlichen Mustern

Frage: Abhängigkeit des Flusses (eines dynamischen Systems / Vektorfeldes) von einem Kontrollparameter μ?

↳ Zahl & Art der Attraktoren kann sich schlagartig bei einem kritischen Wert μ_c ändern $\Rightarrow$ Bifurkation am Bifurkationspunkt μ_c

(„Verzweigung" der Lösungsmannigfaltigkeit)

Notwendige Voraussetzung: Nichtlinearität

Untersuchung mittels linearer Stabilitätsanalyse

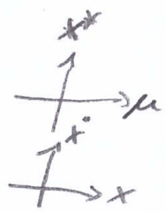
↳ z.B.: Stabilität der Fixpunkte (lokale Bifurkation)

2.1 Eigenwert-Null-Bifurkationen

Dreiklang: (a) Normalform $\dot{x} = f(x)$

(b) Bifurkationsdiagramm

(c) Phasenraumportrait



Idee der Eigenwert-Null-Bifurkationen:

$\lambda < 0 \longrightarrow \lambda > 0$ bei μ_c

stabiler Fixpunkt $\longrightarrow$ instabiler Fixpunkt

$\det \underline{\underline{A}} > 0$ (Knoten) $\longrightarrow$ $\det \underline{\underline{A}} < 0$ (Sattelpunkt)

(i) Sattel-Knoten-Bifurkation

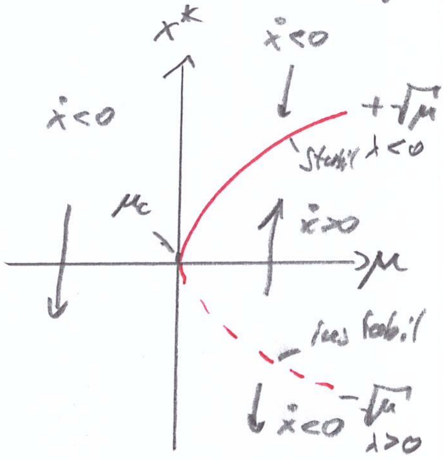
Normalform: $\dot{x} = \mu - x^2$

$\Rightarrow$ FP: $x^* = \pm\sqrt{\mu}$ (existiert nur für $\mu > 0$)

$\Rightarrow \delta x = x - x^*$: $\delta\dot{x} = \underbrace{-2x^*}_{(\underline{D}\underline{F})_{x^*}\text{: hier } 1\times 1}\delta x$

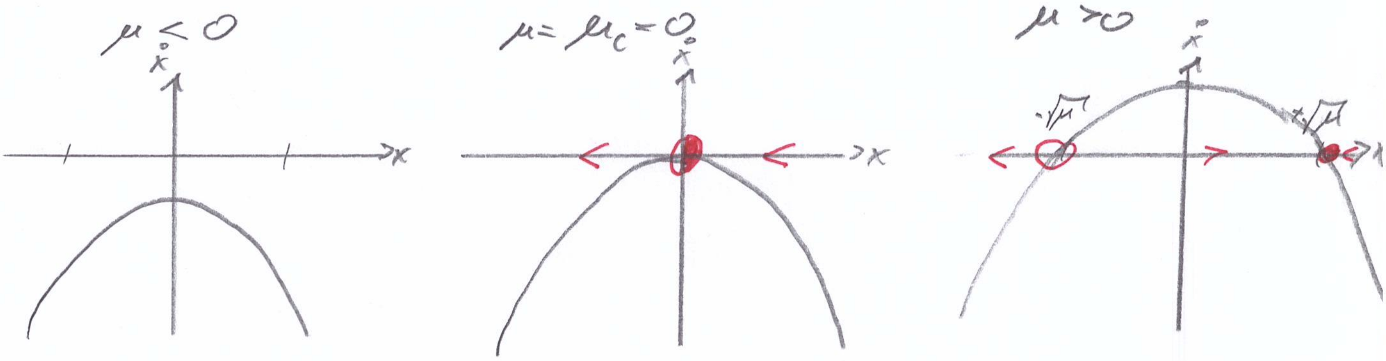
$\Rightarrow x^* = \begin{cases} +\sqrt{\mu} \\ -\sqrt{\mu} \end{cases}$ mit $\lambda = \begin{cases} -2\sqrt{\mu} < 0 \text{ : stabil} \\ +2\sqrt{\mu} > 0 \text{ : instabil} \end{cases}$

