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Continuous control of chaos by self-controlling feedback

K. Pyragas [1,2]
Institute for Theoretical Physics, Technical University of Berlin, W-1000 Berlin, Germany

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Two methods of chaos control with a *small time continuous* perturbation are proposed. The stabilization of unstable periodic orbits of a chaotic system is achieved either by combined feedback with the use of a specially designed external oscillator, or by delayed self-controlling feedback without using of any external force. Both methods do not require an a priori analytical knowledge of the system dynamics and are applicable to experiment. The delayed feedback control does not require any computer analyses of the system and can be particularly convenient for an experimental application.

1. Introduction

Dynamic chaos is a very interesting nonlinear effect which has been intensively studied during the last two decades. The effect is very common; it has been detected in a large number of dynamic systems of various physical nature. In practice, however, this effect is usually undesirable. It restricts the operating range of many electronic and mechanic devices. Ott, Grebogi and Yorke [1] (OGY) have suggested an efficient method of chaos control that can eliminate chaos. The method is based on the idea of the stabilization of unstable periodic orbits (UPOs) embedded within a strange attractor. This is achieved by making a small time-dependent perturbation in the form of feedback to an accessible system parameter. The method turns the presence of chaos into an advantage. Due to the infinite number of different UPOs embedded in a strange attractor, a chaotic system can be tuned to a large number of distinct periodic regimes by switching the temporal programming of *small* parameter perturbation to stabilize different periodic orbits. Recently the OGY method has been successfully applied to some experimental systems [2–4].

[1] Alexander von Humboldt Fellow.
[2] On leave of absence from Institute of Semiconductor Physics, 2600 Vilnius, Lithuania.

An experimental application of the OGY method requires, as a rule, a permanent computer analysis of the state of the system. The changes of the parameter, however, are discrete in time since the method deals with the Poincaré map. This leads to some limitations. The method can stabilize only those periodic orbits whose maximal Lyapunov exponent is small compared to the reciprocal of the time interval between parameter changes. Since the corrections of the parameter are rare and small, the fluctuation noise leads to occasional bursts of the system into the region far from the desired periodic orbit, and these bursts are more frequent for large noise [1]. Therefore, the idea of a *time-continuous* control seems attractive in this context.

The response of chaotic systems to continuous periodic and aperiodic perturbations have been considered in many investigations [5–9] to suppress chaos in the system [5,6], to achieve some desired behaviour [7], to synchronize some subsystems in a complex chaotic system [8,9]. But none of these investigations considered the perturbation in the form of the feedback. The methods developed cannot be applied to the UPO stabilization. They can eliminate the chaos in the system, but the resulting periodic orbits obtained by the methods differ from the UPOs of the initial system and, therefore, they require a comparatively large perturbation.

In the following two methods of permanent control in the form of feedback are suggested. Both methods are based on the construction of a special form of a time-continuous perturbation, which does not change the form of the desired UPO, but under certain conditions can stabilize it. A combined feedback with a periodic external force of a special form is used in the first method. The second method does not require any external force; it is based on a self-controlling delayed feedback. The block diagrams of these methods are shown in fig. 1.

2. External force control

Let us consider a dynamic system which can be simulated by ordinary differential equations. We imagine that the equations are unknown, but some scalar variable can be measured as a system output. We also suppose that the system has an input available for external force. These assumptions can be met by the following model,

$$\frac{dy}{dt} = P(y, \boldsymbol{x}) + F(t), \quad \frac{d\boldsymbol{x}}{dt} = \boldsymbol{Q}(y, \boldsymbol{x}) . \qquad (1)$$

Here y is the output variable and the vector $\boldsymbol{x}$ describes the remaining variables of the dynamic system which are not available or not of interest for observation. It is assumed for simplicity that the input signal $F(t)$ disturbs only the first equation, corresponding to the output variable. We suppose that the

