4. Strukturbildung & Muster

4.1 Einführung

Muster (patterns) findet sich überall

Bsp.:
- Vegetationsdynamik
- Vogelschwärme / Insekten
- Fischschalen
- Anregungswellen
- Physarum polycephalum Zelle
- Ladungsfronten
- Konvektionsmuster
- Rayleigh-Bénard-Zellen
- Taylor-Couette-Strömung
- Marangoni-Effekt

1) Was meint Chat GPT dazu?

2) a) Sucht selbst ein Thema / Bereich und kombiniert die Frage mit "pattern formation".

b) kurz zusammenfassen (wesentliche Punkte, Begriffe, Abbildungen)

Vergleiche: Meixner et al. PRE 55, 6690 (1997)

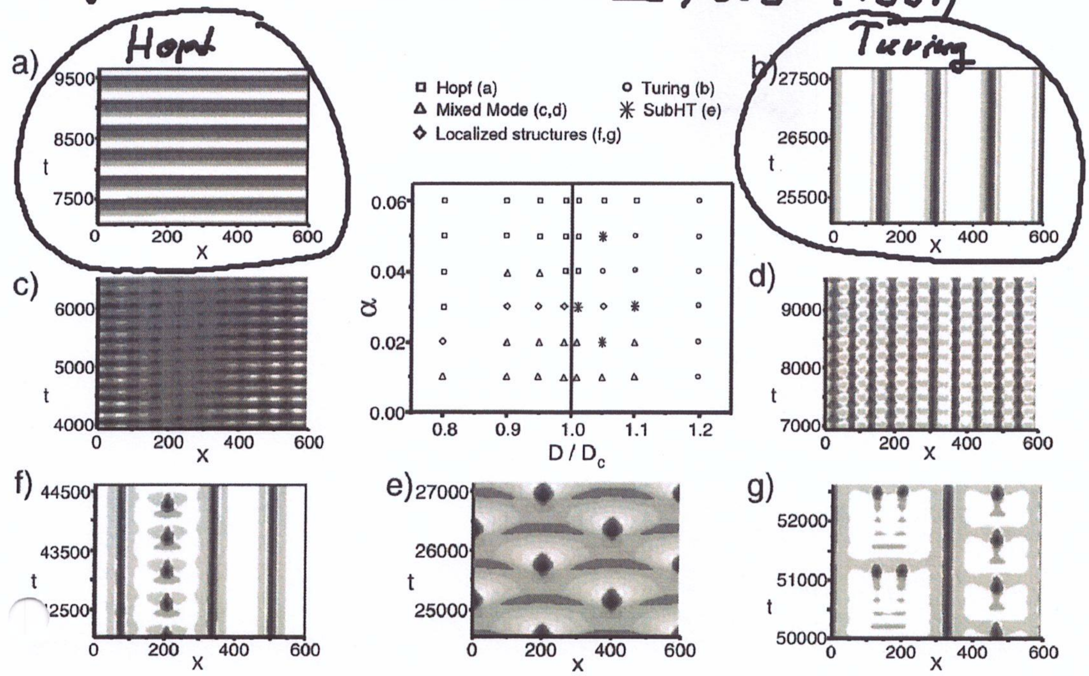

FIG. 5. Regimes of different asymptotic spatiotemporal behavior near the CTHP bifurcation given by the line $D/D_c=1$. The symbols in the $(\alpha, D/D_c)$ control parameter space denote various types of space-time patterns which are illustrated by typical space-time plots of $j(x,t)$ as insets: (a) Hopf oscillations (squares), (b) Turing patterns (dots), (c) and (d) Turing-Hopf mixed modes (triangles), (e) subharmonic Turing-Hopf mode consisting of spatiotemporal spiking (asterisks), and (f) and (g) localized Turing-Hopf structures (diamonds). (For parameters, see Table I.)

