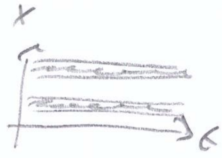
4.4 Beispiele räumlicher Muster



Wiederholung: – Turing: zeitlich konstantes Muster

– Hopf: homogene Oszillationen

Frage: Geht das auch gleichzeitig?

↳ W. Just et al. PRE 64, 026219 (2001)

2 Variablen-Modell:
$$\frac{\partial u}{\partial t} = \alpha\left[j_0 - (u-a)\right] + D\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

$$\frac{\partial a}{\partial t} = \frac{u-a}{1+(u-a)^2} - Ta + \frac{\partial^2 a}{\partial x^2} + \frac{\partial^2 a}{\partial y^2}$$

u: Spannung, a: Ladungsträgerdichte, T: Tunnelrate

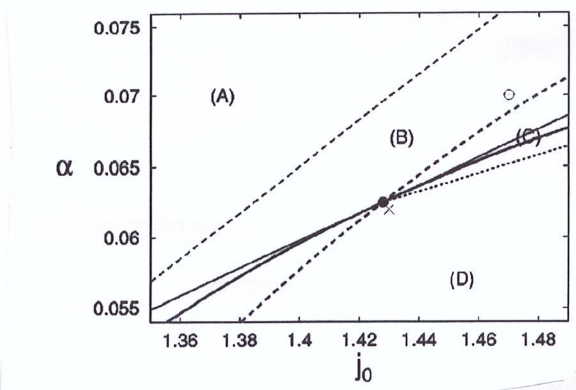


FIG. 3. Different stability regimes in the vicinity of the Turing-Hopf point (full circle) in the (j_0, α) parameter plane. Gray: Turing (broken) and Hopf (full) bifurcation line (see Fig. 1). Existence of Hopf mode [full line, see Eq. (33)], stability of Hopf mode [dotted line, see Eq. (34)], and saddle-node bifurcation of Turing patterns [broken line, see Eq. (60)]. Region (A): trivial solution, region (B): coexistence between trivial solution and Turing pattern, region (C): Turing pattern, region (D): coexistence between Hopf mode and Turing pattern. × and ○ mark the parameter settings used in Figs. 4 and 7, respectively.

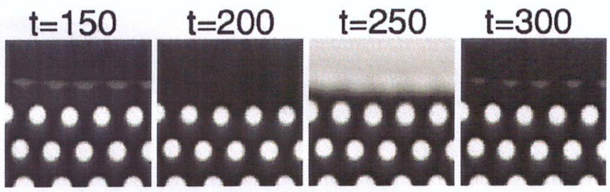


FIG. 5. Coexistence pattern between Turing and Hopf state at $j_0 = 1.43$ and $\alpha = 0.045$. The density plots show the current density $j(r,t)$ at four different times. The time labels refer to Fig. 6.

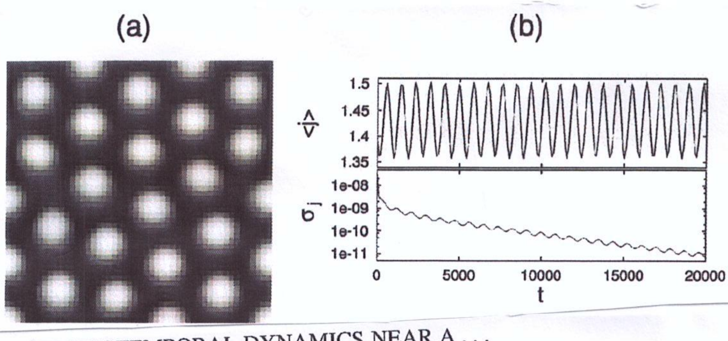


FIG. 4. (a) Density plot of stationary Turing pattern for the current density $j(r,t) = u(r,t) - a(r,t)$. (b) Relaxation of a Hopf mode. Time dependence of the spatial average $\langle j\rangle$ of the current and the corresponding variance $\sigma_j = \langle j^2\rangle - \langle j\rangle^2$. Parameter settings for both parts are the same ($j_0 = 1.43$, $\alpha = 0.062$, $D = 5$, and $T = 0.05$, see Fig. 3), but different initial conditions had been chosen.

