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Minireview

Beyond the odd number limitation of time-delayed feedback control of periodic orbits

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Abstract. We discuss the stabilization of odd-number orbits by time-delayed feedback control. In particular, we review the stabilization of odd-number orbits born in a subcritical Hopf bifurcation or a saddle-node bifurcation of periodic orbits. These examples refute the often invoked odd-number theorem.

1 Introduction

The stabilization of unstable states is a central issue in applied nonlinear science. Starting with the work of Ott, Grebogi, and Yorke [1], a variety of methods have been developed in order to stabilize unstable periodic orbits (UPOs) embedded in a chaotic attractor by employing tiny control forces [2–4]. A particularly simple and efficient scheme is time-delayed feedback as suggested by Pyragas [5], which uses the difference $z(t) - z(t-\tau)$ of a signal z at a time t and a delayed time $t-\tau$. It is an attempt to stabilize periodic orbits of (minimal) period T by a feedback control which involves a time delay $\tau = nT$, for suitable positive integer n. A linear feedback example is

$$\dot{z}(t) = f(\lambda, z(t)) + B[z(t-\tau) - z(t)] \tag{1}$$

where $\dot{z}(t) = f(\lambda, z(t))$ describes a d-dimensional nonlinear dynamical system with bifurcation parameter λ and an unstable orbit of (minimal) period T. The feedback control B is a suitably chosen constant matrix. Typical choices are multiples of the identity or of rotations, or matrices of low rank. More general nonlinear feedbacks are conceivable, of course. The main point, however, is that the Pyragas choice $\tau_P = nT$ of the delay τ eliminates the feedback term on the orbit, and thus recovers the original T-periodic solution $z(t)$. In this sense the method is noninvasive.

Although time-delayed feedback control has been widely used with great success in real world problems in physics, chemistry, biology, and medicine, e.g. [6–18], severe restrictions for the applicability of the method were believed to exist.

It was commonly contended that periodic orbits with an odd number of real Floquet multipliers greater than unity cannot be stabilized by the Pyragas method [19–24], even if the simple scheme (1) is extended by multiple delays in form of an infinite series [25]. To circumvent this restriction more complicated control schemes, like an oscillating feedback [26], half-period delays for special, symmetric orbits [27], or the introduction of an additional, unstable degree of freedom [24,28], have been proposed.

Recently, we have refuted this alleged *odd-number theorem* in Ref. [29] and following works [30–32]. In this review we summarize the main results on stabilization of odd-number orbits by time-delayed feedback control. In Sec. 2 we discuss the counterexample [29–31] of an odd-number orbit born in a subcritical Hopf bifurcation. In Sec. 3 we deal with the stabilization of an odd-number orbit born in a fold bifurcation [32].

2 Stabilization of odd-number orbits close to a subcritical Hopf bifurcation

2.1 Model

Consider the normal form of a subcritical Hopf bifurcation, extended by a time-delayed feedback term

$$\dot{z}(t) = \left[\lambda + i + (1 + i\gamma)|z(t)|^2\right] z(t) + b[z(t-\tau) - z(t)] \tag{2}$$

with $z \in \mathbb{C}$ and real parameters λ and γ. Here the Hopf frequency is normalized to unity. The feedback matrix B is represented by multiplication with a complex number $b = b_R + ib_I = b_0 e^{i\beta}$ with real b_R, b_I, β, and positive b_0. Note that the nonlinearity

$$f(\lambda, z(t)) = \left[\lambda + i + (1 + i\gamma)|z(t)|^2\right] z(t)$$

commutes with complex rotations. Therefore $\exp(i\vartheta)z(t)$ solves (2), for any fixed ϑ, whenever $z(t)$ does. In particular, nonresonant Hopf bifurcations from the trivial steady state solution $z \equiv 0$ at simple imaginary eigenvalues $\eta = i\omega \neq 0$ produce harmonic circular rotating wave solutions $z(t) = z(0)\exp\left(i\frac{2\pi}{T}t\right)$ with period $T = 2\pi/\omega$ even in the nonlinear case and with delay terms. This follows from uniqueness of the emanating Hopf branches.

Transforming Eq. (2) to amplitude and phase variables r, θ using $z(t) = r(t)e^{i\theta(t)}$, we obtain at $b = 0$

$$\dot{r} = \left(\lambda + r^2\right) r \tag{3}$$

$$\dot{\theta} = 1 + \gamma r^2. \tag{4}$$

An unstable periodic orbit (UPO) with $r = \sqrt{-\lambda}$ and period $T = 2\pi/(1 - \gamma\lambda)$ exists for $\lambda < 0$. This is the orbit we will stabilize. We will call it the *Pyragas orbit*. At $\lambda = 0$ a subcritical Hopf bifurcation occurs, and the steady state $z = 0$ loses its stability. The Pyragas control method chooses delays $\tau = \tau_P = nT$. This defines the local *Pyragas curve* in the (λ, τ)-plane for any $n \in \mathbb{N}$

$$\tau_P(\lambda) = \frac{2\pi n}{1 - \gamma\lambda} = 2\pi n(1 + \gamma\lambda + \dots) \tag{5}$$

which emanates from the Hopf bifurcation points $\lambda = 0,\ \tau = 2\pi n,\ z = 0$.

Under further nondegeneracy conditions, the Hopf point $\lambda = 0$, $\tau = nT$ $(n \in \mathbb{N}_0)$ continues to a Hopf bifurcation curve $\tau_H(\lambda)$ for $\lambda < 0$. We determine this *Hopf curve*

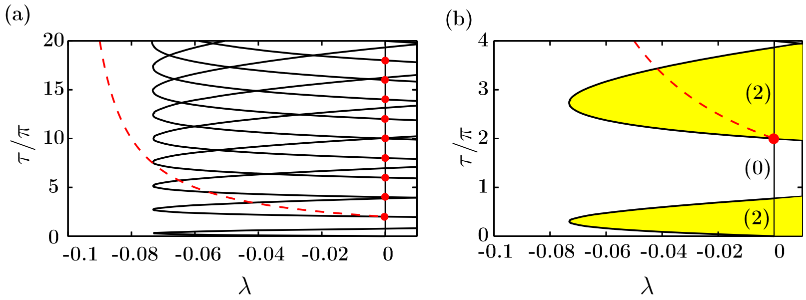

Fig. 1. Pyragas (dashed) and Hopf (solid) curves in the (λ, τ)-plane: (a) Hopf bifurcation curves $n = 0, ..., 10$, (b) Hopf bifurcation curves $n = 0, 1$ in an enlarged scale. Light gray shading marks the domains of unstable $z = 0$ and numbers in parentheses denote the dimension of the unstable manifold of $z = 0$ ($\gamma = -10$, $b_0 = 0.3$, and $\beta = \pi/4$) [29].