$\Rightarrow \lambda = 0$ bei $\mu_c = 0$ (Bifurkationspunkt)



Bifurkationsdiagramm

Phasenportrait:



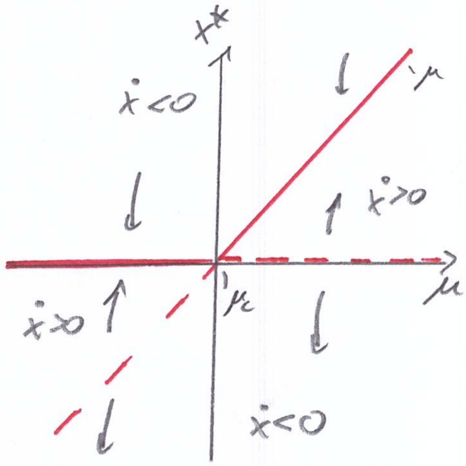
(ii) Transkritische Bifurkation

Normalform: $\dot{x} = \mu x - x^2$

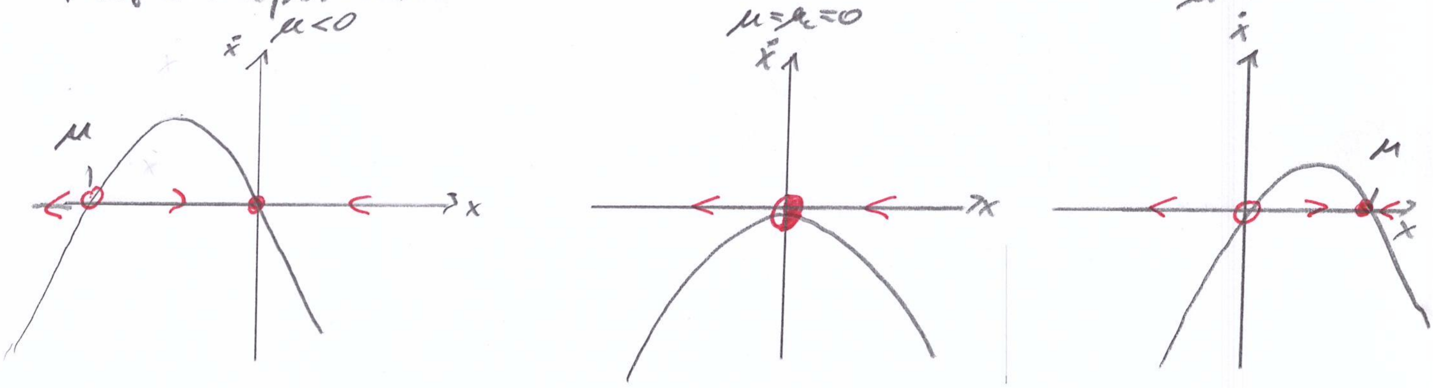
$\Rightarrow$ FP: $x^* = \begin{cases} 0 \\ \mu \end{cases}$ mit $\lambda = \begin{cases} \mu \\ -\mu \end{cases}$

$\delta\dot{x} = (\mu - 2x^*)\delta x$

$\Rightarrow$ Stabilitätswechsel bei $\mu = \mu_c = 0$ $(\lambda = 0)$



Phasenportraits:

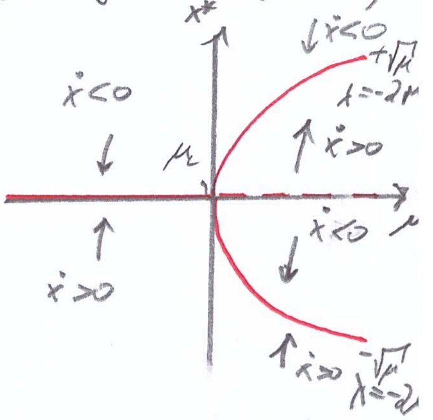


(iii) superkritische Stimmgabel-Bifurkation: (Heugabel-Bifurkation)