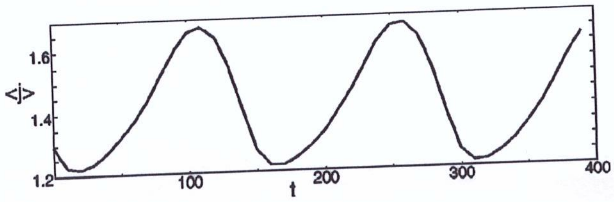


FIG. 6. Time dependence of the spatial average of the current

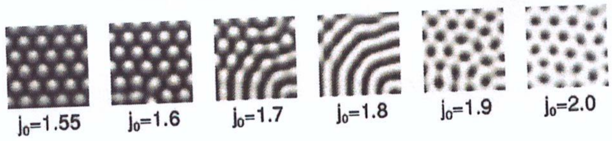
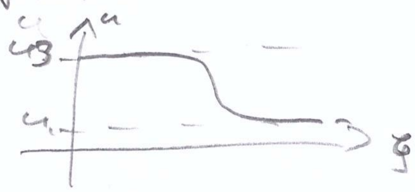


FIG. 8. Time independent patterns appearing for different values of the total current j_0 at $\alpha = 0.075$, $T = 0.05$, and $D = 5$: Transition from hot spots (left) to cold spots (right). Simulations have been performed on a system of size 200×200 with Neumann boundary conditions.

Bemerkg zu Front-Lösung:

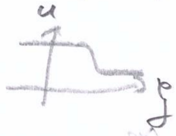
$$\frac{\partial u}{\partial t} = f(u) + D\frac{\partial^2 u}{\partial x^2} \xrightarrow{\xi = x-ct} D u'' + C u' + f(u) = 0$$



Bestimme C allgemein:

Idee: Multiplizieren mit u' und integrieren:

$$\underbrace{D\int_{-\infty}^{\infty} d\xi \, \frac{d^2u}{d\xi^2}\frac{du}{d\xi}}_{= \frac{1}{2} D \left(\frac{du}{d\xi}\right)^2 \Big|_{-\infty}^{\infty} = 0} + C\int_{-\infty}^{\infty} d\xi \left(\frac{du}{d\xi}\right)^2 + \underbrace{\int_{-\infty}^{\infty} d\xi \, f(u)\frac{du}{d\xi}}_{= \int_{u_1}^{u_3} du \, f(u) = A} = 0$$



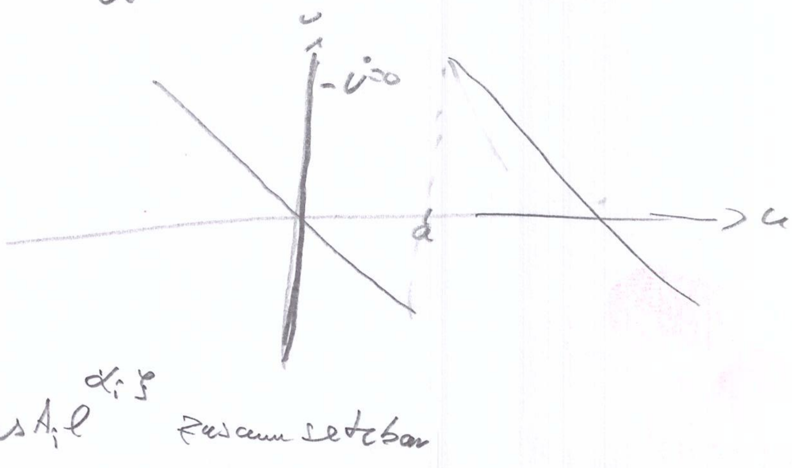
$$\Rightarrow \quad C = \frac{\int_{u_1}^{u_3} du \, f(u)}{\int_{-\infty}^{\infty} d\xi \left(\frac{du}{d\xi}\right)^2}$$

