5. Übung zur Nichtlinearen Dynamik & Strukturbildung

Inhalt: 1. Besprechung des 4. Blattes

2. Komplexe Ginzburg-Landau-Gleichung
   ↳ Benjamin-Feir-Instabilität
3. FitzHugh-Nagumo-Modell
4. Schlögl-Modell

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1. Besprechung des 4. Blattes
   ↳ Lösungen Ausschnitten
   ↳ Code Beispiel

2. Komplexe Ginzburg-Landau Gleichung

1D: $$\frac{\partial}{\partial t} W(x,t) = W(x,t) - (1+ic_2)\,|W(x,t)|^2\, W(x,t) + (1+ic_1)\frac{\partial^2}{\partial x^2} W(x,t)$$

uniforme/homogene Oszillation: $Q=0$

aus allgemeinem Ansatz für ebene Wellen: $W(x,t) = a_Q \exp[i\,(\omega_Q t + Qx)]$

mit $|a_Q|^2 = 1 - Q^2$ und $\omega_Q = -c_2 + (c_2 - c_1)\,Q^2$ folgt:

$|a_0|^2 = 1$, $\omega_0 = -c_2$

$\Rightarrow$ Störung: $W \approx [1 + u(x,t)]\, e^{i\omega_0 t}$.

Einsetzen liefert in linearer Näherung:

$$\frac{\partial}{\partial t} u = -(1+ic_2)(u+u^*) + (1+ic_1)\frac{\partial^2}{\partial x^2} u$$

Weiter mit Fourier-Transformation:

$$u(x,t) = \int_{-\infty}^{\infty} dq\, u_q(t)\, e^{iqx}, \quad u^*(x,t) = \int_{-\infty}^{\infty} dk\, u_k^*(t)\, e^{-ikx}$$

$$\Rightarrow \frac{\partial u_q}{\partial t} = -(1+ic_2)\,(u_q + u_q^*) - (1+ic_1)\, q^2 w_q$$

und $$\frac{\partial u_q^*}{\partial t} = -(1-ic_2)\,(u_q + u_q^*) - (1-ic_1)\, q^2 w_q^*$$

2. FitzHugh-Nagumo Modell

$$\varepsilon \dot{u} = u - \frac{u^3}{3} - v$$

$$\dot{v} = u + a$$

Aktivator (schnell)

Inhibitor (langsam)

a: Bifurkationsparameter

ε: Zeitskalentrennung $\varepsilon \ll 1$

siehe Blatt 5

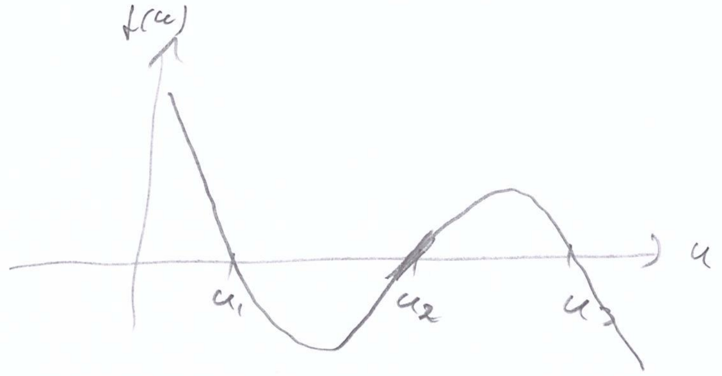
History of FHN

from Papers!

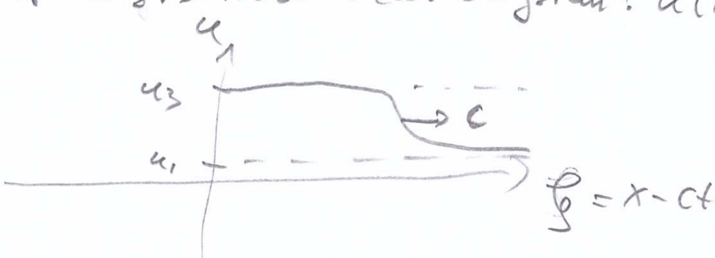
4. Schlögl-Modell:

Dynamik von $u(x,t)$ gegeben durch $\frac{\partial}{\partial t} u(x,t) = f(u) + D \frac{\partial^2}{\partial x^2} u(x,t)$

mit $f(u) = -k\,(u-u_1)(u-u_2)(u-u_3)$



Mit bewegtem Koordinatensystem: $u(x,t) = u(\underbrace{x - \overset{\text{Geschwindigkeit}}{c}t}_{=\xi}) = u(\xi)$



$$\frac{\partial}{\partial t} \underset{\uparrow}{\mapsto} \frac{\partial}{\partial \xi} = \frac{\partial}{\partial(-ct)} \Rightarrow \frac{\partial}{\partial t} \mapsto -c\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial \xi}$$

$\frac{\partial \xi}{\partial t} = -c$

$$\Rightarrow \frac{\partial}{\partial t} u(x,t) = f(u) + D\frac{\partial^2}{\partial x^2} u(x,t)$$

$$-cu'(\xi) = f(u(\xi)) + D\frac{\partial^2}{\partial \xi^2} u(\xi)$$

$$u(\xi \to \pm\infty) = \begin{cases} u_1 \\ u_3 \end{cases} \quad \text{und} \quad \left.\frac{du}{d\xi}\right|_{\pm\infty} = 0$$

$$\Rightarrow \frac{du}{d\xi} = \beta\,(u-u_1)(u-u_3)$$

$$\frac{d^2u}{d\xi^2} = \frac{d}{d\xi}\left(\beta(u-u_1)(u-u_3)\right) = \beta\frac{d}{d\xi}\left(u^2 - u(u_1+u_3) + u_1u_3\right)$$

$$= \beta\left[2uu' - u'(u_1+u_3)\right]$$

$$= 2\beta\, u\,\underbrace{\left(\beta(u-u_1)(u-u_3)\right)} - \beta\underbrace{\left(\beta(u-u_1)(u-u_3)\right)}(u_1+u_3)$$

$$\Rightarrow \quad = (u-u_1)(u-u_3)\left[2\beta^2 u - \beta^2(u_1+u_3)\right]$$

$$\Rightarrow 0 = Cu' + Du'' + f(u)$$

$$= (u-u_1)(u-u_3)\left[C\beta + D2\beta^2 u - D\beta^2(u_1+u_3) - k(u-u_2)\right]$$

$$= (u-u_1)(u-u_3)\left[C\beta - \beta^2 D(u_1+u_3) - ku_2 + u\left(D2\beta^2 - k\right)\right]$$

$u = u_1$ und $u = u_3$ Nullstellen !

3. Nullstelle: $[\ldots] = 0$

$$\Rightarrow \beta = \sqrt{\frac{k}{2D}} \quad \Rightarrow \quad C\sqrt{\frac{k}{2D}} - \frac{k}{2D}D(u_1+u_3) - k\,u_2 = 0$$

$$\Rightarrow C = \sqrt{\frac{Dk}{2}}\left(u_1 + u_3 - 2u_2\right)$$

$$u(\xi) = \frac{1}{2}\left[u_1 + u_3 + (u_1 - u_3)\tanh\left(\frac{1}{2}\sqrt{\frac{k}{2D}}\,(u_3 - u_1)\,\xi\right)\right]$$