Normalform: $\dot{x} = \mu x - x^3 = x(\mu - x^2)$

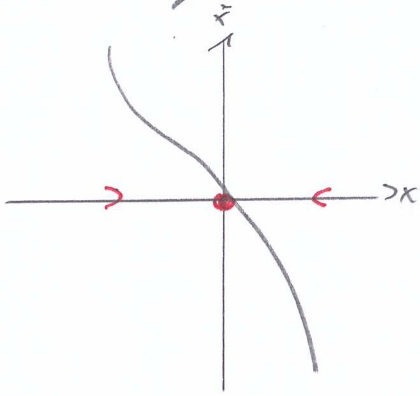
$\Rightarrow$ FP: $x^* = \begin{cases} 0 \\ \pm\sqrt{\mu} \end{cases}$ mit $\lambda = \begin{cases} \mu \\ -2\mu \end{cases}$ ex. für $\mu > 0$

$$\delta\dot{x} = [\mu - 3(x^*)^2]\,\delta x$$

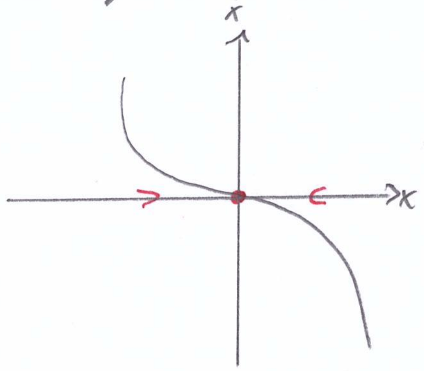


Phasenraumportraits:

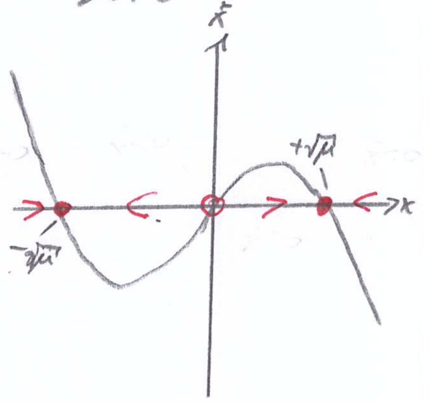
$\mu < 0$



$\mu = \mu_c = 0$



$\mu > 0$

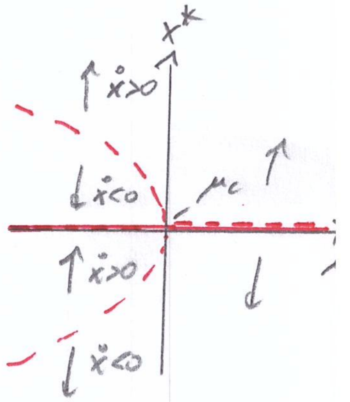


(iv) subkritische Stimmgabel-Bifurkation:

Normalform: $\dot{x} = \mu x + x^3$

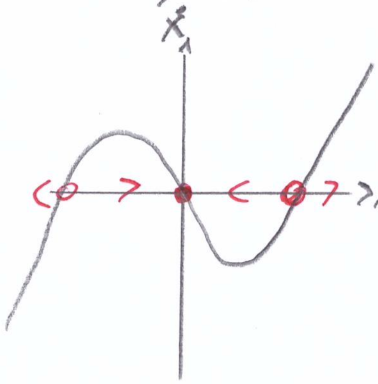
$\Rightarrow$ FP: $x^* = \begin{cases} 0 \\ \pm\sqrt{-\mu} \end{cases}$ mit $\lambda = \begin{cases} \mu \\ 2|\mu| \end{cases}$ ex. für $\mu < 0$

$$\delta\dot{x} = [\mu + 3(x^*)^2]\,\delta x$$

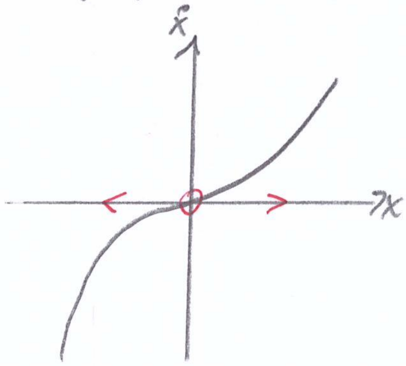


Phasenportraits:

$\mu < 0$



$\mu = \mu_c = 0$



$\mu > 0$