6.8 Systems of linear equations (Gaussian elimination)

Solve $\underline{\underline{A}}\,\underline{x} = \underline{b}$ by transforming $\underline{\underline{A}}$ to row echelon form

$$\left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right) \longrightarrow \dots \longrightarrow \left(\begin{array}{ccc|c} 1 & & * & \tilde{b}_1 \\ 0 & \ddots & & \vdots \\ 0 & \dots\, 0 & 1 & \tilde{b}_m \end{array}\right)$$

Does a solution $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ exist for $m = n$?

(i) $\operatorname{rank} \underline{\underline{A}} = n$: $\operatorname{image}(\underline{\underline{A}}) = \mathbb{K}^n \Rightarrow$ unique solution exists.

(ii) $\operatorname{rank} \underline{\underline{A}} < n$: solution exists (but is not unique) if $\underline{b} \in \operatorname{image}(\underline{\underline{A}})$

— " — does not exist if $\underline{b} \notin \operatorname{image}(\underline{\underline{A}})$

6.9 Inverse matrix

Solution of $\underline{\underline{A}}\,\underline{x} = \underline{b}$ given by $\underline{x} = \underline{\underline{A}}^{-1}\underline{b}$ with $\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \mathbb{1}$

Def.: A square matrix $\underline{\underline{A}} \in \mathbb{K}^{n\times n}$ is called invertible if a matrix $\underline{\underline{A}}^{-1}$ exists with $\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \mathbb{1}$.

condition: $\operatorname{rank} \underline{\underline{A}} = n \Leftrightarrow \det \underline{\underline{A}} \neq 0$

- $\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \underline{\underline{A}}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ Cramer's rule (if $ad-bc \neq 0$)
- Find inverse $\underline{\underline{A}}^{-1}$ by transforming

$$(\underline{\underline{A}} \,|\, \mathbb{1}) \longrightarrow (\mathbb{1} \,|\, \underline{\underline{A}}^{-1})$$

6.10 Determinant

Def: $\underline{\underline{A}} = \{a_{ij}\}_{i,j=1,\dots,n} \in \mathbb{K}^{n\times n}$

We define: for $n=1$: $\det \underline{\underline{A}} = a_{11}$

remove ith row, j-th column

$$\text{for } n \geq 2 : \det \underline{\underline{A}} = \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det(\underline{\underline{A}}_{i1})$$

$$= \sum_{s=1}^{n} (-1)^{r+s} a_{rs} \det(\underline{\underline{A}}_{rs}) \quad \in \mathbb{K}^{(n-1)\times(n-1)}$$

$$= \sum_{r=1}^{n} (-1)^{r+s} a_{rs} \det(\underline{\underline{A}}_{rs})$$

- Sarrus rule : $\underline{\underline{A}} \in \mathbb{K}^{3\times 3}$

$$\det \underline{\underline{A}} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

$-\ +\ -\ +\ -\ +$

- $\det \underline{\underline{A}}^T = \det \underline{\underline{A}}$
- $\det \underline{\underline{A}}\,\underline{\underline{B}} = \det \underline{\underline{A}} \det \underline{\underline{B}}$
- $\det \underline{\underline{A}}^{-1} = \dfrac{1}{\det \underline{\underline{A}}}$
- $\det \underline{\underline{A}} = 0 \iff \operatorname{rank} \underline{\underline{A}} < n$ (singular)
- $\det \underline{\underline{A}} \neq 0 \iff \operatorname{rank} \underline{\underline{A}} = n$ (regular)

ex. $\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 0$ columns linearly dependent

$$2\begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

- $\det \underline{\underline{A}} = \begin{pmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{pmatrix} = \underline{b} \cdot (\underline{c} \times \underline{d})$ (triple product)

6.11 Basiswechsel und Koordinatentransformation

Betrachte $\mathbb{K}$-Vektorraum V mit Basen $B=\{\underline{b}_1,\dots,\underline{b}_n\}$, $C=\{\underline{c}_1,\dots,\underline{c}_n\}$

$$\Rightarrow \underline{v} = \sum_{i=1}^{n} \beta_i \underline{b}_i = \sum_{i=1}^{n} \gamma_i \underline{c}_i$$

derselbe Vektor in 2 Koordinatensystemen: $\underline{v}_B = \begin{pmatrix}\beta_1\\ \vdots \\ \beta_n\end{pmatrix}_B$, $\underline{v}_C = \begin{pmatrix}\gamma_1\\ \vdots \\ \gamma_n\end{pmatrix}$

