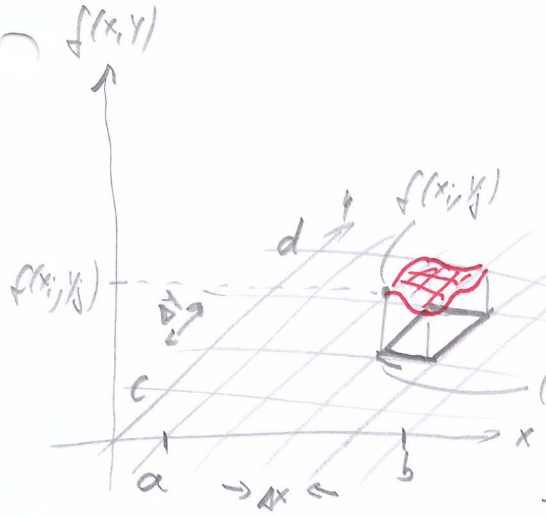
2.8 Integration (of scalar functions) in $\mathbb{R}^n$

Def.: The 2-dimensional integral of $f: \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ (continuous) over $\Omega = [a,b] \times [c,d]$ is defined as:

$$\int_\Omega f(x,y)\,dx\,dy := \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \sum_{j=0}^{\frac{d-c}{\Delta y}-1} \sum_{i=0}^{\frac{b-a}{\Delta x}-1} f(x_i, y_j)\,\Delta x\,\Delta y .$$

$$\left(\begin{array}{llll} \text{with} & x_i = a + i\,\Delta x & \text{and} & N_x = \frac{b-a}{\Delta x} \\ & y_j = c + j\,\Delta y & & N_y = \frac{d-c}{\Delta y} \end{array} \right)$$



$$= \lim_{N_y \to \infty} \lim_{N_x \to \infty} \sum_{j=0}^{N_y-1} \sum_{i=0}^{N_x-1} f(x_i, y_j)\,\Delta x\,\Delta y$$

volume of cuboid: $f(x_i, y_j)\,\Delta x\,\Delta y$

$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Fubini's theorem:

$$\int_\Omega f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

(4) Verallgemeinerung auf $\mathbb{R}^n$:

$f: \Omega \subset \mathbb{R}^n \to \mathbb{R}$, $f = f(x_1, \dots, x_n)$, $\Omega = I_1 \times I_2 \times \dots \times I_n$

stetig in Ω

$$\Rightarrow \int_\Omega f(x_1, \dots, x_n)\, dx_1 \dots dx_n \equiv \int_\Omega f(x_1, \dots, x_n)\, d^n x$$

$$= \int_{I_1} dx_1 \int_{I_2} dx_2 \dots \int_{I_n} dx_n\, f(x_1, \dots, x_n)$$

Bsp./Notation:

(i) Flächenintegrale: $\int_\Omega f(x,y)\, dx\,dy \equiv \int_\Omega f(x,y) \underbrace{dA}_{\text{Flächenelement}}$

(ii) Volumenintegrale: $\int_\Omega f(x,y,z)\, dx\,dy\,dz \equiv \int_\Omega f(\underline{x})\, d^3x = \int_\Omega f(x_1, x_2, x_3)\, d^3x$

$$\equiv \int_\Omega f(x,y,z) \underbrace{dV}_{\text{Volumenelement}}$$

(iii) Dichte: diskret $\rho = \dfrac{\Delta m}{\Delta V}$

↳ Gesamtmasse: $M = \int_{\text{Volumen}} \rho\, dV$

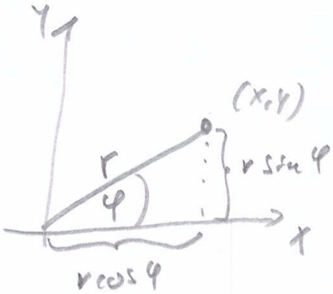
(iv) Volumen einer Teilmenge des $\mathbb{R}^3$: $\Omega \subset \mathbb{R}^3$

$$V = \int_\Omega 1\, d^3x = \int_\Omega dV$$

2.9 Variablen-/Koordinatentransformationen

Idee: Koordinaten, die an die Symmetrie des Systems/Problems angepasst sind. ⇒ Rechnungen vereinfachen

2.9.1 Polarkoordinaten im $\mathbb{R}^2$



$x = x(r,\varphi) = r\cos\varphi \qquad$ mit $r \in [0,\infty)$, $\varphi \in [0,2\pi)$