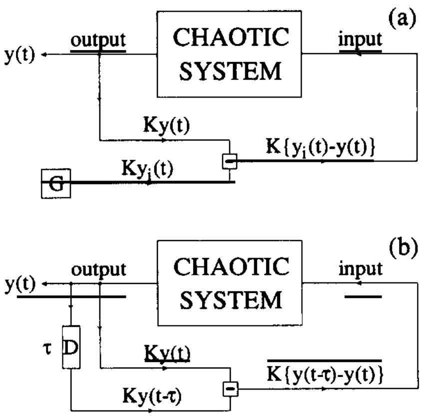

Fig. 1. Block diagram of (a) external force control, and (b) delayed feedback control. G is a special external periodic oscillator, D is a delay line.

considered system without an input signal ($F=0$) has a strange attractor.

It has been demonstrated using a standard method of delay coordinated that a large number of distinct UPOs on a chaotic attractor can be obtained from one scalar signal [10–12]. Applying this method to our system, we can determine from the experimentally measured output signal $y(t)$ various periodic signals of different form $y=y_i(t)$, $y_i(t+T_i)=y_i(t)$ corresponding to different UPOs. Here T_i is the period of the ith UPO. Then we examine these periodic signals and select the one which we intend to stabilize. To achieve this goal we have to design a special external oscillator, which generates the signal proportional to $y_i(t)$. The difference $D(t)$ between the signal $y_i(t)$ and the output signal $y(t)$ is used as a control signal:

$$F(t) = K[y_i(t) - y(t)] = KD(t) . \qquad (2)$$

Here K is an experimentally adjustable weight of the perturbation. The perturbation has to be introduced into the system input as a negative feedback ($K>0$). An experimental realization of such a feedback presents no difficulties for many physical systems. The important feature of perturbation (2) is that it does not change the solution of eq. (1) corresponding to the UPO $y(t)=y_i(t)$. By selecting the weight K, one can achieve the stabilization. When this stabilization is achieved the output signal is very close to $y_i(t)$ and the perturbation $F(t)$ becomes extremely small. Therefore here, as well as in the OGY method, only a *small* external force is used to stabilize the UPOs. We do not intend to prove the validity of this method for the general case, but we have verified it for many chaotic systems such as the Rossler [13], Lorenz [14], Rabinovich and Fabrikant [15], Duffing oscillator [5,16] systems, and others.

The main results presented here are illustrated for the Rossler system:

$$\frac{dx}{dt} = -y - z, \quad \frac{dy}{dt} = x + 0.2y + F(t) ,$$

$$\frac{dz}{dt} = 0.2 + z(x - 5.7) . \qquad (3)$$

Here $F(t)$ is the perturbation defined in eq. (2). For definiteness y is chosen as an output signal. The results do not depend on the choice of output variable.

Figure 2a shows the results of stabilization of the period-five UPO of the Rossler attractor. The origin of the curve F corresponds to the time when the perturbation is switched on. As it was expected, the perturbation becomes small after a transient process and the system comes into the periodic regime corresponding to an initially unstable orbit. To illustrate the validity of the method for other chaotic systems fig. 2b shows the results of stabilization of the period-two UPO of the Lorenz system.

The amplitude of perturbation in a post-transient regime depends on two factors, on the accuracy of the UPO $y_i(t)$ reconstruction, and on the fluctuation noise. In an ideal case the perturbation has to be vanishingly small when the system moves along its periodic orbit, and the stabilization can be achieved with a very small signal of the external oscillator. To investigate the influence of noise, we add terms $\epsilon\xi_x(t)$, $\epsilon\xi_y(t)$, and $\epsilon\xi_z(t)$ to the right-hand sides of eq. (3). The random functions ξ_x, ξ_y, and ξ_z are independent of each other, having mean value 0 and mean-squared value 1. Figure 3 shows the results of the stabilization of the period-one cycle of the Rossler attractor for two different levels of noise. Since the control is permanent, the system does not experience any bursts into the region far from the UPO even for sufficiently large noise. The increase in noise leads to the increase of the amplitude of perturbation and to the smearing-out of the periodic orbit.