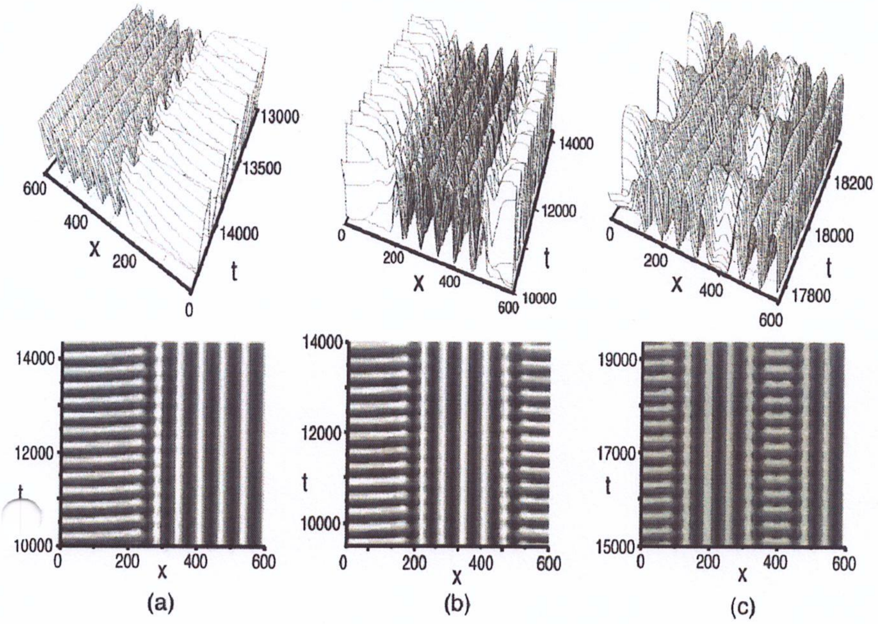

FIG. 6. Localized structures near the CTHP C_2 for $T=0.05$, $\alpha=0.02$, $D=8$ (corresponding to $D/D_c=0.67$), $j_0=3.1$, and different initial conditions: (a) Turing-Hopf front. (b) Turing domain embedded between two Hopf states. (c) Localized Turing-Hopf structures. The current density $j(x,t)$ is shown as a density plot and as a three-dimensional representation.

4.2 Komplex Ginzburg-Landau Gleichg

- Stuart-Landau-Oszillator (Hopf-Normalform)

$$\dot{z} = (\lambda + i\omega \mp (1+i\gamma)|z|^2) z \qquad w(x,t)$$

- harmonischer Oszillator mit Diffusion: $\frac{\partial W}{\partial t} = i\omega_0 W + D \frac{\partial^2}{\partial x^2} W$, $D = d_r + i d_i$

Ansatz: $W(x,t) = \sum_{n=-\infty}^{\infty} W_n(t) \exp\left[i \frac{2\pi n x}{L}\right]$ (Fourier-Reihe)

$\frac{2\pi n}{L}$: Wellenzahl

$\Rightarrow$ Amplitude der n-ten Mode:

$$\frac{\partial}{\partial t} W_n(t) = i\omega_0 W_n(t) - \left(\frac{2\pi n}{L}\right)^2 D\, W_n(t)$$

Lösung: $$W_n(t) = W_n(0) \underbrace{\exp\left[-\left(\frac{2\pi n}{L}\right)^2 d_r t\right]}_{\text{Dämpfung aller Moden } n \neq 0} \exp\Big[i \underbrace{\left(\omega_0 - \left(\frac{2\pi n}{L}\right)^2 d_i\right)}_{\omega_n} t\Big]$$

$\Rightarrow$ nur homogene Mode ($n=0$) überlebt

- Stuart-Landau + Diffusion:

$$\partial_t W = \mu(\sigma_1 + i\omega_1) W \underbrace{- (g_r + i g_i) |W|^2 W}_{\text{vernachlässigbar für klein } |W|\text{: Amplituden}} + (d_r + i d_i)\, \partial_x^2 W$$

$\Rightarrow$ Lösg: $$W_n(t) = W_n(0) \exp\left[\left(\mu\sigma_1 - \left(\frac{2\pi n}{L}\right)^2 d_r\right) t\right] \exp\left[i\, \omega_n t\right], \quad \omega_n = \omega_1 - \left(\frac{2\pi n}{L}\right)^2 d_i$$

Achtung! Entdämpfung für $\mu\sigma_1 - \left(\frac{2\pi n}{L}\right)^2 d_r > 0$

etwa L groß genug $\Rightarrow$ instabile Moden!