next. It is characterized by purely imaginary eigenvalues $\eta = i\omega$ of the transcendental characteristic equation

$$\eta = \lambda + i + b\left(e^{-\eta\tau} - 1\right) \tag{6}$$

which results from the linearization at the trivial steady state $z = 0$ of the delayed system (2). Separating Eq. (6) into real and imaginary parts leads to

$$0 = \lambda + b_0[\cos(\beta - \omega\tau) - \cos\beta] \tag{7}$$

$$\omega - 1 = b_0[\sin(\beta - \omega\tau) - \sin\beta]. \tag{8}$$

Using the trigonometric identity

$$[\cos(\beta - \omega\tau)]^2 + [\sin(\beta - \omega\tau)]^2 = 1 \tag{9}$$

to eliminate $\omega(\lambda)$ from Eqs. (7), (8) yields an explicit expression for the multivalued Hopf curve $\tau_H(\lambda)$ for given control amplitude b_0 and phase β:

$$\tau_H = \frac{\pm \arccos\left(\frac{b_0\cos\beta - \lambda}{b_0}\right) + \beta + 2\pi n}{1 - b_0\sin\beta \mp \sqrt{\lambda(2b_0\cos\beta - \lambda) + b_0^2\sin^2\beta}}. \tag{10}$$

Note that τ_H is not defined in case of $\beta = 0$ and $\lambda < 0$. Thus complex b is a necessary condition for the existence of the Hopf curve in the subcritical regime $\lambda < 0$. Figure 1 displays the family of Hopf curves $n = 0, 1, \ldots$ (solid), Eq. (10), and the Pyragas curve $n = 1$ (dashed), Eq. (5), in the (λ, τ)-plane. In Fig. 1(b) the domains of instability of the trivial steady state $z = 0$, bounded by the Hopf curves, are marked by light gray shading. The dimensions of the unstable manifold of $z = 0$ are given in parentheses along the τ-axis in Fig. 1(b). By construction, the delay τ becomes a multiple n of the minimal period T of the bifurcating Pyragas orbits along the Pyragas curve $\tau = \tau_p(\lambda) = nT$, and the time-delayed feedback term vanishes on these periodic orbits.

The inset of Fig. 2 displays the Hopf and Pyragas curves for different values of the feedback b_0. These choices of b_0 are displayed as full circles in the main figure, which shows the domain of control in the plane of the complex feedback gain b. For $b_0 > b_0^{crit}$ (a) the Pyragas curve runs partly inside the Hopf curve. With decreasing magnitude of b_0 the Hopf curves pull back to the right in the (λ, τ)-plane until the Pyragas curves lies fully outside the instability regime of the trivial steady state (c). At the

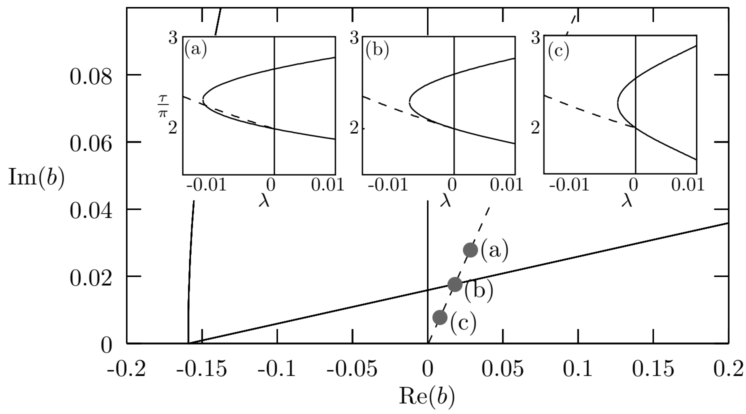

Fig. 2. Change of Hopf curves with varying control amplitude b_0. The main figures shows the complex plane of control gain b. The three values marked by full circles correspond to the insets (a), (b), (c), where the Hopf (solid) and Pyragas (dashed) curves are displayed for $\beta = \frac{\pi}{4}$ and three different choices of b_0: (a) $b_0 = 0.04 > b_0^{crit}$, (b) $b_0 = 0.025 \approx b_0^{crit}$ and (c) $b_0 = 0.01 < b_0^{crit}$; ($\lambda = -0.005$, $\gamma = -10$) [30].

critical feedback value (b) Pyragas and Hopf curve are tangent at ($\lambda = 0, \tau = 2\pi$). Standard exchange of stability results [33], which hold verbatim for delay equations, then assert that the bifurcating branch of periodic solutions locally inherits linear asymptotic (in)stability from the trivial steady state, i.e., it consists of stable periodic orbits on the Pyragas curve $\tau_P(\lambda)$ inside the shaded domains for small $|\lambda|$. Thus local Pyragas stabilization at Hopf bifurcation succeeds in case (a) but fails in case (c). The tangential case (b) is the borderline between local success and local failure.

We stress that an unstable trivial steady state is not a sufficient condition for stabilization of the Pyragas orbit. In fact, the stabilized Pyragas orbit can become unstable again if $\lambda < 0$ is further decreased, for instance in a torus bifurcation. However, there exists an interval for values of λ in our example for which the exchange of stability holds. More precisely, for small $|\lambda|$ unstable periodic orbits possess a single Floquet multiplier $\mu = \exp(\Lambda\tau)$ with $1 < \mu < \infty$, near unity, which is simple. All other nontrivial Floquet multipliers lie strictly inside the complex unit circle. In particular, the (strong) unstable dimension of these periodic orbits is odd, here 1, and their unstable manifold is two-dimensional. Therefore the periodic orbits satisfy the assumption of the alleged odd number limitation: the number of Floquet multipliers exceeding unity is 1, hence odd. Nevertheless, Pyragas stabilization succeeds. This is shown in Fig. 3 panel (a) top, which depicts solutions Λ of the characteristic equation of the periodic solution on the Pyragas curve (see Appendix A). The largest real part is positive for $b_0 = 0$. Thus the periodic orbit is unstable. As the amplitude of the feedback gain increases, the largest real part of the eigenvalue becomes smaller and eventually changes sign at TC. Hence the periodic orbit is stabilized. Note that an infinite number of Floquet exponents are created by the control scheme; their real parts tend to $-\infty$ in the limit $b_0 \to 0$, and some of them may cross over to positive real parts for larger b_0 (dashed line in Fig. 3(a)), eventually terminating the stability of the periodic orbit.

Panel (a) bottom illustrates the stability of the trivial steady state by displaying the largest real part of the eigenvalues η. The interesting region of the top and bottom panel where the periodic orbit becomes stable and the trivial steady state loses stability is magnified in panel (b).