Note one difference between the OGY and the above method. The perturbation in the OGY method is applied only when the state of the system is close to the fixed point, since it uses a linear approximation for the deviations from the fixed point. Here we do not need to wait until the state of the system comes close to the desired periodic orbit. The perturbation can be switched on at any moment. The Rossler system synchronizes with the external oscillator even when the initial conditions are far from the periodic orbit. Then the initial perturbation can

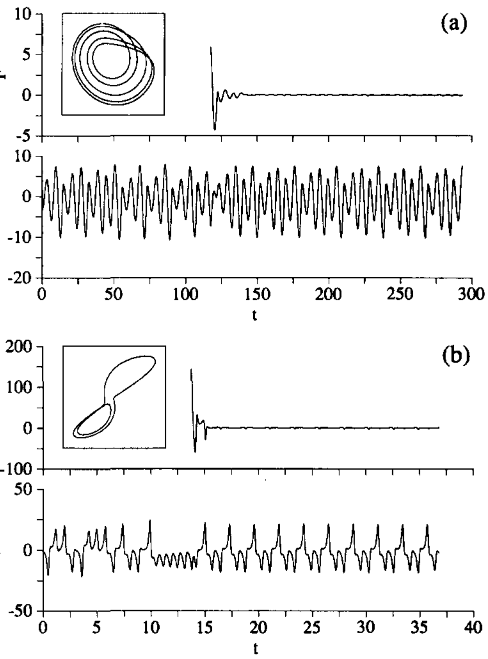

Fig. 2. Dynamics of the output signal $y(t)$ and perturbation $F(t)$ (a) for the Rossler system (eq. (3)), $K=0.4$, $y_i(t)$ corresponds to the period-five cycle, and (b) for the Lorenz system: $dx/dt=10(x-y)$, $dy/dt=-xz+28x-y+F(t)$, $dz/dt=xy-\frac{8}{3}z$, the perturbation $F(t)$ is determined by eq. (2), $y_i(t)$ corresponds to the period-two cycle. The origin of curve F corresponds to the moment of switching on the perturbation. The implement shows the x–y phase portrait of the system in the post-transient regime.

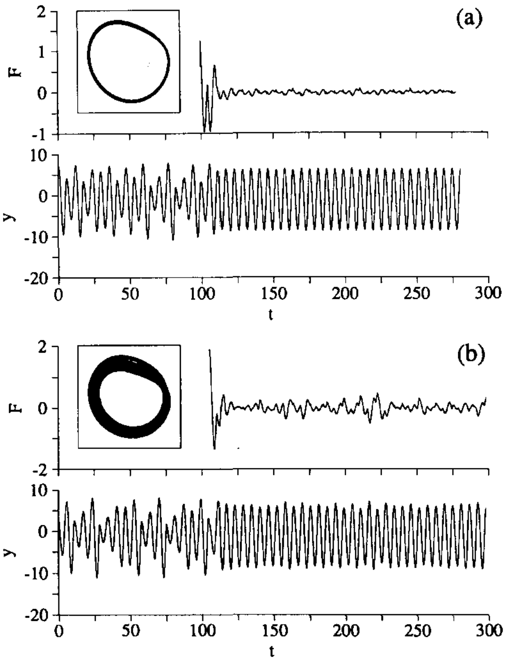

Fig. 3. Results of stabilization of the period-one circle of the Rossler system at two different levels of noise. $K=0.4$; (a) $\epsilon=0.1$; (b) $\epsilon=0.5$.

