4.2 Komplexe Ginzburg-Landau-Gleichung (continued)

$$\frac{\partial}{\partial t} W(x,t) = W(x,t) - (1+iC_2)\,|W(x,t)|^2\, W(x,t) + (1+iC_1)\,\frac{\partial^2}{\partial x^2} W(x,t)$$

$W \in \mathbb{C}$ mit zeitlicher und räumlicher Abhängigkeit

Lösungsansatz (ebene Wellen) $W_Q(x,t) = a_Q \exp[i(\omega_Q t + Qx)]$

mit Wellenzahl Q

$\Rightarrow |a_Q|^2 = 1 - Q^2$ und $\omega_Q = -C_2 + (C_2 - C_1)\,Q^2$

Spezialfall: uniform/homogene Oszillation ($Q=0$): $|a_Q| = 1$, $\omega_0 = -C_2$

Instabilität für $1 + C_1 C_2 < 0$ Benjamin-Feir-Linie

Band von stabilen Moden: $0 < |Q| < \sqrt{\dfrac{2\,|1+C_1C_2|}{1+C_1^2}}$

Reelle Ginzburg-Landau-Gleichung: $\dfrac{\partial W}{\partial t} = \dfrac{\partial^2 W}{\partial x^2} + W - |W|^2 W$
($C_1 = 0 = C_2$)

Spiegelsymmetrie: $W(x,t) = W(-x,t)$

$$\frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial(-x)} = -\frac{\partial}{\partial x}\,,\quad \frac{\partial^2}{\partial x^2} \longleftrightarrow \frac{\partial^2}{\partial x^2}$$

oder auch $W(x,t) \mapsto W(x,t)\, e^{i\phi}$

$$\frac{\partial W}{\partial t}\cancel{e^{i\phi}} = \frac{\partial^2 W}{\partial x^2}\cancel{e^{i\phi}} + W\cancel{e^{i\phi}} - |W|^2 \underbrace{|e^{i\phi}|^2}_{=1} W \cancel{e^{i\phi}}$$

Stationäre Lösung: $W(x,t) = a_0\, e^{iQx}$ mit $Q^2 = 1 - a_0^2$

$\omega_0 = -C_2 + (C_2 - C_1)\,Q^2 = 0$

Einsetzen liefert: $\dfrac{\partial W}{\partial t} = 0 = \underbrace{-Q^2 W}_{\frac{\partial^2}{\partial x^2} W} + W - |a_0|^2 W$

Benjamin-Feir-Instabilität: $1+c_1c_2<0$

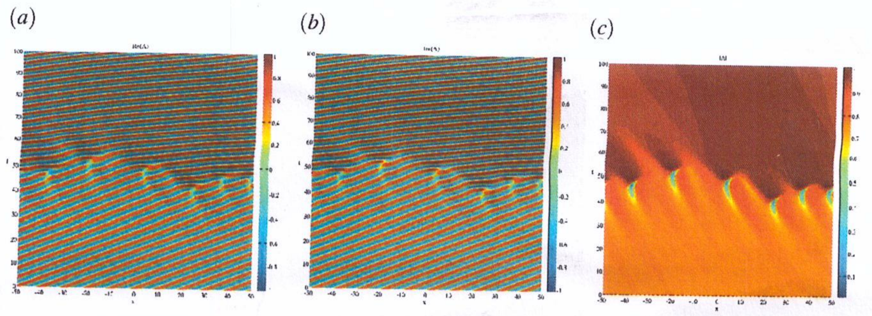
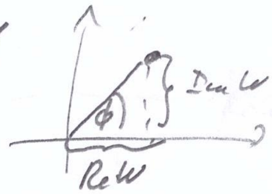


Fig. 5.2 Space-time plots of (a) Re(A), (b) Im(A) and (c) $|A|$ in the case of the Benjamin-Feir instability.

Bei "falscher" Wellenzahl (Q zu groß) ist das Muster instabil und das System geht in eine ebene Welle mit kleinerem Q (größerer Wellenlänge) über. Am Übergang kommt es zu Defekten (Phasensingularitäten) mit $|W|=0$

