PHYSICAL REVIEW LETTERS

VOLUME 71 | 9 AUGUST 1993 | NUMBER 6

Stochastic Resonance without External Periodic Force

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(Received 21 December 1992)

A model of a two-dimensional autonomous system subject to external noise is investigated. Without noise the system has a stable limit cycle in a certain region of control parameter. Various noise-induced effects have been found numerically, such as a noise-induced frequency shift in the presence of the deterministic limit cycle, and noise-induced coherent oscillations in the absence of the deterministic limit cycle. An interesting result is that the stochastic resonance phenomenon appears in a system without an external signal and when the asymptotic state of the deterministic system is stationary.

PACS numbers: 05.40.+j, 05.20.−y

In the last decade the phenomenon of stochastic resonance (SR) has attracted central attention in the study of systems with noise [1–15]. The active role played by noise in generating coherent motion is of main interest. The typical problem in the SR study is to investigate the response of a bistable system to a periodic external force in the presence of noise. However, in many nonlinear systems the coherent motion of the systems is not stimulated by an external force, but by the intrinsic dynamics of the nonlinear systems. For instance, limit cycles are very important objects in various self-organization processes of nonlinear systems in physics, chemistry, biology, and other fields [16]. Therefore, it is interesting to study how noise influences the coherent motion generated by the system dynamics when the external signal is absent.

The limit cycle of a system can be eliminated by adjusting an external control parameter below a critical value [17–19]. Then the asymptotic coherent motion disappears. In this case the intrinsic circulation of the system may still exist, however, and become manifest in the transient process towards the equilibrium state. It will be shown that this transient circulation turns into an asymptotic coherent oscillation by introducing noise. In this case we find a clear stochastic resonance behavior as noise builds up the coherent motion.

To be specific, let us study a simple two-dimensional model in which a limit cycle can be generated or eliminated by adjusting a control parameter b,

$$\begin{aligned} \dot{x} &= g_1(x,y) + c_1(x,y,b)\,, \\ \dot{y} &= g_2(x,y) + c_2(x,y,b)\,, \end{aligned} \tag{1}$$

with

$$\begin{aligned} &g_1 = x(1-x^2-y^2), \quad g_2 = y(1-x^2-y^2)\,, \\ &c_1 = y(x^2-y^2-b), \quad c_2 = -x(x^2-y^2-b)\,, \end{aligned} \tag{2}$$

(g_1,g_2) is the gradient part of the force

$$\begin{aligned} &g_1 = -\frac{\partial u(x,y)}{\partial x}, \quad g_2 = -\frac{\partial u(x,y)}{\partial y}\,, \\ &u(x,y) = -\tfrac{1}{2}(x^2+y^2) + \tfrac{1}{4}(x^2+y^2)^2\,. \end{aligned} \tag{3}$$

The potential $u(x,y)$ has a minimum at the cycle Γ_1: $r^2 = x^2+y^2 = 1$. The circulation part of the force (c_1,c_2) is vertical to the gradient $(g_1c_1+g_2c_2=0)$. The circulation changes its direction on the curve Γ_2: $x^2-y^2=b$. As $b>1$, Γ_1 and Γ_2 do not intersect each other, and then the system has a limit cycle with $r=1$. However, the limit cycle disappears via saddle-node bifurcation as b is lower than the threshold $b_c=1$. In this case, four fixed points appear at the intersections of the two sets Γ_1 and Γ_2, among which two are stable nodes and the other two are saddle points. At the critical parameter b_c the system has an asymptotic heteroclinic orbit, and then the limit cycle has infinite period, or say it disappears.