$y = y(r,\varphi) = r\sin\varphi$

⇒ totale Differenziale: $dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \varphi}d\varphi$

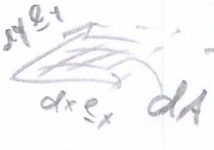
$= \cos\varphi\, dr - r\sin\varphi\, d\varphi$

$dy = \sin\varphi\, dr + r\cos\varphi\, d\varphi$

Zusammengefasst als Jacobi-Matrix $\underline{\underline{J}} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$

Flächenelement: $dA = dx\,dy = \det \underline{\underline{J}}\, dr\, d\varphi = \underbrace{\cos\varphi\, r\cos\varphi - (-r\sin\varphi)\sin\varphi}\, dr\,d\varphi$

Achtung: nicht Produkt aus dx & dy, sondern $(dx\, e_x \times dy\, e_y)$



$= r(\cos^2\varphi + \sin^2\varphi)\, dr\, d\varphi$

$= r\, dr\, d\varphi$

$$\Rightarrow \int_\Omega f(x,y)\,dA = \int_{(x,y)\in\Omega} f(x,y)\,dx\,dy = \int_{(r,\varphi)\in\Omega} f(x(r,\varphi), y(r,\varphi))\, r\, dr\, d\varphi$$

Bsp.: (i) Fläche eines Kreises $\{(r,\varphi) \mid r<R,\ 0\le\varphi<2\pi\}$

$$A = \int_{\text{Kreis}} dA = \int_{(r,\varphi)\in\text{Kreis}} r\,dr\,d\varphi = \int_0^R dr\, r \int_0^{2\pi} d\varphi = \int_0^R dr\, r\, 2\pi = \frac{1}{2}\pi R^2 2\pi$$

$$= \pi R^2$$

(ii) Gauß-Integral: $I = \int_{-\infty}^{\infty} dx \, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$ (s. ÜB 3)

Trick: Berechne $I^2 = \left(\int_{-\infty}^{\infty} dx \, e^{-ax^2}\right)\left(\int_{-\infty}^{\infty} dy \, e^{-ay^2}\right)$

$$= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, e^{-a(x^2+y^2)}$$ (Integral über (x,y)-Ebene)

$x = r\cos\varphi$
$y = r\sin\varphi$

Kettenregel!

$$\to = \int_0^{2\pi} d\varphi \int_0^{\infty} dr \, r \, e^{-ar^2}$$

$$= \int_0^{2\pi} d\varphi \left(-\frac{1}{2a} e^{-ar^2}\right)\Big|_{r=0}^{\infty}$$

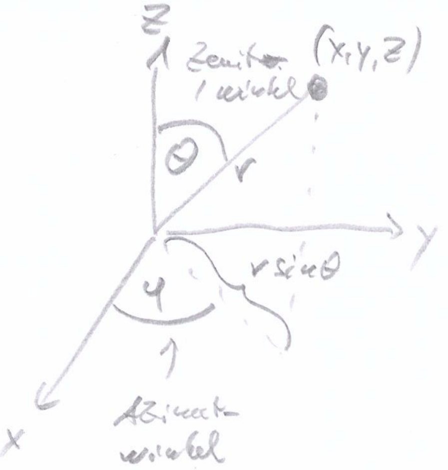
$$= \frac{1}{2a} \int_0^{2\pi} d\varphi$$

$$= \frac{1}{2a} 2\pi = \frac{\pi}{a}$$

$\Rightarrow I = \sqrt{\frac{\pi}{a}}$

normierte Gauß-Funktion: $\int_{-\infty}^{\infty} \sqrt{\frac{a}{\pi}} \, e^{-ax^2} dx = 1$