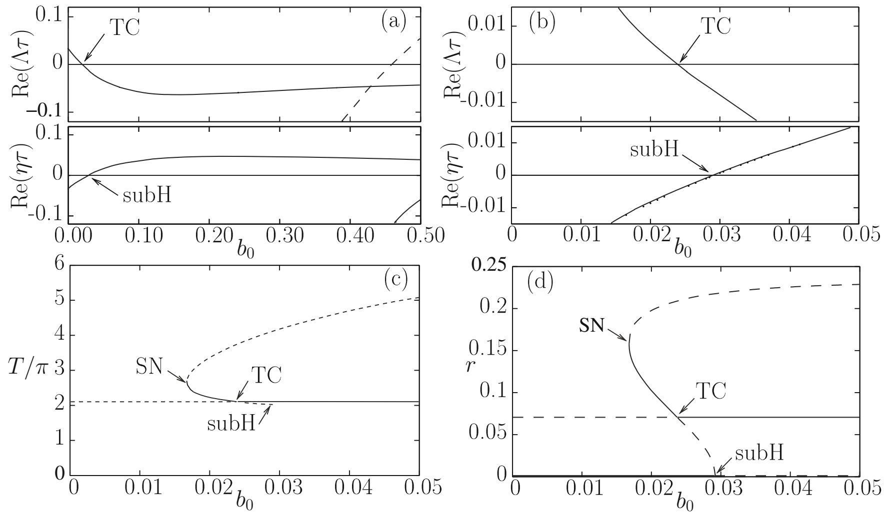

Fig. 3. (a) Top: real part of Floquet exponents Λ of the periodic orbit vs feedback amplitude b_0. Bottom: real part of eigenvalue η of steady state vs. feedback amplitude b_0. (b): Blow-up of (a). (c): Periods and (d): radii of the periodic orbits vs. b_0. The solid and dashed curves correspond to stable and unstable periodic orbits, respectively. Parameters in all panels: $\lambda = -0.005$, $\gamma = -10$, $\tau = \frac{2\pi}{1-\gamma\lambda}$, $\beta = \pi/4$ [30].

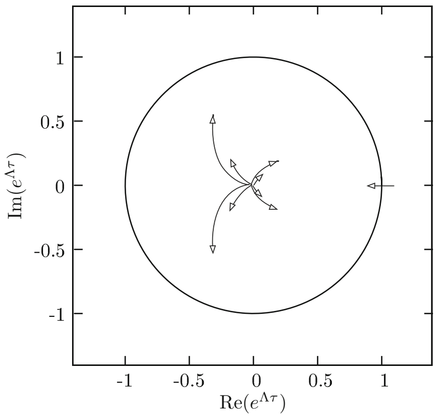

Fig. 4. Floquet multipliers $\mu = \exp(\Lambda\tau)$ in the complex plane with the feedback amplitude $b_0 \in [0, 0.3]$. Arrows indicate the direction of increasing b_0. Same parameters as in Fig. 3 [30].

Figure 4 shows the behavior of the Floquet multipliers $\mu = \exp(\Lambda\tau)$ of the Pyragas orbit in the complex plane with the increasing amplitude of the feedback gain b_0 as a parameter (marked by arrows). There is an isolated real multiplier crossing the unit circle at $\mu = 1$, in contrast to the result stated in [20]. This is caused by a transcritical bifurcation in which the Pyragas orbit collides with a delay-induced stable periodic orbit. In panel (c) and (d) of Fig. 3 the periods and radii of all circular periodic orbits ($r = const$) are plotted versus the feedback strength b_0. For small b_0 only the initial (unstable) Pyragas orbit (T and r independent of b_0) and the trivial steady state $r = 0$ (stable) exist. With increasing b_0 a pair of unstable/stable periodic orbits is created in a saddle-node (SN) bifurcation. The stable one of the two orbits (solid) then exchanges stability with the Pyragas orbit in a transcritical bifurcation (TC), and finally ends in a subcritical Hopf bifurcation (subH), where the steady state $r = 0$ becomes unstable. The Pyragas orbit continues as a stable periodic orbit for larger b_0. Except at TC, the delay-induced orbit has a period $T \neq \tau$ (See Fig. 3(c)). Note that the respective exchanges of stability of the Pyragas orbit (TC) and the steady state (subH) occur at slightly different values of b_0. This is also corroborated by Fig. 3(b). The mechanism of stabilization of the Pyragas orbit by a transcritical bifurcation relies upon the possible existence of such delay-induced periodic orbits with $T \neq \tau$, which was overlooked, e.g., in [20]. Technically, the proof of the odd-number limitation theorem in [20] fails because the trivial Floquet multiplier $\mu = 1$ (Goldstone mode of periodic orbit) was neglected there; $F(1)$ in Eq. (14) in [20] is thus zero and not less than zero, as assumed. At TC, where a second Floquet multiplier crosses the unit circle, this results in a geometrically simple Floquet multiplier $\mu = 1$ of algebraic multiplicity two.

2.2 Conditions on the feedback gain

Next we analyse the conditions under which stabilization of the subcritical periodic orbit is possible. From Fig. 1(b) it is evident that the Pyragas curve must lie inside the yellow region, i.e., the Pyragas and Hopf curves emanating from the point $(\lambda, \tau) = (0, 2\pi)$ must locally satisfy the inequality $\tau_H(\lambda) < \tau_P(\lambda)$ for $\lambda < 0$. More generally, let us investigate the eigenvalue crossings of the Hopf eigenvalues $\eta = i\omega$ along the τ-axis of Fig. 1. In particular we derive conditions for the unstable dimensions of the trivial steady state $z = 0$ near the Hopf bifurcation point $\lambda = 0$ in our model equation (2). On the τ-axis ($\lambda = 0$), the characteristic equation (6) for $\eta = i\omega$ is reduced to

$$\eta = i + b\left(e^{-\eta\tau} - 1\right), \tag{11}$$

and we obtain two series of Hopf points given by

$$0 \leq \tau_n^A = 2\pi n \tag{12}$$

$$0 < \tau_n^B = \frac{2\beta + 2\pi n}{1 - 2b_0 \sin\beta} \quad (n = 0, 1, 2, \dots). \tag{13}$$

The corresponding Hopf frequencies are $\omega^A = 1$ and $\omega^B = 1 - 2b_0 \sin\beta$, respectively. Note that series A consists of all Pyragas points, since $\tau_n^A = nT = \frac{2\pi n}{\omega^A}$. In the series B the integers n have to be chosen such that the delay $\tau_n^B \geq 0$. The case $b_0 \sin\beta = 1/2$, only, corresponds to $\omega^B = 0$ and does not occur for finite delays τ.

We evaluate the crossing directions of the critical Hopf eigenvalues next, along the positive τ-axis and for both series. Abbreviating $\frac{\partial}{\partial\tau}\eta$ by η_τ the crossing direction is given by sign(Re η_τ). Implicit differentiation of (11) with respect to τ at $\eta = i\omega$ implies

$$\mathrm{sign}(\mathrm{Re}\ \eta_\tau) = -\mathrm{sign}(\omega)\,\mathrm{sign}(\sin(\omega\tau - \beta)). \tag{14}$$

We are interested specifically in the Pyragas-Hopf points of series A (marked by dots in Fig. 1) where $\tau = \tau_n^A = 2\pi n$ and $\omega = \omega^A = 1$. Indeed $\mathrm{sign}(\mathrm{Re}\ \eta_\tau) = \mathrm{sign}(\sin\beta) > 0$ holds there, provided we assume $0 < \beta < \pi$, i.e., $b_I > 0$ for the feedback gain. This condition alone, however, is not sufficient to guarantee stability of the steady state for $\tau < 2n\pi$. We also have to consider the crossing direction sign(Re η_τ) along series B, $\omega^B = 1 - 2b_0 \sin\beta$, $\omega^B \tau_n^B = 2\beta + 2\pi n$, for $0 < \beta < \pi$. Equation (14) now implies $\mathrm{sign}(\mathrm{Re}\ \eta_\tau) = \mathrm{sign}((2b_0 \sin\beta - 1)\sin\beta) = \mathrm{sign}(2b_0 \sin\beta - 1)$.