Transformationsmatrix: $\underline{v}_C = \underbrace{{}_C\underline{\underline{M}}_B(\mathrm{id}_V)}_{\text{darstellende Matrix der Identitätsabbildung}} \underline{v}_B =: \underline{\underline{S}}^{-1} \underline{v}_B$

$$\begin{tikzcd}
V \arrow[r, "\mathrm{id}_V"] \arrow[d, "k_B"'] & V \arrow[d, "k_C"] \\
\underline{v}_B \in \mathbb{K}^n \arrow[r, "{}_C\underline{\underline{M}}_B(\mathrm{id}_V)"'] & \underline{v}_C \in \mathbb{K}^n
\end{tikzcd}$$

$${}_C\underline{\underline{M}}_B(\mathrm{id}_V) = \underbrace{\left((\underline{b}_1)_C, \dots, (\underline{b}_n)_C\right)}_{\substack{\text{Bilder der Basisvektoren von } B \\ \text{bzgl. Basis } C}} \in \mathbb{K}^{n\times n}$$

Neu: $L: V \to V$ lineare Abbildung (statt Identität id_V)

$\underline{u} \mapsto \underline{v}$

$$\begin{tikzcd}
V \arrow[r, "L"] \arrow[d, "k_B"'] & V \arrow[r, "\mathrm{id}_V"] \arrow[d, "k_B"] & V \arrow[d, "k_C"] \\
\underline{u}_B \in \mathbb{K}^n \arrow[r, "{}_B\underline{\underline{L}}_B"'] & \underline{v}_B \in \mathbb{K}^n \arrow[r, "{}_C\underline{\underline{M}}_B(\mathrm{id}_V) \,\hat{=}\, \underline{\underline{S}}^{-1}"'] & \underline{v}_C \in \mathbb{K}^n
\end{tikzcd}$$

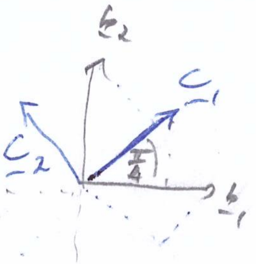
$$\underbrace{\underline{v}_B \overset{!}{=} {}_B\underline{\underline{L}}_B\, \underline{u}_B}$$

$$= \underline{\underline{S}}^{-1} \underline{v}_B = \underline{\underline{S}}^{-1}\, {}_B\underline{\underline{L}}_B\, \underline{u}_B \Leftrightarrow \underline{v}_C = \overbrace{\underline{\underline{S}}^{-1}\, {}_B\underline{\underline{L}}_B}^{{}_C\underline{\underline{L}}_B}\, \underline{u}_B$$

$$\underbrace{\underline{\underline{S}}^{-1}\underline{v}_B}_{\underline{v}_C} = \underbrace{\underline{\underline{S}}^{-1}\,{}_B\underline{\underline{L}}_B\,\underline{\underline{S}}}_{{}_C\underline{\underline{L}}_C}\,\overbrace{\underline{\underline{S}}^{-1}\underline{u}_B}^{\mathbb{1}}$$

$$\underline{v}_C = {}_C\underline{\underline{L}}_C\,\underline{u}_C$$

Bsp.: $\mathbb{R}^2$ mit $B=\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ und $C=\{\underline{c}_1,\underline{c}_2\}$ um $\frac{\pi}{4}$ gedreht



$$\Rightarrow (\underline{c}_1)_B = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\Big[\underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{\underline{b}_1}+\underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{\underline{b}_2}\Big],\quad (\underline{c}_2)_B = \frac{1}{\sqrt{2}}\Big[\underbrace{\begin{pmatrix}-1\\0\end{pmatrix}}_{-\underline{b}_1}+\underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{\underline{b}_2}\Big]$$