2.3.2 Kugelkoordinaten



$$x = x(r,\Theta,\varphi) = r\sin\Theta\cos\varphi$$

$$y = y(r,\Theta,\varphi) = r\sin\Theta\sin\varphi$$

$$z = z(r,\Theta,\varphi) = r\cos\Theta$$

mit $r \in [0,\infty)$, $\Theta \in [0,\pi]$, $\varphi \in [0,2\pi)$

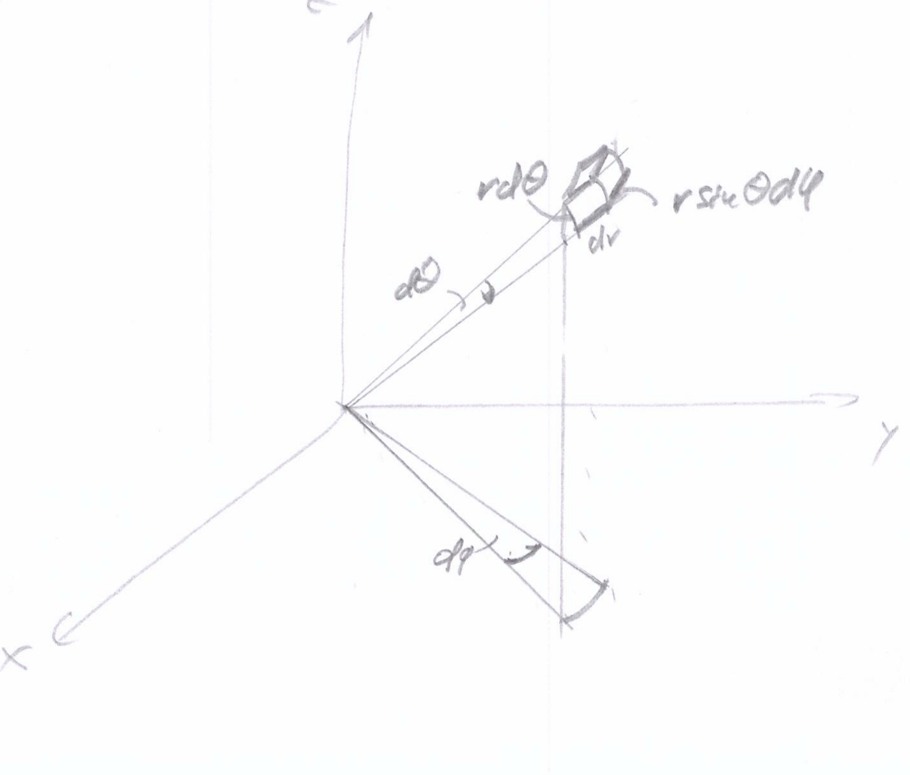
Jacobi-Matrix: $\underline{\underline{J}} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \Theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \Theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial r} \end{pmatrix} = \begin{pmatrix} \sin\Theta\cos\varphi & r\cos\Theta\cos\varphi & -r\sin\Theta\sin\varphi \\ \sin\Theta\cos\varphi & r\cos\Theta\sin\varphi & r\sin\Theta\cos\varphi \\ \cos\Theta & -r\sin\Theta & 0 \end{pmatrix}$

$\Rightarrow \det \underline{\underline{J}} = r^2\sin\Theta \Rightarrow dV = dx\,dy\,dz = r^2\sin\Theta\,dr\,d\Theta\,d\varphi$

später

Volumenelement

$$\int_\Omega f(x,y,z)\,dV = \int_{(x,y,z)\in\Omega} f(x,y,z)\,dx\,dy\,dz = \int_{(r,\Theta,\varphi)\in\Omega} f\big(x(r,\Theta,\varphi), y(r,\Theta,\varphi), z(r,\Theta,\varphi)\big)\,r^2\sin\Theta\,dr\,d\Theta\,d\varphi$$



Bsp.: (i) Volumen einer Kugel: $\{(r,\theta,\varphi) \mid r\leq R,\ 0\leq\theta\leq\pi,\ 0\leq\varphi<2\pi\}$

$$V = \int_{\text{Kugel}} dV = \int_{(r,\theta,\varphi)\in\text{Kugel}} r^2 \sin\varphi \, dr\, d\theta\, d\varphi$$

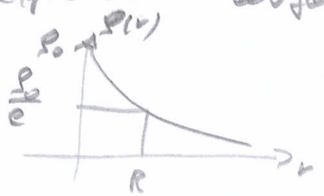
$$= \int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\varphi\, r^2 \sin\theta$$

$$= \underbrace{\int_0^R dr\, r^2}_{\frac{1}{3}R^3} \underbrace{\int_0^\pi d\theta \sin\theta}_{-\cos\theta\big|_0^\pi = +2} \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}$$

$$= \frac{4\pi}{3} R^3$$

(ii) Gesamtmasse einer exponentiell abfallenden Dichte:

$$\rho(r) = \rho_0\, e^{-\frac{r}{R}}$$



$$M = \int_{\text{ganzen Raum}} \rho(r)\, dV$$

$$= \underbrace{\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi}_{\int d\Omega = 4\pi} \underbrace{\int_0^\infty dr\, r^2 \rho_0 e^{-\frac{r}{R}}}_{= \rho_0 R^3 \int_0^\infty du\, u^2 e^{-u}}$$