The specific form of Eqs. (2) is not important. The key points are the system shows coherent oscillation in a certain parameter region (here $b>1$) while the limit cycle can be eliminated by adjusting an external control parameter below a certain threshold (here $b<1$). Actually, the asymptotic motion of the system (1) can be explicitly solved. As $t\to\infty$, the system eventually approaches the cycle Γ_1. The solution on Γ_1 can be analytically given:

$$t-t_0=\int_{\theta_0}^{\theta}\frac{d\theta}{b-\cos(2\theta)}\,, \tag{4}$$

where we have $x=r\cos(\theta)$, $y=r\sin(\theta)$. From Eq. (4) it is clear that the asymptotic motion is a limit cycle indeed as $b>1$ with the period $\int_0^{2\pi}d\theta/[b-\cos(2\theta)]$. As $b\to 1$ the period goes to infinity, and the trajectory eventually approaches (and leaves) the two heteroclinic points $x=\pm 1$, $y=0$. If $b<1$ the system monotonously approaches one of the attractors at $x_1=\sqrt{(1+b)/2}$, $y_1=-\sqrt{(1-b)/2}$ or $x_2=-x_1$, $y_2=-y_1$.

Including noise, we may get the corresponding Langevin equations (LEs)

$$\begin{aligned}\dot{x}&=g_1+c_1+Q_1(t)\,,\\ \dot{y}&=g_2+c_2+Q_2(t)\,,\end{aligned} \tag{5}$$

with $\langle Q_i(t)\rangle=0$, $\langle Q_i(t)Q_j(t')\rangle=D\delta_{ij}\delta(t-t')$; $i,j=1,2$. There has not been an analytic approach to deal with these coupled LEs, especially for intermediate values of D. Therefore in the following we will treat these equations numerically.

We simulate the LEs by a simple Euler forward procedure. The data are taken in a time period $T=5000$. The correctness of all the following results are confirmed by changing the time step and the total time. The spectra of the time series are obtained by the last Fourier transformation, and each plot is provided by the average of 500 runs.

First we fix $b=1.05$ at which the asymptotic motion of the deterministic system is a limit cycle on Γ_1. Figure 1 shows the power spectrum $\langle S(\omega)\rangle=\langle|y(\omega)|^2\rangle T$ for various values of D. $\langle|y(\omega)|^2\rangle$ is the average of the power spectra of the 500 different time series $\langle y(\omega)\rangle$, i.e., $\langle|y(\omega)|^2\rangle=\sum_{i=1}^{500}|y_i(\omega)|^2/500$. For small noise [Fig. 1(a), $D=0.00003$] the spectrum peak is very high and sharp. As D increases [Fig. 1(b), $D=0.05$], the peak becomes lower and less sharp, and the frequency of the spectrum peak is considerably shifted. For sufficiently large D [Fig. 1(c), $D=0.9$], we have a typical pure noise spectrum. The peak of finite frequency disappears, and the coherent motion of the deterministic system is almost completely destroyed by the strong noise. Two points

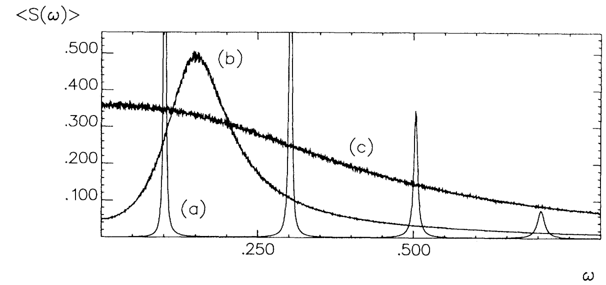

FIG. 1. The averaged power spectrum of $y(t)$ for $b=1.05$ and various D. A limit cycle exists for the deterministic system. (a) $D=0.00003$. (b) $D=0.05$. (c) $D=0.9$.

should be emphasized for Fig. 1: First, without noise there is a delta function in the spectrum at the frequency of the limit cycle. The inclusion of noise reduces the delta function to a peak of finite height. Second, the center of the peak ω_p (which will be regarded as the characteristic frequency of the system) is modified by noise: from Figs. 1(a) to 1(b) ω_p is shifted to a large extent. These two points are essentially different from the periodically forced system where a delta function must appear in the output spectrum at the input frequency, and the position of this frequency is not shifted no matter how large the noise is. However, as the phase of the input signal is subject to noise some phenomena similar to our case can also be observed (see Ref. [12]). The profile of Fig. 1(b) is typical for a limit cycle system subject to noise. The peak indicates the existence of coherent motion, and the width of the peak shows the influence of the random force.