be rather large. However, we do not expect that this will be the case for all dynamic systems. More complicated periodically driven dynamic systems along with the stabilized UPO can have alternative stable solutions belonging to different basins of initial conditions. Such multistability can be an undesired feature for the purpose considered here. Large initial values of the perturbation can be also undesired for some experiments. In many cases both these problems can be solved by restriction of the perturbation. Introducing some nonlinear element into the feedback circuit it is possible to achieve the saturation of the perturbation $F(t)$ for large values of the deviation $D(t)$:

$$\begin{aligned} F(t) &= -F_0, & KD(t) &\leqslant -F_0, \\ &= KD(t), & -F_0 &< KD(t) < F_0, \\ &= F_0, & KD(t) &\geqslant F_0. \end{aligned} \tag{4}$$

Here $F_0 > 0$ is the saturating value of the perturbation. Although in proximity to the UPO both perturbations (2) and (4) work identically, they lead to different transient processes. Figure 4 illustrates the influence of restriction (4) to the system dynamics. The perturbation in this case is always small including the transient process. However, the transient process on average is now much longer. The system "waits" until the trajectory comes close to the periodic orbit and only then synchronizes with an external oscillator. As in the OGY method the mean duration of the transient process increases rapidly with the decrease of F_0. The efficiency of restriction (4) to eliminate the multistability will be illustrated in section 3.

To analyze the local stability of the system we have calculated the maximal Lyapunov exponent of the UPOs using the linearization of system (3) with respect to small deviations from the corresponding UPOs. The dependence of the leading Lyapunov exponent λ of the period-one and period-two orbits on the parameter K is shown in fig. 5. The negative values of $\lambda(K)$ determine the interval of K corresponding to the stabilized UPO. The period-one UPO is stable in the finite interval $K=[K_{\min}, K_{\max}]$, but the period-two UPO has an infinite interval $K=[K_{\min}, \infty]$ of stabilization. Here the values $K_{\min}$ and $K_{\max}$ define the threshold of the stabilization: $\lambda(K_{\min})=\lambda(K_{\max})=0$. The Lyapunov exponent $\lambda(K)$ of both orbits have minima at some value of $K=K_{\rm op}$ providing an optimal control. Note that for all values of $K>0$ the perturbation decreases the Lyapunov exponent of the initial system, $\lambda(K)<\lambda(0)$, but not for all values of K this perturbation is sufficiently efficient to invert the sign of λ. The presence of the minimal threshold of the stabilization is well understood. The weight K of the perturbation has to be sufficiently large to compensate the divergence of the trajectories close to the UPO. A rather large value of K deteriorates the control. This is related to the fact that the perturbation disturbs only one variable of the system. For large K the changes of this variable are very fast and the remaining variables have no time to follow these changes. To support this assumption we have considered multivariable control. A perturbation in the form of eq. (2) with corresponding variables have been added to each equation of the system (3). As a result the monotonously

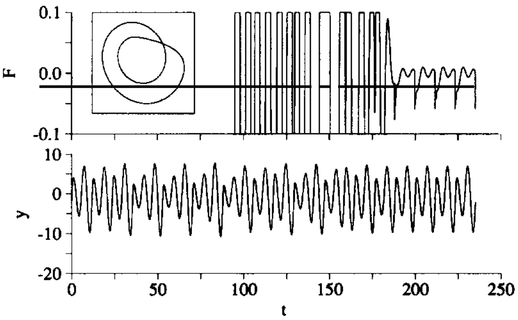

Fig. 4. Results of stabilization of the period-two circle of the Rossler system with the restricted perturbation (4). $K=0.4$, $F_0=0.1$.

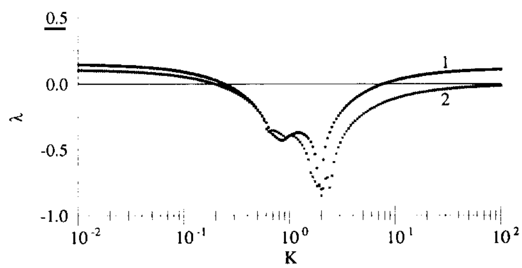

Fig. 5. Dependence of the Lyapunov exponents of the period-one and period-two UPOs of the Rossler system on the weight K of the perturbation.

decreasing characteristics $\lambda(K)$ for both orbits have been obtained.