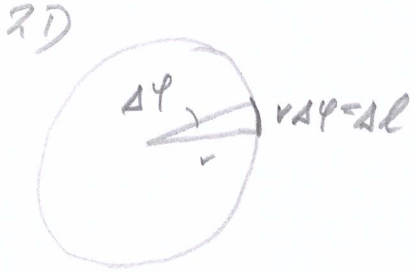
$\int d\Omega$: Raumwinkel

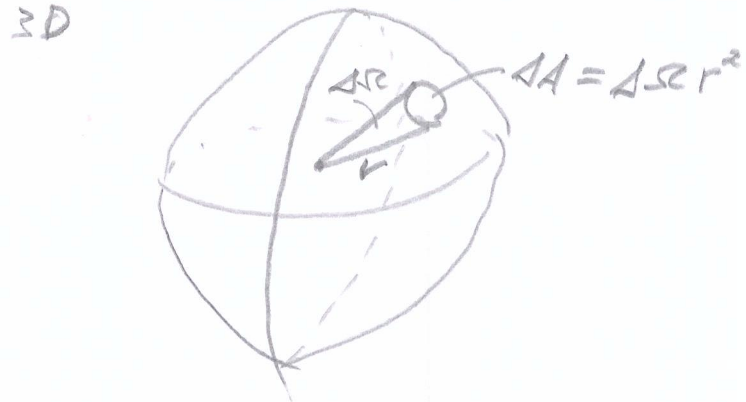
$u = \frac{r}{R}$, $R\,du = dr$

$\int_0^\infty du\, u^2 e^{-u} = 2!$ (vgl. 1.10)

$$= 8\pi \rho_0 R^3$$

Raumwinkel:





gesamter Raumwinkel:

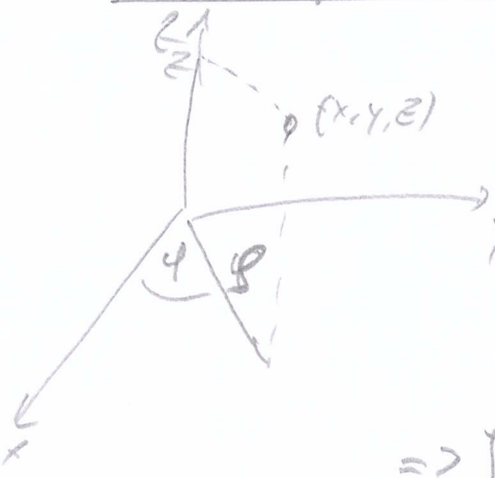
$$\Omega = \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi = 4\pi = \frac{\text{Kugeloberfläche}}{r^2} = \frac{4\pi r^2}{r^2}$$

$$\Rightarrow dV = dA\,dr = r^2\,dr\,d\Omega \underset{\substack{\uparrow \\ d\Omega = \sin\theta\, d\theta\, d\varphi}}{=} r^2 \sin\theta\, dr\, d\theta\, d\varphi$$

$$\int f(r)\, dV = \int d\Omega \int dr\, f(r) = 4\pi \int dr\, f(r)$$

unabhängig von Winkeln θ, φ

2.9.3 Zylinderkoordinaten



$$x = x(\varrho, \varphi) = \varrho \cos\varphi$$

$$y = y(\varrho, \varphi) = \varrho \sin\varphi$$

$$z = z$$

$$\Rightarrow \underline{\underline{J}} = \begin{pmatrix} \cos\varphi & -\varrho\sin\varphi & 0 \\ \sin\varphi & \varrho\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \det \underline{\underline{J}} = \varrho(\cos^2\varphi + \sin^2\varphi) = \varrho$$

$\rightarrow$ Volumenelement: $dV = dx\,dy\,dz = \varrho\, d\varrho\, d\varphi\, dz$

Bsp.: Zylindervolumen:

$$V = \int_{\text{Zylinder}} dV = \int_{(\varrho,\varphi,z)\in\text{Zylinder}} \varrho\, d\varrho\, d\varphi\, dz = \int_0^R d\varrho\, \varrho \int_0^{2\pi} d\varphi \int_0^{\overset{\text{Höhe}}{\downarrow}H} dz$$

$$= \frac{1}{2}R^2 \quad 2\pi \quad H$$

$$= \pi R^2 H$$