In Fig. 2, we reduce b to the critical value ($b=1$). Without noise the system will asymptotically approach to $y(t)=0$, $x(t)=\pm 1$, then no coherent oscillation exists for the deterministic system. Therefore the oscillation of $y(t)$ can be regarded as an order parameter to measure the level of the coherent motion. An interesting fact is that the inclusion of noise stimulates the coherent motion, recovering the limit cycle again. For small noise [Fig. 2(a), $D=0.00003$], one finds a small peak at very small frequency. If we increase noise [Fig. 2(b), $D=0.05$], the

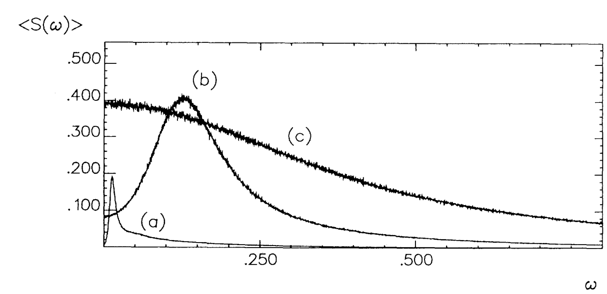

FIG. 2. The same as in Fig. 1 with b replaced by $b=1$, then the deterministic limit cycle vanishes without noise. (a) $D=0.00003$. (b) $D=0.05$. (c) $D=0.9$.

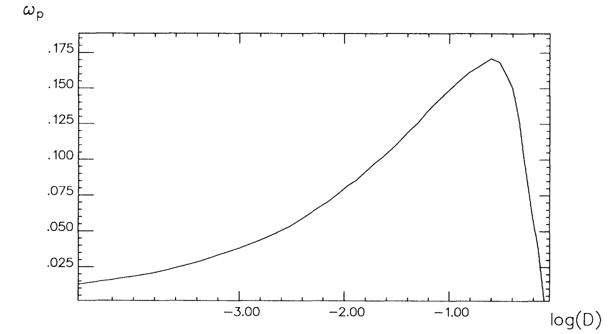

FIG. 3. The center frequency of the spectrum peak ω_p vs $\log(D)$ with $b=1$ (the same in Figs. 4 and 5).

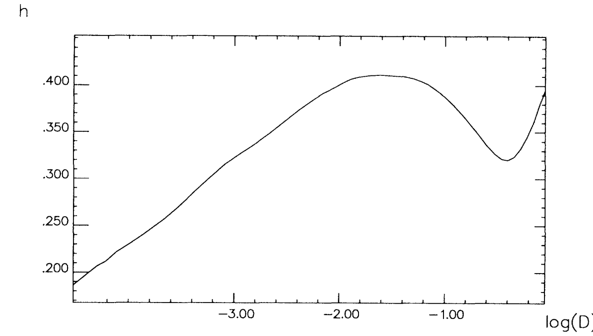

FIG. 4. The spectrum peak height h plotted against $\log(D)$. A resonancelike curve is obvious. However, h goes up again for very large D.

peak moves towards larger ω, and the height of the peak increases. The resemblance of Figs. 2(b) and 1(b) convinces us that in Fig. 2(b) there exists indeed a strong coherent motion stimulated purely by noise. This coherent motion can be called noise-induced collective oscillation. For very large noise [Fig. 2(c), $D=0.9$], the peak of finite frequency disappears like in Fig. 1(c).