Fall: $1+c_1c_2 \lesssim 0$ schwach instabil : $|W| \simeq 1$

Phasenvariable: $\tan\phi = \dfrac{\mathrm{Im}\,W}{\mathrm{Re}\,W}$



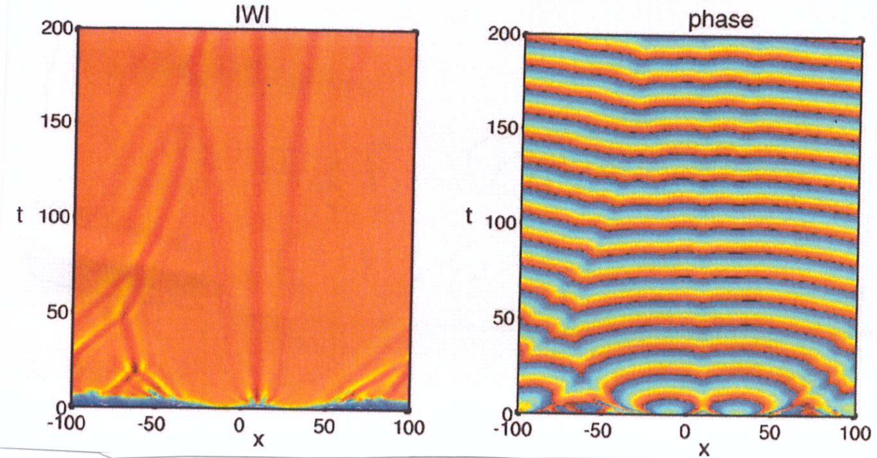


Fig 5

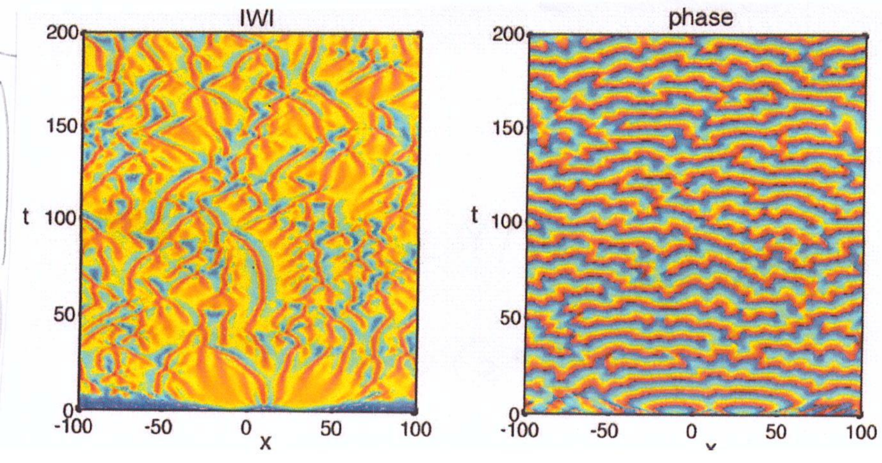
$c_1 = -4$

$c_2 = 0.5$

V. Garcia-Morales, K. Krischer, Contemporary Physics 53, 79 (2012)

Defekt: bei x_0

$$\lim_{\substack{x \to x_0^+ \\ \uparrow \\ \text{von rechts}}} \frac{\mathrm{Im}\,W(x,t)}{\mathrm{Re}\,W(x,t)} \neq \lim_{\substack{x \to x_0^- \\ \text{von links}}} \frac{\mathrm{Im}\,W(x,t)}{\mathrm{Re}\,W(x,t)}$$



$c_1 = -4$

$c_2 = 1$

Fig 6

Raum zeitliches Chaos: $c_1=0$, $c_2=-3$

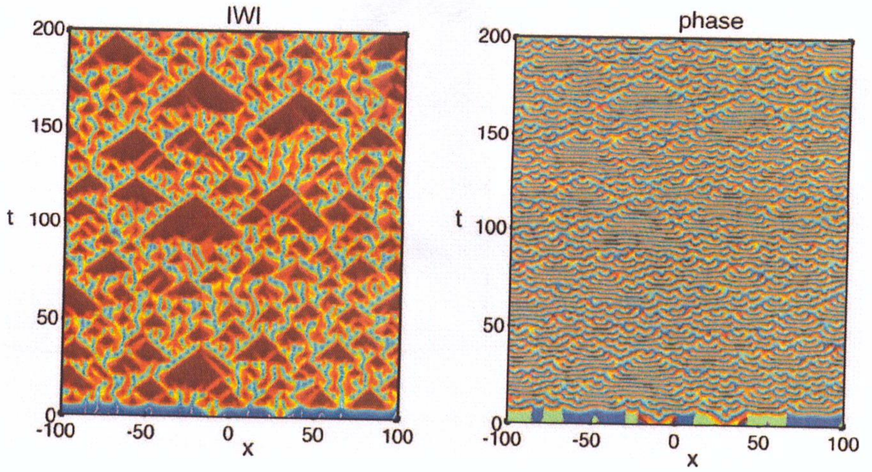


Figure 7. Spatiotemporal evolution of the absolute value $|W|$ of the complex amplitude (left) and phase (right) in a situation of spatiotemporal chaos: $c_1 = 0$, $c_2 = -3$.