To compensate for the destabilization of $z = 0$ upon each crossing of any point $\tau_n^A = 2\pi n$, we must require stabilization ($\text{sign}(\text{Re}\,\eta_\tau) < 0$) at each point τ_n^B of series B. If $b_0 \geq 1/2$, this requires $0 < \beta < \arcsin(1/(2b_0))$ or $\pi - \arcsin(1/(2b_0)) < \beta < \pi$. The distance between two successive points τ_n^B and τ_{n+1}^B is $2\pi/\omega^B > 2\pi$. Therefore, there is at most one τ_n^B between any two successive Hopf points of series A. Stabilization requires exactly one such τ_n^B, specifically: $\tau_{k-1}^A < \tau_{k-1}^B < \tau_k^A$ for all $k = 1, 2, \ldots, n$. This condition is satisfied if, and only if,

$$0 < \beta < \beta_n^*, \tag{15}$$

where $0 < \beta_n^* < \pi$ is the unique solution of the transcendental equation

$$\frac{1}{\pi}\beta_n^* + 2nb_0 \sin\beta_n^* = 1. \tag{16}$$

This holds because the condition $\tau_{k-1}^A < \tau_{k-1}^B < \tau_k^A$ first fails when $\tau_{k-1}^B = \tau_k^A$. Equation (15) represents a necessary but not yet sufficient condition that the Pyragas choice $\tau_P = nT$ for the delay time will stabilize the periodic orbit.

To evaluate the remaining condition, $\tau_H < \tau_P$ near $(\lambda, \tau) = (0, 2\pi)$, we expand the exponential in the characteristic equation (6) for $\omega\tau \approx 2\pi n$, and obtain the approximate Hopf curve for small $|\lambda|$:

$$\tau_H(\lambda) \approx 2\pi n - \frac{1}{b_I}(2\pi n b_R + 1)\lambda. \tag{17}$$

Recalling (5), the Pyragas stabilization condition $\tau_H(\lambda) < \tau_P(\lambda)$ is therefore satisfied for $\lambda < 0$ if, and only if,

$$\frac{1}{b_I}\left(b_R + \frac{1}{2\pi n}\right) < -\gamma. \tag{18}$$

Equation (18) defines a domain in the plane of the complex feedback gain $b = b_R + ib_I = b_0 e^{i\beta}$ bounded from below (for $\gamma < 0 < b_I$) by the straight line

$$b_I = \frac{1}{-\gamma}\left(b_R + \frac{1}{2\pi n}\right). \tag{19}$$

Equation (16) represents a curve $b_0(\beta)$, i.e.,

$$b_0 = \frac{1}{2n \sin\beta}\left(1 - \frac{\beta}{\pi}\right), \tag{20}$$

which forms the upper boundary of a domain given by the inequality (15). Thus Eqs. (19) and (20) describe the boundaries of the domain of control in the complex plane of the feedback gain b in the limit of small λ. Figure 5 depicts this domain of control for $n = 1$, i.e., a time delay $\tau = 2\pi/(1 - \gamma\lambda)$. The lower and upper solid curves correspond to Eq. (19) and Eq. (20), respectively. The grayscale displays the numerical result of the largest real part, wherever < 0, of the Floquet exponent, calculated from linearization of the amplitude and phase equations around the periodic orbit (Appendix A). Outside the shaded areas the periodic orbit is not stabilized. With increasing $|\lambda|$ the domain of stabilization shrinks, as the deviations from the linear approximation (17) become larger. For sufficiently large $|\lambda|$ stabilization is no longer possible, in agreement with Fig. 1(b). Note that for real values of b, i.e., $\beta = 0$, no stabilization occurs at all. Hence, stabilization fails if the feedback matrix B is a multiple of the identity matrix.

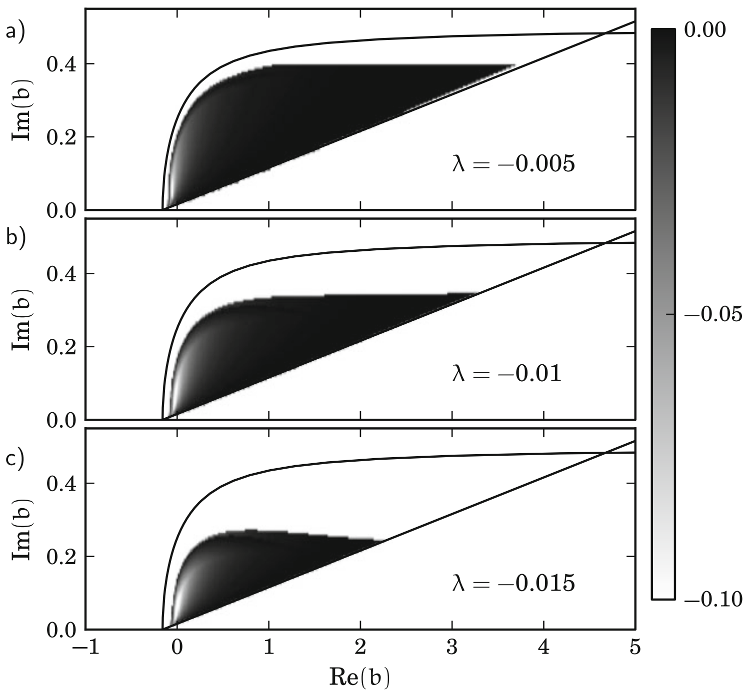

Fig. 5. Domain of control in the plane of the complex feedback gain $b = b_0 e^{i\beta}$ for three different values of the bifurcation parameter λ. The solid curves indicate the boundary of stability in the limit $\lambda \nearrow 0$, see (19), (20). The shading shows the magnitude of the largest (negative) real part of the Floquet exponents of the periodic orbit ($\gamma = -10$, $\tau = \frac{2\pi}{1-\gamma\lambda}$) [29].

3 Stabilization of odd-number orbits close to a fold-bifurcation

In the previous section we have shown how the unstable periodic orbit born in a subcritical Hopf bifurcation can be stabilized using time-delayed feedback control. Although this single counterexample [29] is sufficient to refute the odd-number limitation, it is important, for applications, to investigate under which other circumstances odd-number orbits can be stabilized. In this section we show that an odd-number orbit born in a saddle-node bifurcation of limit cycles can also be stabilized by time-delayed feedback.

We do this by analytical treatment of a generic model for fold bifurcations of harmonic circular rotating waves. We derive necessary and sufficient conditions for successful control. In particular, we show that the stabilization can be achieved by delayed feedback with arbitrarily small control amplitude provided the phase of the control is chosen appropriately.

3.1 Analysis of folds of rotating waves

3.1.1 Properties of the fold system without control

As a normal form paradigm for fold bifurcation of rotating waves we consider planar systems of the form

$$\dot{z} = g(\lambda, |z|^2)z + ih(\lambda, |z|^2)z. \quad (21)$$

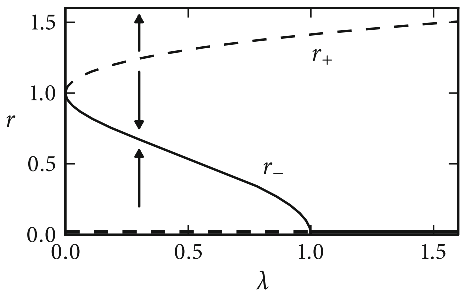

Fig. 6. Bifurcation diagram of rotating waves (solid line: stable; dashed line: unstable) of Eqs. (21) and (24). Arrows indicate (in-)stability according to Eq. (22) [32].