In the following figures we fix $b=1$ and consider the influence of different values of the noise strength D on the features of the averaged power spectra. In Fig. 3, we plot ω_p against $\log(D)$. ω_p increases as D increases for low values of D until a maximum is reached and then ω_p decreases. After a certain critical value of D we have $\omega_p=0$ identically, then the peak centers at $\omega=0$, and the system is completely governed by strong noise.

It is interesting to see the dependence of the noise-induced coherent motion on the noise strength. Figure 4 plots the peak height (h) against $\log(D)$. One finds an obvious peak at the optimal noise strength. However, the $h-\log(D)$ curve does not always correctly represent the coherent motion. At very large D, the spectrum peak at $\omega=0$ certainly increases by increasing D. However, in this case h no more represents the strength of the coherent motion. For a more appropriate representation we plot, in Fig. 5, $\beta=h(\Delta\omega/\omega_p)^{-1}$ vs $\log(D)$ where $\Delta\omega$ is the width of the peak at the height $h_1=e^{-1/2}h$, and thus $\Delta\omega/\omega_p$ reasonably corresponds to the relative width of the peak, which is in fact the familiar quality factor of a signal. β represents therefore the degree of the coherence and is actually the signal-to-noise ratio (SNR) of the output. The $\beta-\log(D)$ curve shows clearly a stochastic resonance maximum.

For the case of b a little smaller than 1 we get the results qualitatively the same as in Fig. 2 to Fig. 5. The peak in Fig. 5, however, moves to the left as b decreases. For the too small b, the SR effect disappears. The situation in this case will be discussed in a regular paper [20].

The reason for the SR peak in Eq. (5) is physically clear. In our system, noise plays a twofold role. On the one hand, it stimulates coherent motion of the system, and transfers the transient circulation of the deterministic system to an asymptotic collective oscillation. On the other hand, noise spoils naturally the coherent motion activated by itself. In Fig. 5 in the region $D<D_m$ (D_m is the position of the maximum of β) the first tendency dominates and we have a monotonously increasing $\beta-\log(D)$ curve there. In the region $D>D_m$, the second one dominates, and then the curve decreases. As a result a resonancelike behavior occurs. Actually, the competition of these seemingly opposite roles played by noise is also the key point in the original SR cases.

Actually, the noise induced oscillation has been observed in various systems due to different mechanisms [21–25]. Here, we extend the discussion to more general two-dimensional limit cycle systems. The main new point in this Letter is that we analyze in detail the influence of noise on the features of the noise-induced coherent motion, such as the behavior of the noise-induced frequency shift, the strength of the coherent motion and the SNR of the output. Most importantly we find for the first time *the SR phenomenon for systems whose deterministic dynamics is autonomous, i.e., SR without external signal.*

The conclusions obtained in our model can be generally extended to a wide range of more complicated systems with limit cycles or transient circulations [17–19] where

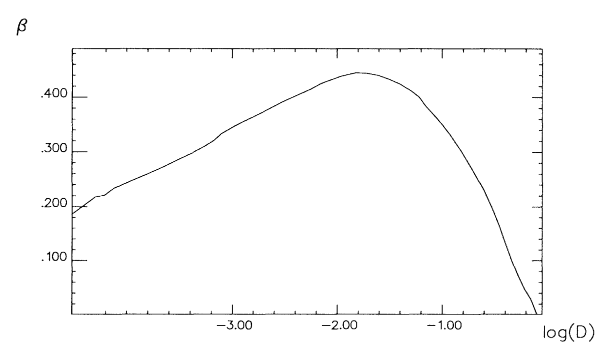

FIG. 5. The SNR $\beta=h(\Delta\omega/\omega_p)^{-1}$ vs $\log(D)$. A stochastic resonance maximum can be seen.

the SR without external signal may have interesting practical applications.

G.H. would like to thank the support of the Institute of Theoretical Physics and Synergetics, University of Stuttgart for his visit, where this collaboration began. C.Z.N. acknowledges the support of the Deutsche Forschungsgemeinschaft (DFG).

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