4. Vector calculus / analysis

4.1 (Conservative) vector fields

- Vectorfield: $\underline{v} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n,\ \underline{x} \mapsto \underline{v}(\underline{x}) = \begin{pmatrix} v_1(\underline{x}) \\ \vdots \\ v_n(\underline{x}) \end{pmatrix}$

  $\uparrow$ potentially $\dim \Omega < n$

- C^s-vectorfield: $v_1, \dots, v_n$ s-times continuously differentiable.

- Conservative vectorfield: $\underline{v}$ is gradient of scalar potential ϕ

$$\underline{v}(\underline{x}) = \underline{\nabla} \phi(\underline{x})$$

4.2 Derivations of vectorfields

- (Rotation) curl of $\underline{v}(\underline{x}) = \begin{pmatrix} v_1(\underline{x}) \\ v_2(\underline{x}) \\ v_3(\underline{x}) \end{pmatrix}$:

$$\operatorname{rot} \underline{v} \equiv \underline{\nabla} \times \underline{v} := \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix}$$

$$(\operatorname{rot} \underline{v})_i = \sum_{jk=1}^{3} \varepsilon_{ijk} \partial_j v_k$$

- Divergence of $\underline{v}(\underline{x})$: $\operatorname{div} \underline{v}(\underline{x}) = \underline{\nabla} \cdot \underline{v} := \frac{\partial}{\partial x_1} v_1 + \frac{\partial}{\partial x_2} v_2 + \frac{\partial}{\partial x_3} v_3$

(iii) Def: Der Laplace-Operator ist definiert als: $\Delta := \mathrm{div}\,\mathrm{grad}$

Also: $\Delta\phi(x_1,x_2,x_3) = \underline{\nabla}\cdot\left(\underline{\nabla}\phi\right) = (\partial_{x_1},\partial_{x_2},\partial_{x_3})\cdot\begin{pmatrix}\partial_{x_1}\phi\\ \partial_{x_2}\phi\\ \partial_{x_3}\phi\end{pmatrix}$

$$= \frac{\partial^2}{\partial x_1^2}\phi + \frac{\partial^2}{\partial x_2^2}\phi + \frac{\partial^2}{\partial x_3^2}\phi$$

↳ Laplace-Gleichung: $\Delta\phi = 0$ (wichtig bei Transportprozesse, Potenzial theorie...)

Achtung: $\Delta\underline{A} = \frac{\partial^2\underline{A}}{\partial x^2} + \frac{\partial^2\underline{A}}{\partial y^2} + \frac{\partial^2\underline{A}}{\partial z^2} = \mathrm{grad}\,\mathrm{div}\underline{A} - \mathrm{rot}\,\mathrm{rot}\underline{A} = \underline{\nabla}(\underline{\nabla}\cdot\underline{A}) - \underline{\nabla}\times(\underline{\nabla}\times\underline{A})$

Bem: • div, rot, grad sind linear: $\mathrm{rot}(\underline{u}+\underline{v}) = \mathrm{rot}\,\underline{u} + \mathrm{rot}\,\underline{v}$

$\mathrm{div}(\underline{u}+\underline{v}) = \mathrm{div}\,\underline{u} + \mathrm{div}\,\underline{v}$

$\mathrm{grad}(\phi+\psi) = \mathrm{grad}\,\phi + \mathrm{grad}\,\psi$

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• Produktregel: $\mathrm{div}(\phi\,\underline{v}) = \phi\,\mathrm{div}\,\underline{v} + (\mathrm{grad}\,\phi)\cdot\underline{v}$

$\mathrm{rot}(\phi\underline{A}) = \underline{\nabla}\phi\times\underline{A} + \phi(\underline{\nabla}\times\underline{A}) = \mathrm{grad}\phi\times\underline{A} + \phi\,\mathrm{rot}\underline{A}$

4.3 Gradienten- & Wirbelfelder

Satz: Gradientenfelder $\underline{v} = \underline{\nabla}\phi$ sind wirbelfrei.