Was passiert bei 2D? $w(x,y,t)$:

$$\frac{\partial}{\partial t} w = w - (1+ic_2)|w|^2 w + (1+ic_2)\left(\frac{\partial^2}{\partial x^2} w + \frac{\partial^2}{\partial y^2} w\right)$$

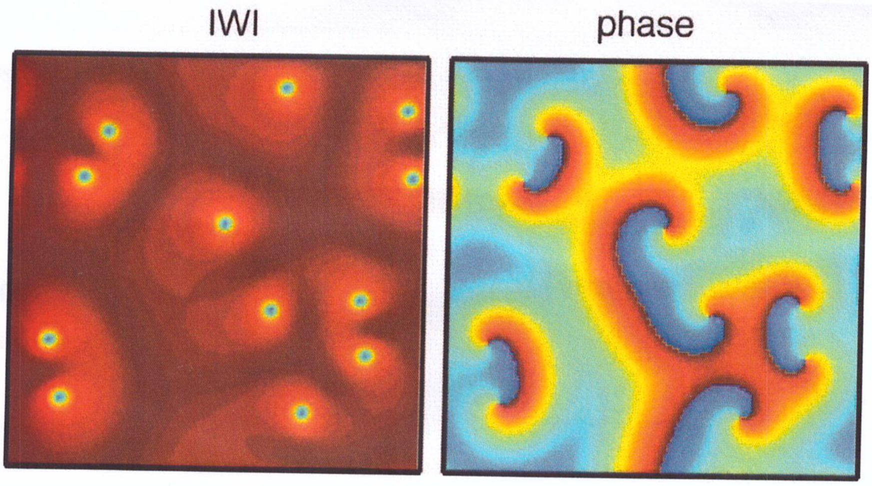


Figure 8. Spatial distribution of the absolute value $|W|$ of the complex amplitude (left) and phase (right) obtained after a transient from the 2D CGLE on a rectangular domain of 76 × 76 size in a situation of spatiotemporal chaos: $c_1 = 0$, $c_2 = 1$.

Defekte als Zentren von Spiralwellen in der Phase.

⇒ topologische Ladung $\quad n_{top} = \frac{1}{2\pi} \oint \nabla\phi \cdot ds$

Contour-Integral (geschlossene Kurve)

⇒ 1x im Kreis sammelt Phase 2π auf.

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) W(x,y,t) = \Delta\, W(x,y,t)$$

mit Laplace-Operator Δ

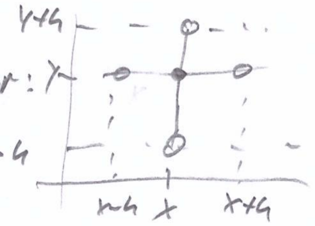
siehe Wellengleichung: $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u$

$$\ddot{u} = c^2 \Delta u$$

$$\Delta u = \operatorname{div}\operatorname{grad} u = \operatorname{div}\begin{pmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{pmatrix} = \begin{pmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\end{pmatrix}\begin{pmatrix}\frac{\partial u}{\partial x}\\ \frac{\partial u}{\partial y}\end{pmatrix} = \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$$

Diskretisierung: $\Delta u(x,y) = \lim_{h\to 0}\frac{1}{h^2}\left[u(x-h,y)+u(x+h,y)+u(x,y-h)+u(x,y+h)-4u(x,y)\right]$

Auf einem Gitter:



Differenzenquotient koppelt benachbarte Gitterpunkte

$\Rightarrow$ Verallgemeinerung: allgemeiner Kopplungskern:

$$\frac{\partial W}{\partial t} = W-(1+ic_2)|W|^2 W + (1+ic_1)\int H(|x-x'|)\left[W(x')-W(x)\right]dx'$$

(i) $H(|x-x'|) = \delta^{(2)}(x-x')$ liefert komplexe Ginzburg-Landau-Gleichung

$$\int \delta^{(n)}(x-x')\, f(x')\,dx' = (-1)^n \frac{d^n f(x)}{dx^n} \Rightarrow \int \delta^{2}(x-x')\, f(x')\,dx' = \frac{d^2 f(x)}{dx^2}$$

(ii) $H(|x-x'|) = \frac{K}{2\sqrt{D}}\exp\left[-\frac{|x-x'|}{\sqrt{D}}\right]$