The experimental application of this method can be divided into two stages. In the first, preparatory, stage the output signal should be investigated and the oscillator generating a periodical signal proportional to $y_i(t)$ should be designed. In the second stage the control is achieved simply by combining the scheme shown in fig. 1a. A combined feedback using the difference between an output signal and the signal of the external oscillator performs here a self-controlling function.

3. Delayed feedback control

The complexity of the experimental realization of the above method is mainly in the design of a special periodic oscillator. The second method which we have considered has no such shortcoming. The idea of this method consists in substituting the external signal $y_i(t)$ in eq. (2) for the delayed output signal $y(t-\tau)$. In other words, we use a perturbation of the form

$$F(t)=K[y(t-\tau)-y(t)]=KD(t)\ . \tag{5}$$

Here τ is a delay time. If this time coincides with the period of the ith UPO $\tau=T_i$ then the perturbation becomes zero for the solution of system (1) corresponding to this UPO $y(t)=y_i(t)$. This means that the perturbation in the form (5) as well as in te form (2) does not change the solution of system (1) corresponding to the ith UPO. Choosing an appropriate weight K of the feedback one can achieve the stabilization. The results of such a stabilization for the Rossler system and for the Duffing oscillator are shown in fig. 6. These results are very similar to those in the case of an external force control. However, an experimental realization is simpler in this case. No external perturbation or computer is needed for this control. This control is achieved by the use of the output signal, which is fed in a special form into the system input. The difference between the delayed output signal and the output signal itself is used as a control signal. This feedback performs the function of self-control. Only a simple delay line is required for this feedback. To achieve the stabilization of the desired UPO, two parameters, namely, the time of delay τ and the weight K of the feedback, should be adjusted in experiment. The amplitude of the feedback signal can be considered as a criterion of UPO stabilization. When the system moves along its UPO this amplitude is extremely small. The dependence of this amplitude on the delay time for the Rossler system is illustrated in fig. 7a. Excluding the transient process, the dispersion of the perturbation $\langle D^2(t)\rangle$ has been calculated for each value of τ with 20 different initial conditions, and the corresponding 20 values of this dispersion for each τ have been depicted. The resulting figure represents the sequence of resonance curves with very deep minima. These minima are located at the points of delay time coinciding with the periods of the UPO $\tau=T_i$. The phase portraits for these values of delay time are shown in figs. 7b1, 7b4, 7b8. They correspond to initially unstable period-one, -two and -three cycles. The resonance curves are separated by additional minima intervals, corresponding to the steady-state so-

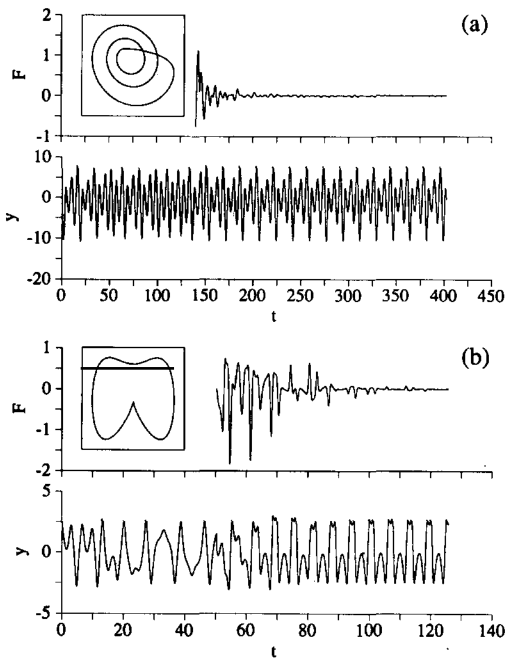

Fig. 6. Results of the stabilization of (a) the period-three cycle of the Rossler system, $K=0.2$, $\tau=17.5$, and (b) the period-one cycle of the nonautonomous Duffing oscillator: $dx/dt=y$, $dy/dt=x-x^3-dy+f\cos(\omega t)+F(t)$, $f=2.5$, $\omega=1$, $d=0.2$, $K=0.4$, $\tau=2\pi/\omega$, in the case of delayed feedback control with the use of a perturbation in the form of eq. (5).