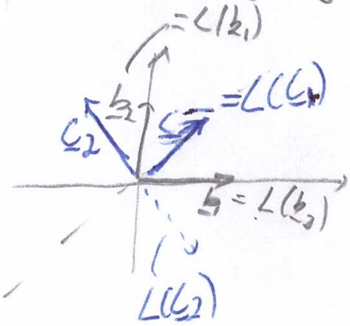
$$\Rightarrow {}_C\underline{\underline{M}}_B = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} =: \underline{\underline{S}}^{-1}$$

Basistransformationsmatrizen

$$(\underline{b}_1)_C = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix},\quad (\underline{b}_2)_C = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}_C \Rightarrow {}_B\underline{\underline{M}}_C = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$$

$$\left[\ \underline{\underline{S}}^{-1}\underline{\underline{S}} = \frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\ \right]$$

L: Spiegelung an Diagonalen $y=x$



$$\left.\begin{aligned} L(\underline{b}_1) &= \underline{b}_2\\ L(\underline{b}_2) &= \underline{b}_1\end{aligned}\right\}\quad {}_B\underline{\underline{L}}_B = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$\begin{aligned}
{}_C\underline{\underline{L}}_C &= \underline{\underline{S}}^{-1}\,{}_B\underline{\underline{L}}_B\,\underline{\underline{S}}\\
&= \frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\\
&= \frac{1}{2}\begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix} = \underbrace{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}_{\text{schön, einfach}}
\end{aligned}$$

$\leftarrow \underline{c}_1$ auf sich selbst ✓

$\leftarrow \underline{c}_2$ auf $-\underline{c}_2$ ✓

$$L : V \to W$$
$$\underline{v} \mapsto \underline{w}$$

$$\begin{tikzcd}
V \arrow[r, dash] \arrow[d, "K_A"] & V \arrow[r, "L"] \arrow[d, "K_B"] & W \arrow[r, "id_W"] \arrow[d, "K_C"] & W \arrow[d, "K_D"] \\
\underline{v}_A \in \mathbb{K}^n & \underline{v}_B \in \mathbb{K}^n \arrow[l, "\underline{\underline{T}}^{-1}"] \arrow[r, "\underset{C=B}{L}"'] & \underline{w}_C \in \mathbb{K}^m \arrow[r, "\underline{\underline{S}}^{-1}"'] & \underline{w}_D \in \mathbb{K}^m
\end{tikzcd}$$

$\underline{v}_B = \underline{\underline{T}}^{-1} \underline{v}_A$

$\underline{v}_A = \underline{\underline{T}}\, \underline{v}_B$

$$\underline{w} = L(\underline{v})$$

$$\Leftrightarrow \underline{w}_C = \underset{C=B}{L}\, \underline{v}_B$$

$$\underline{w}_D = \underline{\underline{S}}^{-1} \underline{w}_C$$

$$\Leftrightarrow \underline{\underline{S}}^{-1} \underline{w}_C = \underline{\underline{S}}^{-1} \underset{C=B}{L}\, \underline{v}_B \qquad \left(\underline{w}_D = \underset{D=B}{L}\, \underline{v}_B \right)$$

$$\Leftrightarrow \underline{w}_D = \underbrace{\underline{\underline{S}}^{-1} \underset{C=B}{L}\, \underline{\underline{T}}^{-1}}\, \underbrace{\underline{\underline{T}}\, \underline{v}_B}$$

Transformiere $B \to A$

$$\Leftrightarrow \underline{w}_D = \underset{D=A}{L}\, \underline{v}_A$$

Berücksichtige Koordinatenänderung $B \to A$ bei L

darstellende Matrix von L bzgl Basen A und D

6.12 Eigenwertproblem

Frage/Idee: Finde Basis, für die eine gegebene Matrix Diagonalform bekommt

$$ {}_B\underline{\underline{L}}_B = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \Rightarrow {}_B\underline{\underline{L}}_B \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} \lambda_1 v_1 \\ \vdots \\ \lambda_n v_n \end{pmatrix} $$

Def: Sei $\underline{\underline{A}} \in \mathbb{K}^{n\times n}$. $\lambda \in \mathbb{K}$ heißt Eigenwert von $\underline{\underline{A}}$, wenn

es Vektor $\underline{v} \in \mathbb{K}^n \setminus \{\underline{0}\}$ gibt mit $\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}$.