Here $z(t)$ is a scalar complex variable, g and h are real valued functions, and λ is a real parameter. Systems of the form (21) are again S^1-equivariant, i.e., $e^{i\theta}z(t)$ is a solution whenever $z(t)$ is, for any fixed $e^{i\theta}$ in the unit circle S^1. In polar coordinates $z = re^{i\theta}$ of amplitude r and phase θ this manifests itself by absence of θ from the right hand sides of the resulting differential equations

$$\begin{aligned} \dot{r} &= g(\lambda, r^2)r, \\ \dot{\theta} &= h(\lambda, r^2). \end{aligned} \tag{22}$$

In particular, all periodic solutions of Eq. (21) are indeed rotating waves, alias harmonic, of the form

$$z(t) = re^{i\omega t}$$

for suitable nonzero real constants r, ω. Specifically, this requires $\dot{r} = 0$, $\dot{\theta} = \omega$:

$$\begin{aligned} 0 &= g(\lambda, r^2), \\ \omega &= h(\lambda, r^2). \end{aligned} \tag{23}$$

Fold bifurcations of rotating waves are generated by the nonlinearities

$$\begin{aligned} g(\lambda, r^2) &= \left(r^2 - 1\right)^2 - \lambda, \\ h(\lambda, r^2) &= \gamma(r^2 - 1) + \omega_0. \end{aligned} \tag{24}$$

Our choice of nonlinearities is generic in the sense that $g(\lambda, r^2)$ is the normal form for a nondegenerate fold bifurcation [34] at $r^2 = 1$ and $\lambda = 0$. See Fig. 6 for the resulting bifurcation diagram. We fix coefficients $\gamma, \omega_0 > 0$.

Using Eqs. (23) and (24), the amplitude r and frequency ω of the rotating waves then satisfy

$$r^2 = 1 \pm \sqrt{\lambda}, \quad \omega = \omega_0 + \gamma(r^2 - 1) = \omega_0 \pm \gamma\sqrt{\lambda}. \tag{25}$$

The signs $\pm$ correspond to different branches $r = r_\pm(\lambda)$ in Fig. 6, $+$ unstable and $-$ stable.

3.1.2 Fold system with delayed feedback control

Our goal is to investigate delay stabilization of the fold system (21) by the same delayed feedback term as before:

$$\dot{z} = f(\lambda, |z|^2)z + b_0 e^{i\beta}\left[z(t-\tau) - z(t)\right], \tag{26}$$

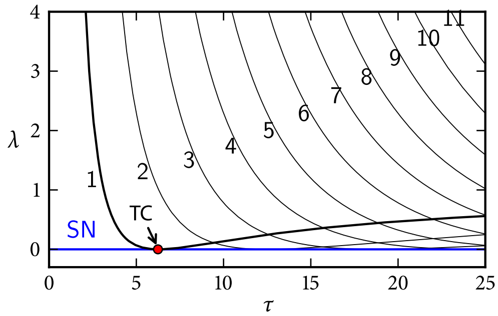

Fig. 7. The Pyragas curves $\lambda = \lambda(\tau)$, corresponding to the unstable branch in Fig. 6, in the parameter plane (τ, λ); see Eq. (29). Parameters: $\gamma = \omega_0 = 1$ [32].

with real positive control amplitude b_0, delay τ, and real control phase β. Here we have used the abbreviation $f = g + ih$. The Pyragas choice requires the delay τ to be an integer multiple k of the minimum period T of the periodic solution to be stabilized:

$$\tau_p = nT. \tag{27}$$

This choice guarantees that periodic orbits of the original system (21) with period T are reproduced exactly and noninvasively by the control system (26). The minimum period T of a rotating wave $z = re^{i\omega t}$ is given explicitly by $T = 2\pi/\omega$. Using Eqs. (25), Eq. (27) becomes

$$\tau_p = \frac{2\pi n}{\omega_0 \pm \gamma\sqrt{\lambda}}, \tag{28}$$

or, equivalently,

$$\lambda = \lambda(\tau_p) = \left(\frac{2\pi n - \omega_0 \tau_p}{\gamma \tau_p}\right)^2. \tag{29}$$

In the following we select only the branch of $\lambda(\tau)$ corresponding to the τ-value with the + sign, which is associated with the unstable orbit. Condition (29) then determines the n-th *Pyragas curve* in parameter space (τ, λ) where the delayed feedback is noninvasive, indeed. The fold parameter $\lambda = 0$ corresponds to $\tau = 2\pi n/\omega_0$, along the n-th Pyragas curve. See Fig. 7 for the Pyragas curves in the parameter plane (τ, λ).

For the delay stabilization system (26) we now consider τ as the relevant bifurcation parameter. We restrict our study of Eq. (26) to $\lambda = \lambda(\tau)$ given by the Pyragas curve (29), because $\tau_p = nT$ is the primary condition for noninvasive delayed feedback control.

We begin with the trivial case $b_0 = 0$ of vanishing control, somewhat pedantically; see Sec. 3.1.1. For each $\lambda = \lambda(\tau)$, we encounter two rotating waves given by

$$r^2 = 1 \pm \frac{2\pi n - \omega_0 \tau}{\gamma \tau}, \quad \omega = \omega_0 \pm \frac{2\pi n - \omega_0 \tau}{\tau}. \tag{30}$$

The two resulting branches form a transcritical bifurcation at $\tau = 2\pi n/\omega_0$. At this stage, the transcriticality looks like an artefact, spuriously caused by our choice of the Pyragas curve $\lambda = \lambda(\tau)$. Note, however, that only one of the two crossing branches features minimum period T such that the Pyragas condition $\tau = nT$ holds. This happens along the branch

$$r^2 = 1 + \frac{2\pi n - \omega_0 \tau}{\gamma \tau}, \quad \omega = 2\pi n/\tau,$$

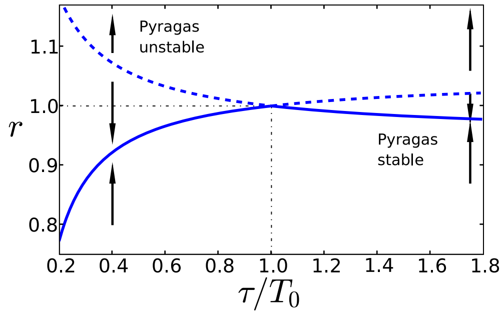

Fig. 8. Bifurcation diagram of rotating waves of Eq. (26) at vanishing control amplitude $b_0 = 0$. Parameters: $T_0 = 2\pi/\omega_0$, $\omega_0 = 1$, $\gamma = 10$ [32].

see Fig. 8. We call this branch, which corresponds to '+' in Eq. (30) the *Pyragas branch.* The other branch has minimum period T with

$$nT = \frac{\pi n}{\omega_0 \tau - \pi n} \tau \neq \tau,$$

except at the crossing point $\omega_0 \tau = 2\pi n$. The minus-branch therefore violates the Pyragas condition for non-invasive control, even though it has admittedly been generated from the same fold bifurcation.