Beweis: $\mathrm{rot}\,\underline{v} = \underline{\nabla}\times\underline{v} = \underline{\nabla}\times(\underline{\nabla}\phi)$

i-te Komponente: $[\underline{\nabla}\times\underline{\nabla}\phi]_i = \varepsilon_{ijk}\,\partial_j(\underline{\nabla}\phi)_k$

$$= \underbrace{\varepsilon_{ijk}}_{\text{antisymm in } j,k}\ \underbrace{\partial_j\partial_k}_{\text{symm in } j,k}\phi = 0$$

Also: (i) $\mathrm{rot}\,\mathrm{grad} = \underline{\nabla}\times\underline{\nabla} = 0$

(ii) Umkehrung gilt auch: Jedes wirbelfreie Feld $\underline{v}$ ($\mathrm{rot}\underline{v}=0$) hat ein Potenzial ϕ ($\underline{v} = \underline{\nabla}\phi$)

$\Rightarrow$ $\underline{v}$ ist konservativ $\Leftrightarrow$ $\mathrm{rot}\,\underline{v} = 0$

Bsp.: $\underline{A}(\underline{x}) = \begin{pmatrix} -y^3 z \\ x e^z \\ e^{2y} \end{pmatrix} \Rightarrow$ Berechne $\Delta \underline{A}$ (Laplace-Operator)

$$\Rightarrow \Delta \underline{A} = \frac{\partial}{\partial x}\begin{pmatrix} 0 \\ e^z \\ 0 \end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix} -3y^2 z \\ 0 \\ 2e^{2y} \end{pmatrix} + \frac{\partial}{\partial z}\begin{pmatrix} -y^3 \\ x e^z \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -6yz \\ 0 \\ 4e^{2y} \end{pmatrix} + \begin{pmatrix} 0 \\ xe^z \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -6yz \\ xe^z \\ 4e^{2y} \end{pmatrix}$$

$$\underline{\nabla}(\underline{\nabla} \cdot \underline{A}) = \underline{\nabla}\left(\frac{\partial}{\partial x}A_1 + \frac{\partial}{\partial y}A_2 + \frac{\partial}{\partial z}A_3\right) =$$

$$= \underline{\nabla}\left(\frac{\partial}{\partial x}(-y^3 z) + \frac{\partial}{\partial y}(xe^z) + \frac{\partial}{\partial z}(e^{2y})\right)$$

$$= \underline{\nabla}(0) = \underline{0}$$

$$-\underline{\nabla} \times (\underline{\nabla} \times \underline{A}) = \underline{\nabla} \times \begin{pmatrix} 2e^{2y} - xe^z \\ -y^3 - 0 \\ e^z + 3y^2 z \end{pmatrix} = \underline{\nabla} \times \begin{pmatrix} 2e^{2y} - xe^z \\ -y^3 \\ e^z + 3y^2 z \end{pmatrix}$$

$$= -\begin{pmatrix} 6yz - 0 \\ -xe^z - 0 \\ 0 - 4e^{2y} \end{pmatrix} = \begin{pmatrix} -6yz \\ xe^z \\ 4e^{2y} \end{pmatrix}$$

$$\Rightarrow \Delta \underline{A} = \text{grad div} \underline{A} - \text{rot rot} \underline{A}$$

Satz: Wirbelfelder $\underline{v} = \text{rot}\,\underline{A}$ sind quellenfrei.

Beweis: $\text{div}\,\underline{v} = \text{div}(\text{rot}\,\underline{A}) = \underline{\nabla} \cdot \underline{\nabla}\underline{A} = \sum_i \partial_i (\underline{\nabla} \times \underline{A})_i$

$$= \sum_{ijk} \partial_i \, \varepsilon_{ijk} \partial_j A_k$$

$$= 0$$

Also: (i) div rot = 0

(ii) Umkehrung gilt auch: Jedes quellenfreie Feld hat ein Vektorpotential.

z.B: $\underline{B} = \underline{\nabla} \times \underline{A}$

Anwendung: Maxwell-Gleichungen (homogene)

(a) $\underline{\nabla} \cdot \underline{B} = 0$ (b) $\underline{\nabla} \times \underline{E} + \frac{\partial \underline{B}}{\partial t} = 0$

mit $\underline{B} = \underline{\nabla} \times \underline{A}$: $\underline{\nabla} \times \underline{E} + \frac{\partial}{\partial t} \underline{\nabla} \times \underline{A} = 0$

Satz v. Schwarz: $\underline{\nabla} \times \left(\underline{E} + \frac{\partial \underline{A}}{\partial t} \right) = 0$

wirbelfrei: $\underline{E} + \frac{\partial \underline{A}}{\partial t} = -\underline{\nabla} \phi$ Konvention

$\Rightarrow$ $\underline{B} = \underline{\nabla} \times \underline{A}$

$\underline{E} = -\underline{\nabla}\phi - \frac{\partial \underline{A}}{\partial t}$ ($\phi, \underline{A}$ nicht eindeutig → Eichtransformationen)