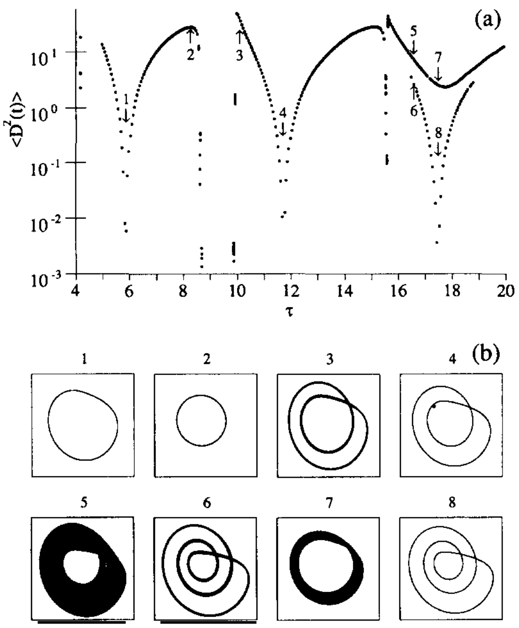

Fig. 7. (a) Dependence of the dispersion of perturbation on delay time and (b) the x–y phase portraits of the Rossler system in the post-transient regime for some values of the delay time. $K=0.2$.

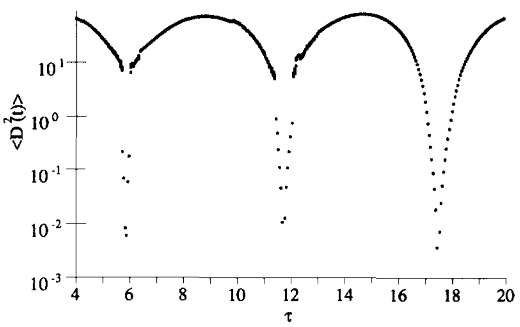

Fig. 8. The same as in the fig. 7a, but for the case of restricted delayed perturbation. $F_0=0.1$.

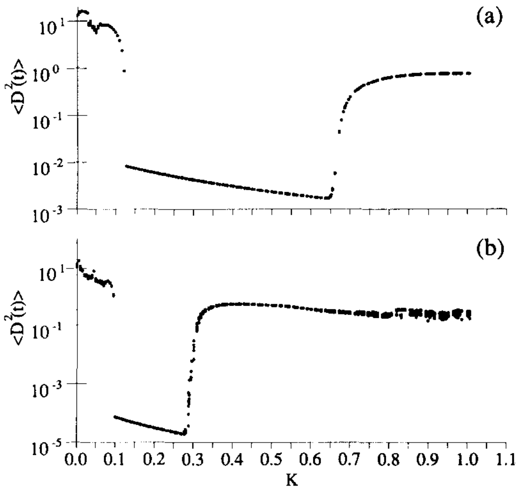

Fig. 9. Dependence of the dispersion of the perturbation on K for two values of the delay time τ, coinciding with the periods of the first two periodic orbits: (a) $\tau=5.9$, (b) $\tau=11.75$.

lution of the Rossler system, i.e. and unstable fixed point. Therefore, the method can stabilize unstable fixed points as well as UPO. When the delay time differs considerably from the period of the UPO, the output oscillations of the system can be chaotic (figs. 7b3, 7b5–7) of periodic (fig. 7b2). The periodic orbits obtained far from resonance (fig. 7b2) differ considerably from the UPOs. They correspond to new periodical solutions of the system caused by a large perturbation. The periods of these orbits differ from the delay time τ.

