1.8 Fundamental theorem of calculus

(i) Let f be a continuous real-valued function on a closed interval I. Then, for all $x_0 \in I$, the function $F: I \to \mathbb{R}$ defined as $F(x) = \int_{x_0}^{x} f(\tilde{x}) d\tilde{x}$ is differentiable with $F'(x) = f(x)$.

(ii) Let $f: [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$ and F an antiderivative in (a,b). Then, $\int_a^b f(x)\,dx = F(b) - F(a)$

- antiderivative not unique (differ by a constant)
- $F(x) = \int f(x) dx$ indefinite integral

1.9 Rules of integration

(i) same boundaries / limits of integration: $\int_a^a f(x) dx = 0$

(ii) swapping boundaries: $\int_a^b f(x)\,dx = -\int_b^a f(x) dx$

(iii) additivity of boundaries: $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$

(iv) linearity: $\int_a^b (\lambda f(x) + \mu g(x)) dx = \lambda \int_a^b f(x) dx + \mu \int_a^b g(x) dx$

(v) partial integration: $\int_a^b f(x) g(x) dx = f(x) g(x) \Big|_a^b - \int_a^b f(x) g'(x) dx$

(vi) substitution:

$$\int_a^b f(x)\,dx = \int_{u^{-1}(a)}^{u^{-1}(b)} f(u(t)) \frac{du}{dt} dt, \quad \int_a^b f(u(x)) \frac{du}{dx} dx = \int_{u(a)}^{u(b)} f(t) dt$$

1.10 Examples & tricks

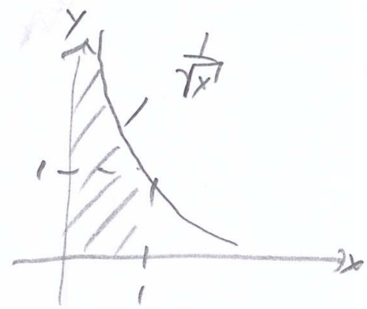
1.11 Uneigentliche Integrale

Integration | von Funktionen mit Singularitäten
| mit unbeschränktem Integrationsbereich

Fazit: Manchmal geht's, manchmal nicht!?

Bsp: (i) $\int_0^1 \frac{1}{\sqrt{x}}\,dx := \lim_{a\to 0} \int_a^1 \frac{1}{\sqrt{x}}\,dx$

divergiert für $x \to 0$

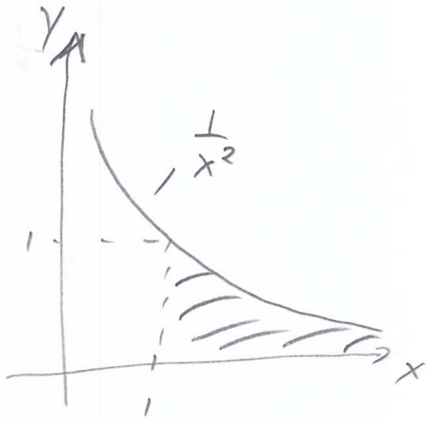


$$= \lim_{a\to 0} 2\sqrt{x}\Big|_a^1$$

$$= \lim_{a\to 0} \left(2 - 2\sqrt{a}\right)$$

$$= 2$$

(ii) $\int_1^\infty \frac{1}{x^2}\,dx := \lim_{a\to\infty} \int_1^a \frac{1}{x^2}\,dx$

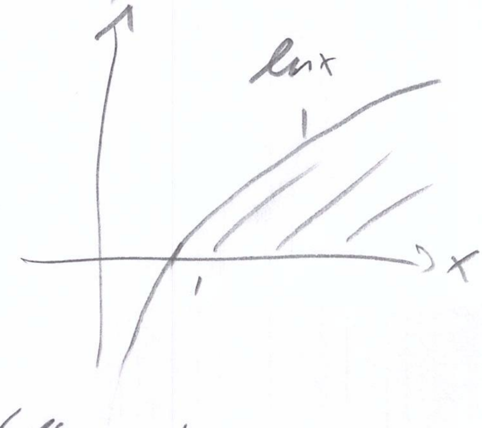


$$= \lim_{a\to\infty} -\frac{1}{x}\Big|_1^a$$

$$= \lim_{a\to\infty} \left(-\frac{1}{a} + \frac{1}{1}\right)$$

$$= 1$$

(ii) $\int_1^\infty \frac{1}{x}\,dx = \lim_{a\to\infty} \int_1^a \frac{1}{x}\,dx$



$$= \lim_{a\to\infty} \ln x\Big|_1^a$$

$$= \lim_{a\to\infty} \left(\ln a - \underbrace{\ln 1}_{0}\right)$$

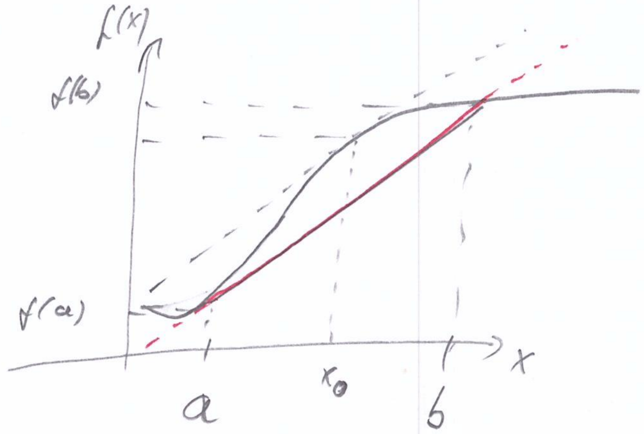
$= \lim_{a\to\infty} \ln a$ existiert nicht/divergent

1.12 Mittelwertsätze

(i) Differenzialrechnung:

Sei $f:[a,b]\to\mathbb{R}$ stetig in $[a,b]$ und diffbar in $]a,b[$.

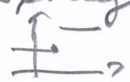
Dann gilt: $\exists\, x_0 \in\,]a,b[\; : f'(x_0) = \dfrac{f(b)-f(a)}{b-a}$



Durchschnittliche Steigung liegt bei x_0 an.

↳ Geschwindigkeitskontrolle durch Intervallmessung

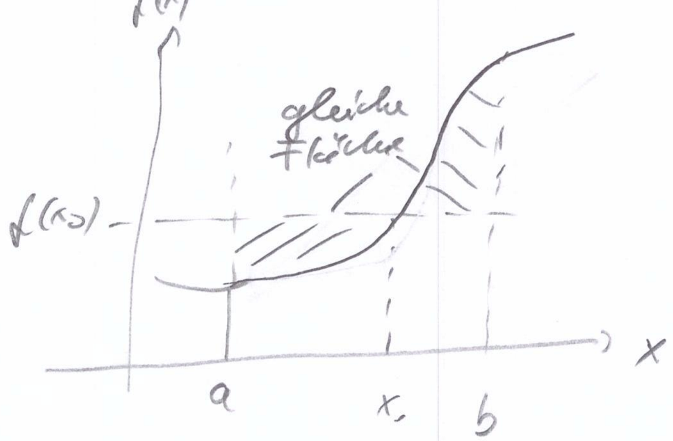
- mehrere Stellen x_0 möglich
- Klappt nicht bei Knicken & Sprüngen



(ii) Integralrechnung:

Sei $f:[a,b]\to\mathbb{R}$ stetig. Dann gilt:

$$\exists\, x_0 \in [a,b] : \int_a^b f(x)\,dx = f(x_0)(b-a)$$



$f(x_0)$ bestimmt die Höhe des Rechtecks der Breite $b-a$

- Klappt nicht bei Sprüngen