- >0 für $A>0$: Zündfront wandert (u_1 metastabil)
- $=0$ für $A=0$: equal area rule
- <0 für $A<0$: Löschfront (u_3 metastabil)

Approximation des FitzHugh-Nagumo-Modells

$$\left.\begin{aligned} \varepsilon \frac{\partial u}{\partial t} &= u - \frac{u^3}{3} - v + \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial v}{\partial t} &= u + a \end{aligned}\right\} \longrightarrow \begin{aligned} \frac{\partial u}{\partial t} &= -u - H(u-a) + \frac{\partial^2 u}{\partial x^2} \\ \frac{\partial v}{\partial t} &= \varepsilon u \end{aligned}$$

$$H(u-a) = \begin{cases} 0 & u \le a \\ 1 & u > a \end{cases}$$



$\Rightarrow$ Stückweise linear $\Rightarrow$ Lösg aus $A_i e^{\lambda_i \xi}$ zusammensetzbar

John Rinzel, Joseph B. Keller Biophys. Journal 13, 1313 (1973)

$\rho \propto Z$:

$$\begin{aligned}
\dot{\alpha} &= s(\eta - \eta\alpha + \alpha - q\alpha^2), \\
\dot{\eta} &= s^{-1}(-\eta - \eta\alpha + f\rho), \\
\dot{\rho} &= w(\alpha - \rho).
\end{aligned} \tag{2.5}$$

The parameters s, w and q are determined from the rates of the reactions, see [29, 25].

Krug et al. introduced in [30] the modified Oregonator model, which describes the light sensitivity of the Belousov-Zhabotinsky reaction. For this purpose, the reaction scheme (2.3) was extended by a simple reaction, corresponding to the light-induced bromide flow

$$\xrightarrow{\Phi} Y,$$

which leads to the modified three-component Oregonator model, given by

$$\begin{aligned}
\epsilon\dot{x} &= x(1-x) + y(q-x), \\
\epsilon'\dot{y} &= \phi + fz - y(q+x), \\
\dot{z} &= x - z.
\end{aligned} \tag{2.6}$$

The parameter ϕ accounts for the light intensity. The following parameter values were suggested: $q = 2 \times 10^{-3}, f = 2.1, \epsilon = 0.05, \epsilon' = \epsilon/8$. With this set of parameters, it was found that for $\phi = 1.762 \times 10^{-3}$ the stable equilibrium in Eq. (2.6) undergoes a Hopf bifurcation, thus giving access to both excitable (monostable) and oscillatory reaction kinetics upon variation of the parameter ϕ near the bifurcation value.

Often, one can exploit the smallness of the parameter ϵ' and set the left-hand side of the second equation in Eq. (2.6) equal zero. In this case the model can be further reduced to the so-called two-component version of Oregonator, which reads

$$\begin{aligned}
\dot{u} &= \frac{1}{\epsilon}\left[u - u^2 - (fv + \phi)\frac{u-q}{u+q}\right], \\
\dot{v} &= u - v.
\end{aligned} \tag{2.7}$$

We would like to mention that Eq. (2.7) is qualitatively similar to the FitzHugh-Nagumo equation [9, 10], which describes the propagation of the action potential in the squid axons.