The problem of multistability arises for the Rossler system with delayed perturbation. As can be seen in fig. 7b, the Rossler system for large values of the delay time has two stable solutions depending on initial conditions. The phase portraits 7b5 and 7b6 as well as 7b7 and 7b8 have been obtained for the same values of the delay time but with different initial conditions. As has been mentioned in the previous section this problem can be avoided by restriction of the perturbation. The influence of restriction (4) on the results presented in fig. 7a can be seen from fig. 8. Due to the restriction the upper branch of points in proximity to the period-three resonance disappears. An asymptotical behaviour of the system becomes unambiguous for all values of K. The windows of K corresponding to the stabilization of the fixed point also disappear. This is because the trajectories of an unperturbed Rossler attractor do not reach the fixed point and to reach it a large perturbation is needed.

The dependences of the dispersion $\langle D^2(t)\rangle$ and the Lyapunov exponents λ on K for the two first periodic orbits are shown in figs. 9 and 10. In the case of delayed feedback each of the two orbits can be stabilized in a finite interval of K. These intervals are

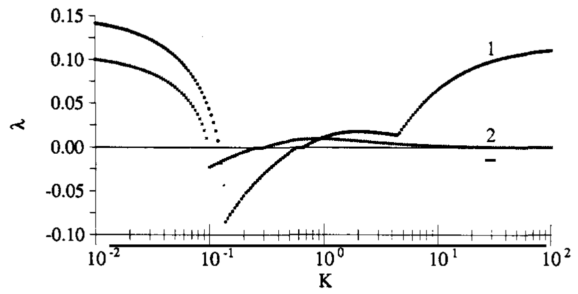

Fig. 10. Dependence of the Lyapunov exponents of the two first periodic orbits of the Rossler system on K in the case of delayed feedback: (1) $\tau=5.9$, (2) $\tau=11.75$.

much narrower than those obtained with an external force control. This means that the delayed feedback control is more sensitive to the fitting of the parameters. The external force control is more efficient since the perturbation always tends to attract the current trajectory to the desired periodic orbit, determined beforehand. The delayed feedback perturbation tends to decrease the distance between the current trajectory and the delayed trajectory which in the process of stabilization does not coincide exactly with the UPO.

4. Discussion and conclusions

Note that the perturbation in both forms (2) and (5) expands the dimension of the originally low-dimensional system. In the case of an external force control the perturbation increases the dimension by one, as any external periodical signal $y_i(t)$ can be presented by one additional ordinary differential equation. The delayed feedback perturbation increases the dimension to infinity. Therefore, one can conclude that the stabilization in both methods is achieved through additional degrees of freedom introduced in the system with the perturbation. The perturbation does not change the projections of the UPOs on an original low-dimensional phase space. The additional degrees of freedom change only the Lyapunov exponent of the UPOs, so that they become stable.

This can be illustrated with a simple analytical example. An unperturbed ($F_n=0$) one-dimensional logistic map

$$x_{n+1}=4x_n(1-x_n)+F_n \tag{6}$$

has the unstable fixed point $x_n=\frac{3}{4}$ with the eigenvalue $\lambda=-2$. The perturbation in the form of a delay $F_n=K(x_{n-1}-x_n)$ does not change the x-coordinate of this fixed point, but increases the dimension of the map to two. The analysis of this two-dimensional map shows that the absolute values of both eigenvalues of the fixed point are less than 1 in the interval of the parameter $K=[-1, -0.5]$. Therefore, for these values of K a "one-dimensional" unstable fixed point turns into a "two-dimensional" stable fixed point. A more detailed theory of this stabilization is in progress and will be reported elsewhere.

In conclusion, we have shown that the UPO of a chaotic system can be stabilized by a *small time-continuous* perturbation. The permanent control is noise resistant. The stabilization can be achieved by the use of a specially designed external periodic oscillator, or by the use of delayed self-controlling feedback without use of any external force. The multistability of the system under control can be avoided by restriction of the perturbation. An experimental realization of the second method is very simple and this method should be applicable to a wide variety of systems.

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