6.13 Diagonalisation

- $L: V \to V$ with representing ~~matrix~~ $\underset{C=C}{L}$ (w.r.t. basis C)
- $B = \{\underline{b}_1, \dots, \underline{b}_n\}$ alternative basis of eigenvectors of $\underset{C=C}{L}$: $\underset{C=C}{L}\,\underline{b}_i = \lambda_i \underline{b}_i$
- $\underset{B=B}{L} = \underline{\underline{S}}^{-1} \underset{C=C}{L}\, \underline{\underline{S}} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ with $\underline{\underline{S}} = \begin{pmatrix} | & & | \\ \underline{b}_1 & \dots & \underline{b}_n \\ | & & | \end{pmatrix}$

Thm: $\underline{\underline{A}}$ diagonalisable $\Leftrightarrow$ There is a basis of eigenvectors

7 Differential equations

7.1 Example

$\ddot{z}(t) = -g \Rightarrow z(t) = -\frac{1}{2} g t^2 + v_0 t + z_0$, v_0, z_0 initial conditions

$z_0 = z(0)$, $v_0 = \dot{z}(0)$

7.2 Nomenclature

- ordinary DEq
- partial DEq
- DEq. of n-th order
- general solution contain n constants (determined by initial or boundary values)
- linear DEq. $a_n(x) \frac{d^n y}{dx^n} + \dots + a_1(x) \frac{dy}{dx} = b(x)$
- inhomogeneous DEq. : $b(x) \neq 0$
- homogeneous DEq : $b(x) = 0$ ($y(t) = 0$ is trivial solution)

7.3 Solving Differential equations

(i) direct integration

(ii) separation of variables

(iii) ansatz

(iv) variation of constants (linear inhomogeneous DEq)

(v) power series

7.4 Linear Inhomogeneous DEqs

- $L[y(x)] = b(x) \Rightarrow y(t) = \underbrace{y_h(x)}_{\substack{L[y_h(x)]=0 \\ \text{homogeneous solution}}} + \underbrace{y_p(x)}_{\substack{L[y_p(x)]=b(x) \\ \text{particular solution}}}$

- ex: $\ddot{y}(t) + \omega_0^2 y(t) = f_0 \cos(\Omega t)$

(a) homogeneous: $\ddot{y}(t) + \omega_0^2 y(t) = 0 \Rightarrow y_h(t) = C_1 \sin(\omega_0 t) + C_2 \cos(\omega_0 t)$

$$= A \sin(\omega_0 t + \varphi)$$

(b) particular solution: ansatz: $y_p(t) = \alpha \cos(\omega t)$

$$\Rightarrow \omega = \Omega, \quad \alpha = \frac{f_0}{\omega_0^2 - \Omega^2} \Rightarrow y(t) = A \sin(\omega_0 t + \varphi) + \frac{f_0}{\omega_0^2 - \Omega^2} \cos(\Omega t)$$

what about $\omega_0 = \Omega$?

Nachfrag: $y'(x) = y(x) + x^2$

(a) $y'(x) = y(x) \Rightarrow y_h(x) = ce^{x}$

(b) $c \to c(x) \Rightarrow c'(x) = x^2 e^{-x}$

partielle Integration:

$$c(x) = \int x^2 e^{-x} dx = -x^2 e^{-x} + \int 2x e^{-x} dx$$

$$\int f g' = f g - \int f' g$$

$$= -x^2 e^{-x} + 2 \int x e^{-x} dx$$

$$\int f g'$$

$$= -x^2 e^{-x} + 2 \left(-x e^{-x}\right) + 2 \int e^{-x} dx$$

$$f g \quad - \int f' g$$

$$= -x^2 e^{-x} - 2x e^{-x} - 2 e^{-x} + d$$

$$= -(x^2 + 2x + 2) e^{-x} + d$$ ✓

F.3 (v) Potenzreihenansatz: y+

Bsp.: $(1-x^2)y''(x) - 4xy'(x) - 2y(x) = 0$

Kategorie: linear, homogene DGL mit nicht konstanten Koeffizienten

Lösung: $y(x) = \frac{1}{1-x^2} = (1-x^2)^{-1}$ $|x|<1$

Test: $y'(x) = + \frac{2x}{(1-x^2)^2}$

$$y''(x) = \frac{2}{(1-x^2)^2} + \frac{2x\,2(1-x^2)\,2x}{(1-x^2)^3}$$

$$\Rightarrow \cancel{(1-x^2)} \frac{2}{(1-x^2)^{\cancel{2}}} + \cancel{(1-x^2)} \frac{8x^2(1-x^2)}{(1-x^2)^{\cancel{3}}} - 4x \frac{2x}{(1-x^2)^2} - \frac{2}{(1-x^2)} = 0$$

$$\cancel{\frac{2}{1-x^2}} + \cancel{\frac{8x^2}{1-x^2}} - \cancel{\frac{8x^2}{1-x^2}} - \cancel{\frac{2}{1-x^2}} = 0 \quad \checkmark$$

Potenzreihe: (Entwicklung um $x_0=0$) $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$$\Rightarrow y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\Rightarrow y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Einsetzen:

$$0 = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1) a_n x^n - 4\sum_{n=1}^{\infty} n a_n x^n - 2\sum_{n=0}^{\infty} a_n x^n$$

Indexverschiebung:

$$0 = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^n - 4\sum_{n=1}^{\infty} n a_n x^n - 2\sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=2}^{\infty} x^n \Big[(n+2)(n+1)a_{n+2} - \big(n(n-1) + 4n + 2\big) a_n\Big]$$

$$+ 2a_2 + 6a_3 x - 4a_1 x - 2a_0 - 2a_1 x$$

$$= \sum_{n=2}^{\infty} x^n \Big[(n+2)(n+1)a_{n+2} - (n^2+3n+2) a_n\Big] + 6(a_3 - a_1)x + 2a_2 - 2a_0$$

Idee: Potenzreihe gleich Null $\left(\sum_{k=0}^{\infty} 0 x^k\right)$, wenn alle Koeffizienten Null sind:

$\hookrightarrow x^0: \quad 0 = 2a_2 - 2a_0 \Rightarrow a_2 = a_0$

$x^1: \quad 0 = 6(a_3 - a_1) \Rightarrow a_3 = a_1$

$n \geq 2: \quad 0 = \underbrace{(n+2)(n+1)}_{n^2+3n+2} a_{n+2} - (n^2+3n+2) a_n \Rightarrow a_{n+2} = a_n$

$\Rightarrow$ alle a_n aus a_0 und a_1 bestimmt (Integrationskonstanten!)

Etwa: $a_0 \neq 0, a_1 = 0$: $a_n = \begin{cases} a_0 & n \text{ gerade} \\ 0 & n \text{ ungerade} \end{cases}$

$$\Rightarrow \quad y(x) = \sum_{k=0}^{\infty} a_0 x^{2k} = a_0 \sum_{k=0}^{\infty} (x^2)^k \underset{\substack{\uparrow \\ \text{geometrische} \\ \text{Reihe}}}{=} \frac{a_0}{1-x^2} \qquad |x| < 1$$

allgemeine Lösung: $y(x) = \dfrac{a_0 + a_1 x}{1 - x^2}, \quad |x| < 1$