$\underline{v}$ heißt Eigenvektor zum Eigenwert λ.

Der Untervektorraum $Eig_{\underline{\underline{A}}}(\lambda) = \{\underline{v} \in \mathbb{K}^n \mid \underline{\underline{A}}\,\underline{v} = \lambda \underline{v}\}$

heißt Eigenraum zum Eigenwert λ.

Berechnung: Suche λ, $\underline{v} \neq \underline{0}$ mit $\underline{\underline{A}}\,\underline{v} - \lambda \underline{\underline{\mathbb{1}}}_n \underline{v} = \underline{0}$

$\Leftrightarrow (\underline{\underline{A}} - \lambda \underline{\underline{\mathbb{1}}}_n)\underline{v} = \underline{0}$ lineares Gleichungssystem!

$\Rightarrow$ nichttriviale Lösung ($\underline{v} \neq \underline{0}$), wenn $\det(\underline{\underline{A}} - \lambda \underline{\underline{\mathbb{1}}}_n) = 0$

Def: $\chi_{\underline{\underline{A}}}(x) = \det(\underline{\underline{A}} - x \underline{\underline{\mathbb{1}}}_n)$ heißt charakteristisches Polynom von $\underline{\underline{A}}$.

Es gilt: λ ist Eigenwert von $\underline{\underline{A}} \Leftrightarrow \chi_{\underline{\underline{A}}}(\lambda) = 0$ (Nullstelle von $\chi_{\underline{\underline{A}}}(x)$)

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $= \det(\underline{\underline{A}} - \lambda \underline{\underline{\mathbb{1}}})$

Bsp.: (i) $\underline{\underline{A}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: λ Eigenwert $\Leftrightarrow |\underline{\underline{A}} - \lambda \underline{\underline{\mathbb{1}}}| = 0$

$\Leftrightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$

$\Leftrightarrow \lambda^2 = 1 \Rightarrow \lambda_1 = 1$, $\lambda_2 = -1$

Eigenvektor zu $\lambda_1 = 1$: $(\underline{\underline{A}} - \lambda_1 \underline{\underline{\mathbb{1}}})\underline{v}_1 = \underline{0}$

$$ \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} = \underline{0} \Leftrightarrow v_1^{(1)} = v_2^{(1)} \Rightarrow Eig_{\underline{\underline{A}}}(1) = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} $$

$\underline{v} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Eigenvektor zu $\lambda_2 = -1$: $(\underline{\underline{A}} + 1\,\mathbb{1})\,\underline{v}_2 = \underline{0}$

$$\Leftrightarrow \begin{pmatrix} +1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1^{(2)} \\ v_2^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow v_1^{(2)} = -v_2^{(2)}$$

$$\Rightarrow \underline{v}_2 = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \Rightarrow \mathrm{Eig}_{\underline{\underline{A}}}(-1) = \mathrm{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

(ii) Rotation um $\frac{\pi}{4} = \varphi$ (also 45°)

$$\underline{\underline{A}} = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\Rightarrow 0 \overset{!}{=} \chi_{\underline{\underline{A}}}(\lambda) = |\underline{\underline{A}} - \lambda\,\mathbb{1}| = \begin{vmatrix} \frac{1}{\sqrt{2}} - \lambda & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} - \lambda \end{vmatrix}$$

$$= \left(\frac{1}{\sqrt{2}} - \lambda\right)^2 + \frac{1}{2}$$

keine Nullstellen in $\mathbb{R}$, aber $\mathbb{C}$: $\lambda_{1,2} = \frac{1}{\sqrt{2}}(1 \pm i)$

- Im Komplexen ($\underline{\underline{A}} \in \mathbb{C}^{n\times n}$) faktorisiert: $\chi_{\underline{\underline{A}}}(x) = (\lambda_1 - x)^{k_1} \cdots (\lambda_r - x)^{k_r}$

  mit $k_1 + \ldots + k_r = n$.

  k_i heißt algebraische Vielfachheit

- $\dim \mathrm{Eig}_{\underline{\underline{A}}}(\lambda)$ heißt geometrische Vielfachheit.

  $\dim \mathrm{Eig}_{\underline{\underline{A}}}(\lambda) > 1$: λ ist entartet.