Our strategy for Pyragas control of the unstable part of the Pyragas branch is now simple. For a nonzero control amplitude b_0, the Pyragas branch persists without change, due to the noninvasive property $\tau = nT$ along the Pyragas curve $\lambda = \lambda(\tau)$. The minus-branch, however, will be perturbed slightly for small $b_0 \neq 0$. If the resulting perturbed transcritical bifurcation

$$\tau = \tau_c \tag{31}$$

moves to the left, i.e., below $2\pi n/\omega_0$, then the stability region of the Pyragas branch has invaded the unstable region of the fold bifurcation. Again this refutes the alleged odd number limitation of Pyragas control, see Fiedler et al. [29] and references therein.

Let $\tau = \tau_c$ denote the transcritical bifurcation point on the Pyragas curve $\lambda = \lambda(\tau)$; see Eq. (29). Let $z(t) = r_c e^{i\omega_c t}$ denote the corresponding rotating wave, and abbreviate $\varepsilon \equiv r_c^2 - 1$. In Appendix B, we obtain conditions for the transcritical bifurcation in Eq. (26). As a result, the following relations between the control amplitude b_c at the bifurcation and ε, τ_c are shown:

$$b_c = -\varepsilon \frac{\omega_0 + \gamma\varepsilon}{n\pi(\gamma \sin\beta + 2\varepsilon\cos\beta)} \tag{32}$$

and

$$b_c = -\frac{2\pi n - \omega_0 \tau_c}{\tau_c \left(\frac{1}{2}\gamma^2 \tau_c \sin\beta + (2\pi n - \omega_0 \tau_c)\cos\beta\right)}. \tag{33}$$

As follows from Eqs. (32) and (33), for small ε, alias for τ_c near $2n\pi/\omega_0$, *the optimal control phase is* $\beta = -\pi/2$ in the limit $\varepsilon \to 0$. Indeed for fixed n, ω_0, γ and for $\varepsilon \to 0$ this control phase β allows for stabilization with the smallest amplitude $|b_c|$. For $\beta = -\pi/2$ Eqs. (32) and (33) simplify to

$$b_c = \frac{\varepsilon}{n\pi}\left(\frac{\omega_0}{\gamma} + \varepsilon\right) \tag{34}$$

and

$$b_c = \frac{2}{(\gamma\tau_c)^2}\left(2n\pi - \omega_0\tau_c\right), \tag{35}$$

respectively. For small $b_0 > 0$ we also have the expansions

$$\varepsilon = -\left(n\pi \frac{\gamma}{\omega_0} \sin\beta\right) b_0 + \cdots \tag{36}$$

and

$$\tau_c = \frac{2\pi n}{\omega_0} + \left(\frac{1}{2\omega_0}\left(\frac{2n\pi\gamma}{\omega_0}\right)^2 \sin\beta\right) b_0 + \cdots . \tag{37}$$

for the location of the transcritical bifurcation. In particular we see that *odd number delay stabilization can be achieved by arbitrary small control amplitudes* b_0 *near the fold, for* $\gamma > 0$ *and* $\sin\beta < 0$. Note that the stability region of the Pyragas curve increases if $\varepsilon = r_c^2 - 1 > 0$; see Fig. 6. *For vanishing phase angle of the control,* $\beta = 0$*, in contrast, delay stabilization cannot be achieved by arbitrarily small control amplitudes* b_0*, near the fold in our system (26).*

Even far from the fold at $\lambda = 0$, $\tau = 2n\pi/\omega_0$ the above formulas (32)–(35) hold and indicate a transcritical bifurcation from the (global) Pyragas branch of rotating waves of Eq. (26), along the Pyragas curve $\lambda = \lambda(\tau)$. This follows by analytic continuation. Delay stabilization, however, may fail long before $\tau = \tau_c$ is reached. In fact, nonzero purely imaginary Floquet exponents may arise, which destabilize the Pyragas branch long before $\tau = \tau_c$ is reached. This interesting point remains open.

A more global picture of the orbits involved in the transcritical bifurcation may be obtained by numerical analysis. Rewriting Eq. (26) in polar coordinates $z = re^{i\theta}$ yields

$$\dot{r} = [(r^2-1)^2 - \lambda]r + b_0[\cos(\beta + \theta(t-\tau) - \theta)\, r(t-\tau) - r\cos\beta] \tag{38}$$

$$\dot{\theta} = \gamma(r^2-1) + \omega_0 + b_0[\sin(\beta + \theta(t-\tau) - \theta)\, r(t-\tau)/r - \sin\beta]. \tag{39}$$

To find all rotating wave solutions we make the ansatz $r = \text{const}$ and $\dot{\theta} = \omega = \text{const}$ and obtain

$$\begin{aligned} 0 &= (r^2-1)^2 - \lambda + b_0[\cos(\beta - \omega\tau) - \cos\beta] \\ \omega &= \gamma(r^2-1) + \omega_0 + b_0[\sin(\beta - \omega\tau) - \sin\beta]. \end{aligned}$$

Eliminating r we find a transcendental equation for ω

$$0 = -\gamma^2\lambda + \gamma^2 b_0[\cos(\beta - \omega\tau) - \cos\beta] + (\omega - \omega_0 - b_0[\sin(\beta - \omega\tau) - \sin\beta])^2 .$$

One can now solve this equation numerically for ω and insert the result into

$$r = \left(\frac{\omega - \omega_0}{\gamma} - \frac{b_0}{\gamma}[\sin(\beta - \omega\tau) - \sin\beta] + 1\right)^{\frac{1}{2}}$$

to obtain the allowed radii (discarding imaginary radii).

The orbit which stabilizes the Pyragas branch in the transcritical bifurcation may be the minus-branch or another delay induced orbit which is born in a fold bifurcation, depending on the parameters. Figure 9 displays the different scenarios and the crossover in dependence on the control amplitude b_0. The value of γ is chosen as $\gamma = 9$, 10.5, 10.6, and 13 in panels (a), (b), (c), and (d), respectively. Evidently the Pyragas orbit is stabilized by a transcritical bifurcation T_1. As the value of γ increases, a pair of a stable and an unstable orbit generated by a fold bifurcation F_1 approaches the minus-branch (see Fig. 9(a)). On this branch, fold bifurcations (F_2 and F_3) occur as shown in Fig. 9(b). At $\gamma = 10.6$, the fold points of F_1 and F_2 touch