Transformiere: $t \to \frac{t}{\mu\sigma_1}$, $x \to \left(\frac{d_r}{\mu\sigma_1}\right)^{\frac{1}{2}} x$

$W \to \left(\frac{\mu\sigma_1}{g_r}\right)^{1/2} \exp\left(i\,\omega_1 t/\sigma_1\right) W$, $\quad c_1 = \frac{d_i}{d_r}$, $\quad c_2 = \frac{g_i}{g_r}$

$$\boxed{\frac{\partial}{\partial t} W = W - (1+ic_2)\,|W|^2 W + (1+ic_1)\frac{\partial^2}{\partial x^2} W}$$

Komplexe Ginzburg-Landau-Gleichung

Lösg: $W_Q = a_Q \exp\left[i(\omega_Q t + Qx)\right]$ Q: Wellenzahl

$$\Rightarrow \quad i\omega_Q W = W - (1+ic_2)\,|a_Q|^2 W + (1+ic_1)(-Q^2) W$$

Re: $0 = 1 - |a_Q|^2 - Q^2 \Rightarrow |a_Q|^2 = 1 - Q^2$

Im: $\omega_Q = -c_2 |a_Q|^2 - c_1 Q^2$

$$= -c_2(1-Q^2) - c_1 Q^2$$

$$= -c_2 + (c_2 - c_1) Q^2$$

Uniform/homogen Oszillatorzustand ($Q=0$): $|a_0| = 1$, $\omega_0 = -c_2$

Check Stabilität: kleine Störg: $W \approx [1 + w(x,t)]\exp(i\omega_0 t)$

$$\partial_t w = -(1+ic_2)(w+w^*) + (1+ic_1)\partial_x^2 w$$

$\Rightarrow$ $\partial_t w_q = -(1+ic_2)(w_q + w_q^*) + (1+ic_1)\, q^2 w_q$ mit $q = \frac{2\pi n}{L}$

$\uparrow$ linearer Ordnung

$$\partial_t w_q^* = -(1-ic_2)(w_q + w_q^*) + (1-ic_1)\, q^2 w_q^*$$

$$\Rightarrow \frac{\partial}{\partial t}\begin{pmatrix} w_q \\ w_q^* \end{pmatrix} = \begin{pmatrix} -(1+ic_2) - (1+ic_1)q^2 & -(1+ic_2) \\ -(1-ic_2) & -(1-ic_2) - (1-ic_1)q^2 \end{pmatrix}\begin{pmatrix} w_q \\ w_q^* \end{pmatrix}$$

$\Rightarrow$ Eigenwerte: $\lambda_\pm^{(q)} = -(1+q^2)\left\{1 \pm \sqrt{1 - \frac{q^2\left[(1+c_1^2)q^2 + 2(1+c_1c_2)\right]}{(1+q^2)}}\right\}$

Instabilität für $\boxed{1 + c_1 c_2 < 0}$ Benjamin-Feir-Linie

Band von instabilen Moden: $0 < |q| < \sqrt{\frac{2\,|1+c_1c_2|}{1+c_1^2}}$

- Hinweise / Tipps zur Nachbereitung / Vorbereitung:
  - ↳ Benjamin-Feir-Instabilität
  - ↳ Amplituden-/Phasen-Gleichung
  - ↳ Muster in der CGLE
  - ↳ mitbewegendes Bezugssystem
  - ↳ Transformation $\underline{x} \to \underline{x} - \underline{v}t$
  - ↳ Eigenwerte nachvollziehen