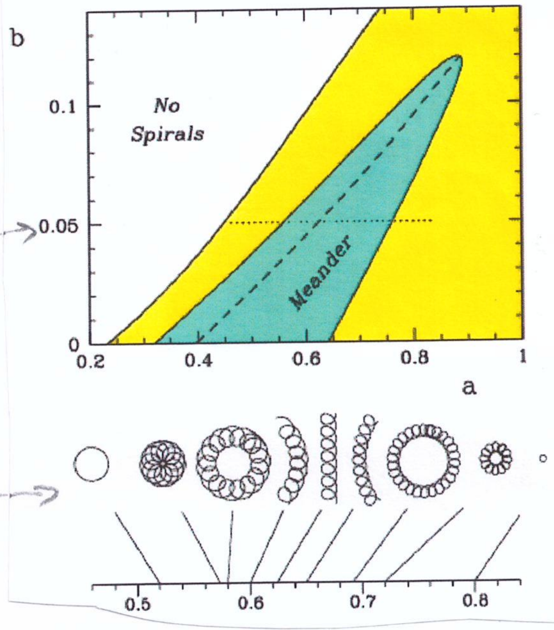
For spatially extended Belousov-Zhabotinsky reaction we must account for diffusion:

$$\begin{aligned}
\partial_t u &= \frac{1}{\epsilon}\left[u - u^2 - (fv + \phi)\frac{u-q}{u+q}\right] + D\Delta u, \\
\partial_t v &= u - v.
\end{aligned} \tag{2.8}$$

Barkley-Modell: $\frac{\partial u}{\partial t} = \frac{1}{\varepsilon} u(1-u)\left(u - \frac{v+b}{a}\right) + \Delta u$

$$\frac{\partial v}{\partial t} = u - v$$

Dwight Barkley: Physica D $\underline{49}$, 61 (1991)
+Dowle IJBC $\underline{7}$, 2529 (1997)



b=0.05

Oregonator-Modell: (2D)

$$\frac{\partial u}{\partial t} = \frac{1}{\varepsilon}\left[u(1-u) - (fv+\phi)\frac{u-q}{u+q}\right] + D\Delta u$$

$$\frac{\partial v}{\partial t} = u - v$$

Dissertation Grigory Bordyugov (2006)

V. Zykov et al.: PRL $\underline{92}$, 018304 (2004)

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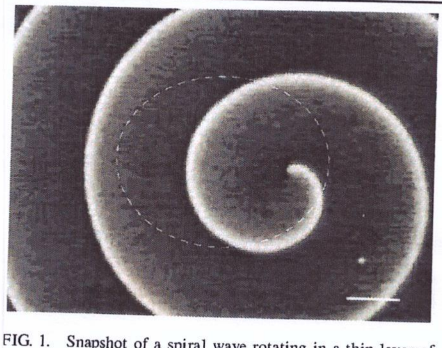

FIG. 1. Snapshot of a spiral wave rotating in a thin layer of the BZ reaction. The dashed line indicates the boundary of

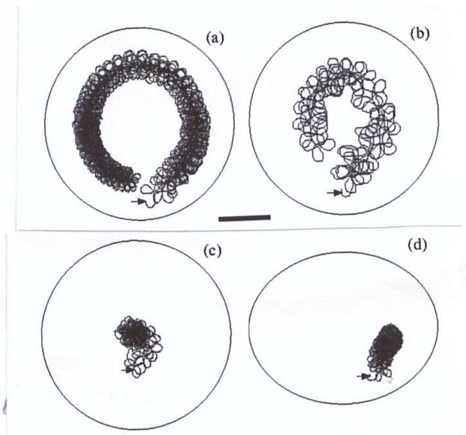

FIG. 2. Resonant drift of a spiral wave induced by a global feedback with $k_{fb} = -1.5$, $B_0 = 25$, and $I_0 = 70$. (a)–(c) Circular domain of radius $R = \lambda$; (d) elliptical domain with large axis $a = 2\lambda = 4$ mm and small axis $b = a/1.25$. In (a) and (d), the time delay is $\tau = 0$, in (b) $\tau/T_\infty = 0.32$, and in (c) $\tau/T_\infty = 0.5$. Initial spiral tip locations are marked by arrows. Scale bar: 1 mm.