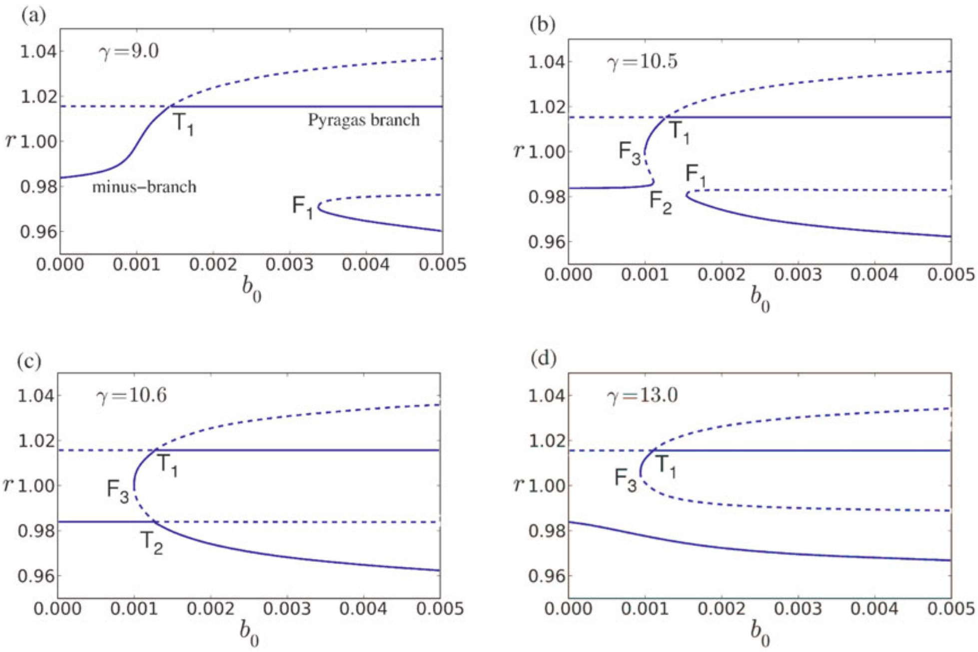

Fig. 9. Radii of stable (solid) and unstable (dashed) rotating wave solutions in dependence on b_0 for different γ. Parameters: $\omega_0 = 1$, $\lambda = 0.001$, $\beta = -\pi/2$ [32].

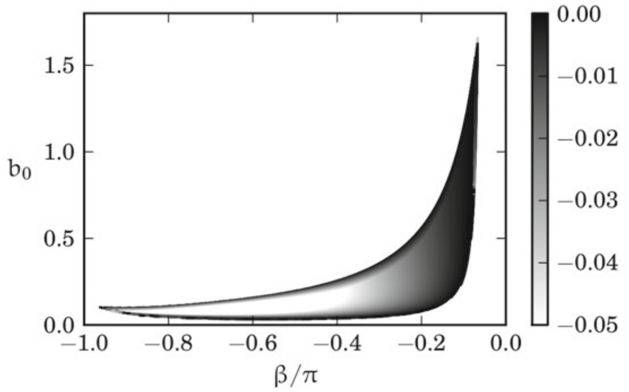

Fig. 10. Domain of stability of the Pyragas orbit. The grayscale shows only negative values of the largest real part of the Floquet exponents. Parameters: $\omega_0 = 1$, $\lambda = 0.0001$, $\gamma = 0.1$ [32].

in a transcritical bifurcation T_2 and annihilate (see Figs. 9(c) and (d)). Thus, for further increase of γ, one is left with the stable minus-branch and the unstable orbit, which was generated at the fold bifurcation F_3. In all panels the radius of the Pyragas orbit remains unaffected by the noninvasive control. The radius of the minus-branch, however, is altered because the delay time does not match the orbit period.

Figure 10 shows the region in the (β, b_0)-plane where the Pyragas orbit is stable, for a set of parameters. The grayscale shows only negative values of the largest real part of the Floquet exponents. One can see that the orbit is most stable for feedback phases $\beta \approx -\pi/2$ which agrees with the previous analytic results for small λ. The picture was obtained by linear stability analysis of Eqs. (38) and (39) and numerical solution of the transcendental eigenvalue problem for the Floquet exponents similar to the discussion in Appendix A [32].

4 Conclusion

In conclusion, we have discussed the stabilization of odd-number orbits by time-delayed feedback control. Such unstable periodic orbits are common in many nonlinear non-chaotic systems, whereas the original work of Pyragas considered unstable periodic orbits embedded in a chaotic attractor. Our presented strategy has also been used by Postlethwaite and Silber to design a feedback gain matrix in order to stabilize odd-number orbits in the chaotic Lorenz system [35]. In general, however, periodic orbits embedded in a strange attractor do not necessarily belong to the class of odd-number orbits, since they may or may not possess unstable complex Floquet multipliers. Moreover stabilization of odd-number orbits, in its present form, may be subject to a quantitative limitation. For planar harmonic orbits this has been investigated in [36]. Instabilities due to simple real Floquet multipliers exceeding $\exp(9) = 8103.1\ldots$, in fact, could not be stabilized by linear delayed feedback. This limitation may pertain, in particular, to periodic orbits of long period in chaotic regions.

In Sec. 2 we have discussed the first counterexample [29] which refuted the odd-number theorem. We have analysed the generic example of the normal form of a subcritical Hopf bifurcation, which is paradigmatic for a large class of nonlinear systems. We have worked out explicit analytical conditions for stabilization of the periodic orbit generated by a subcritical Hopf bifurcation in terms of the amplitude and the phase of the feedback control gain[1]. Our results underline the crucial role of a non-vanishing phase of the control signal for stabilization of periodic orbits violating the odd-number limitation. The feedback phase is readily accessible and can be adjusted, for instance, in laser systems, where subcritical Hopf bifurcation scenarios are abundant and Pyragas control can be realized via coupling to an external Fabry-Perot resonator [18]. The importance of the feedback phase for the stabilization of steady states in lasers [18] and neural systems [37], as well as for stabilization of periodic orbits by a time-delayed feedback control scheme using spatio-temporal filtering [38], has been noted recently. Here, we have shown that the odd-number limitation does not hold in general, which opens up fundamental questions as well as a wide range of applications. The result will not only be important for practical applications in physical sciences, technology, and life sciences, where one might often desire to stabilize periodic orbits with an odd number of positive Floquet exponents, but also for tracking of unstable orbits and bifurcation analysis using time-delayed feedback control [39].

In Sec. 3 we have discussed the stabilization of odd-number orbits born in a saddle-node bifurcation of limit cycles. We have studied the normal form of this bifurcation and derived stabilization conditions on the feedback gain. Similar results have also been obtained for a rate-equation model of a three-section semiconductor laser with all-optical delayed feedback [32]. It consists of a tandem laser coupled to a Michelson interferometer. For suitably chosen parameter values, this model has a fold bifurcation. Numerical bifurcation analysis has established successful control in the vicinity of this bifurcation.

Another example for the stabilization of an odd-number orbit is provided by the stabilization of complex spatio-temporal dynamics near a subcritical Hopf bifurcation in a reaction-diffusion system [40].

[1] For the complex conjugate values of b, stabilization of the periodic orbit can be shown by analogous arguments.

It has also been shown that odd-number orbits in coupled Stuart-Landau oscillators can be stabilized by time-delayed coupling [41]. Our results have been extended to generic subcritical Hopf bifurcations in n-dimensional dynamical systems by Brown et al. [42].

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Appendix A: Calculation of Floquet exponents

The Floquet exponents of the Pyragas orbit can be calculated explicitly by rewriting Eq. (2) in polar coordinates $z = r\, e^{i\theta}$

$$\dot{r} = (\lambda + r^2)\, r + b_0[\cos(\beta + \theta(t-\tau) - \theta)\, r(t-\tau) - \cos(\beta)\, r] \tag{40}$$

$$\dot{\theta} = 1 + \gamma r^2 + b_0 \left[\sin(\beta + \theta(t-\tau) - \theta)\, \frac{r(t-\tau)}{r} - \sin(\beta)\right] \tag{41}$$

and linearizing around the periodic orbit according to $r(t) = r_0 + \delta r(t)$ and $\theta(t) = \Omega t + \delta\theta(t)$, with $r_0 = \sqrt{-\lambda}$ and $\Omega = 1 - \gamma\lambda$ (see Eq. (3)). This yields

$$\begin{pmatrix} \dot{\delta r}(t) \\ \dot{\delta\theta}(t) \end{pmatrix} = \begin{bmatrix} -2\lambda - b_0 \cos\beta & b_0 r_0 \sin\beta \\ 2\gamma r_0 - b_0 \sin\beta\, r_0^{-1} & -b_0 \cos\beta \end{bmatrix} \begin{pmatrix} \delta r(t) \\ \delta\theta(t) \end{pmatrix} \tag{42}$$

$$+ \begin{bmatrix} b_0 \cos\beta & -b_0 r_0 \sin\beta \\ b_0 \sin\beta r_0^{-1} & b_0 \cos\beta \end{bmatrix} \begin{pmatrix} \delta r(t-\tau) \\ \delta\theta(t-\tau) \end{pmatrix}. \tag{43}$$

With the ansatz

$$\begin{pmatrix} \delta r(t) \\ \delta\theta(t) \end{pmatrix} = u\, \exp(\Lambda t), \tag{44}$$

where u is a two-dimensional vector, one obtains the autonomous linear equation

$$\begin{bmatrix} -2\lambda + b_0 \cos\beta\, (e^{-\Lambda\tau} - 1) - \Lambda & -b_0 r_0 \sin\beta\, (e^{-\Lambda\tau} - 1) \\ 2\gamma r_0 + b_0 r_0^{-1} \sin\beta\, (e^{-\Lambda\tau} - 1) & b_0 \cos\beta\, (e^{-\Lambda\tau} - 1) - \Lambda \end{bmatrix} u = 0. \tag{45}$$

The condition of vanishing determinant then gives the transcendental characteristic equation

$$0 = \left(-2\lambda + b_0 \cos\beta\, (e^{-\Lambda\tau} - 1) - \Lambda\right)\, \left(b_0 \cos\beta\, (e^{-\Lambda\tau} - 1) - \Lambda\right) \tag{46}$$

$$-b_0 r_0 \sin\beta\, (e^{-\Lambda\tau} - 1)\, \left(2\gamma r_0 + b_0 r_0^{-1} \sin\beta\, (e^{-\Lambda\tau} - 1)\right) \tag{47}$$

for the Floquet exponents Λ which can be solved numerically.

Appendix B: Condition for transcritical bifurcation

In this Appendix, we derive conditions (32) and (33) at which the transcritical bifurcation in system (26) occurs. To derive Eq. (32) we could proceed by brute force: linearize the control system (26) along the Pyragas branch, in polar coordinates, derive the characteristic equation in a co-rotating coordinate frame, eliminate the trivial zero characteristic root, and determine $\tau = \tau_c$, $r = r_c$, and $b_0 = b_c$ such that a nontrivial zero characteristic root remains. Instead, we will proceed locally in a two-dimensional center manifold of the fold, following the arguments in Just *et al.* [31]. Any periodic solution in the center manifold of Eq. (26) is a rotating wave $z(t) = re^{i\omega t}$.

Hence, let us compute the rotating waves of the system (26), globally. Substituting $z(t) = re^{i\omega t}$ into Eq. (26) and decomposing into real and imaginary parts, we obtain

$$0 = g(\lambda, r^2) + 2b_0 \sin\frac{\omega\tau}{2} \sin\left(\beta - \frac{\omega\tau}{2}\right), \tag{48}$$

$$\omega = h(\lambda, r^2) - 2b_0 \sin\frac{\omega\tau}{2} \cos\left(\beta - \frac{\omega\tau}{2}\right). \tag{49}$$

With $\varepsilon = r^2 - 1$ and our choices (24) for g and h, these equations become

$$0 = \varepsilon^2 - \lambda(\tau) + 2b_0 \sin\frac{\omega\tau}{2} \sin\left(\beta - \frac{\omega\tau}{2}\right), \tag{50}$$

$$\omega = \gamma\varepsilon + \omega_0 - 2b_0 \sin\frac{\omega\tau}{2} \cos\left(\beta - \frac{\omega\tau}{2}\right). \tag{51}$$

For small enough b_0, we can solve Eq. (51) for $\omega = \omega(\varepsilon)$ and insert into Eq. (50):

$$0 = G(\tau, \varepsilon). \tag{52}$$

Here $G(\tau, \varepsilon)$ abbreviates the right hand side of Eq. (50) with $\omega = \omega(\varepsilon)$ substituted for ω. The condition for a transcritical bifurcation in the system with control then reads

$$0 = \frac{\partial}{\partial\varepsilon} G(\tau_c, \varepsilon), \tag{53}$$

in addition to Eq. (52). It simplifies matters significantly that this calculation has to be performed along the Pyragas branch only, where $\omega\tau = 2\pi\tau/T = 2\pi k$; see Eq. (27). Therefore Eq. (53) becomes

$$\begin{aligned} 0 &= \frac{\partial}{\partial\varepsilon} G(\tau_c, \varepsilon) \\ &= 2\varepsilon + b_0\tau_c \cos k\pi \sin\left(\beta - k\pi\right) \omega'(\varepsilon) \\ &= 2\varepsilon + b_0\tau_c\omega'(\varepsilon) \sin\beta. \end{aligned} \tag{54}$$

To obtain the derivative ω' of ω with respect to ε we have to differentiate Eq. (51) implicitly, at $\omega\tau = 2k\pi$

$$\omega' = \gamma - b_0\tau\omega' \cos\beta.$$

Solving for ω', for small b_0, yields

$$\omega' = \frac{\gamma}{1 + b_0\tau\cos\beta} = \frac{\gamma}{1 + b_0\frac{2k\pi}{\omega_0 + \gamma\varepsilon}\cos\beta}. \tag{55}$$

Here we have used $\omega\tau = 2k\pi$ and $\omega = \omega_0 + \gamma\varepsilon$. Plugging Eq. (55) into Eq. (54), the control amplitude b_0 enters linearly, and we obtain

$$\begin{aligned} 0 &= \varepsilon\left(\omega_0 + \gamma\varepsilon\right)\left(1 + b_0\frac{2k\pi}{\omega_0 + \gamma\varepsilon}\cos\beta\right) + b_0 k\pi\gamma\sin\beta \\ &= \varepsilon\left(\omega_0 + \gamma\varepsilon + b_0 2k\pi\cos\beta\right) + b_0 k\pi\gamma\sin\beta. \end{aligned} \tag{56}$$

Solving for b_0, we obtain the required expression (32) for the value of the control amplitude, at which the transcritical bifurcation occurs.

The equivalent condition (33) follows from Eq. (32) by straightforward substitution of Eq. (28) and $-\sqrt{\lambda} = r^2 - 1 = \varepsilon$.

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