2.5 High(er)dimensional Taylor expansion

reminder: $f: \mathbb{R} \to \mathbb{R}$: $f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0) \underbrace{(x-x_0)^k}_{\Delta x^k}$

generalization to $f: \mathbb{R}^n \to \mathbb{R}$ using vector notation:

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \Delta x = \begin{pmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{pmatrix}, \quad x_0 \in \mathbb{R} \longrightarrow \underline{x}_0 \in \mathbb{R}^n, \quad \frac{d}{dx} \to \underline{\nabla} = \begin{pmatrix} \partial_{x_1} \\ \vdots \\ \partial_{x_n} \end{pmatrix}$$

$$\Rightarrow \quad f(\underline{x}) = \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{\left(\sum_{i=1}^{n} \Delta x_i \frac{\partial}{\partial x_i} \right)^k}_{(\Delta \underline{x} \cdot \underline{\nabla})^k} f(\underline{x}_0)$$

scalar product!

up to order 2: $k = 0, 1, 2$

transposed: $\Delta \underline{x}^T = (x_1, \ldots, x_n)$

$$f(\underline{x}) = f(\underline{x}_0) + \underline{\nabla} f(\underline{x}_0) \cdot \Delta \underline{x} + \frac{1}{2} \Delta \underline{x}^T \underline{\underline{H}}(\underline{x}_0) \Delta \underline{x} + O(|\Delta x|^3)$$

with the Hessian / Hesse matrix (of 2^{nd} derivatives)

$$\underline{\underline{H}}(\underline{x}_0) = \begin{pmatrix} \partial_{x_1} \partial_{x_1} f(\underline{x}_0) & \cdots & \partial_{x_1} \partial_{x_n} f(\underline{x}_0) \\ \vdots & & \vdots \\ \partial_{x_n} \partial_1 f(\underline{x}_0) & \cdots & \partial_{x_n} \partial_{x_n} f(\underline{x}_0) \end{pmatrix}$$

2.6 Extreme values

reminder: $f: \mathbb{R} \to \mathbb{R}$ with extreme value/extremum at a: $f'(a) = 0$

↳ $f''(a) > 0$:) : Minimum (positive curvature)

↳ $f''(a) < 0$:(: Maximum (negative curvature)

↳ $f''(a) = 0$: anything is possible (including saddles)

generalization to $f: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto f(x,y)$

$$df(\underline{a}) = \frac{\partial}{\partial x} f(\underline{a})\, dx + \frac{\partial}{\partial y} f(\underline{a})\, dy = 0 \Rightarrow \frac{\partial}{\partial x} f(\underline{a}) = 0 \wedge \frac{\partial f}{\partial y}(\underline{a}) = 0$$

↳ Minimum: $\partial_x^2 f(\underline{a}) > 0$, $\partial_y^2 f(\underline{a}) > 0$, $D := (\partial_x^2 f)(\partial_y^2 f) - (\partial_x \partial_y f)^2 > 0$

$\partial_x \partial_x f = \frac{\partial^2}{\partial x^2} f$

↳ Maximum: $\partial_x^2 f(\underline{a}) < 0$, $\partial_y^2 f(\underline{a}) < 0$, $D > 0$

↳ Saddle: $D < 0$

↳ unclear (higher orders needed): $D = 0$

$\boxed{\begin{array}{l} n > 2: \\ df(\underline{a}) = 0 \\ \text{Eigenvalues of } \underline{\underline{H}}(\underline{a}): \\ \hookrightarrow \text{all negative: Maximum} \\ \hookrightarrow \text{all positive: Minimum} \end{array}}$

2.7 Extremum with constraints

idea: extremum of $f(x,y)$ for a constraint $g(x,y) = 0$

↳ $\underline{\nabla} f(x,y) = -\lambda \underline{\nabla} g(x,y)$ (parallel gradients)

Lagrange multiplier

2.8 Integralrechnung (skalare Funktionen) in $\mathbb{R}^n$

(1) Definition: ($\mathbb{R}^2$)

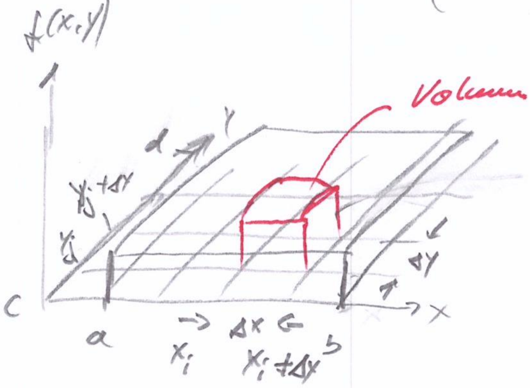
Sei $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^2$ stetig in Ω (etwa $[a,b] \times [c,d]$). Wir definieren das **zweidimensionale Integral** als

$$\underbrace{\int_\Omega f(x,y)\,dx\,dy}_{\text{Volumen unter } f(x,y)} := \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \sum_{j=0}^{\frac{d-c}{\Delta y}-1} \sum_{i=0}^{\frac{b-a}{\Delta x}-1} f(x_i, y_j)\,\Delta x\,\Delta y$$

↳ Anzahl der Stützstellen: $N_x = \frac{b-a}{\Delta x}$ bzw. $N_y = \frac{d-c}{\Delta y}$

$\left(\Delta x = \frac{b-a}{N_x}\right)$ $\left(\Delta y = \frac{d-c}{N_y}\right)$

Volumen des kleinen Quaders $f(x_i, y_j)\,\Delta x\,\Delta y$



$x_i = a + i\Delta x$

$y_j = c + i\Delta y$

(2) Satz von Fubini:

Notation: $\int_\Omega f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy \equiv \int_c^d dy \int_a^b dx\, f(x,y)$

$$\int_\Omega f(x,y)\,dx\,dy = \lim_{N_y \to \infty} \sum_{j=0}^{N_y-1} \Big(\underbrace{\lim_{N_x \to \infty} \sum_{i=0}^{N_x-1} f(x_i, y_j)\,\Delta x}_{\int_a^b (x, y_j)\,dx} \Big) \Delta y$$

$$= \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$

$$= \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

(3) Beispiele:

(i) $\Omega = [0,2]\times[0,1]$ Rechteck, $f(x,y) = xy + y^2$

$$\Rightarrow \int_\Omega f(x,y)\,dx\,dy = \int_0^2 dx \int_0^1 dy\,(xy+y^2)$$

$$= \int_0^2 dx \left(\frac{1}{2}xy^2 + \frac{1}{3}y^3\right)\Bigg|_{y=0}^{y=1}$$

$$= \int_0^2 dx \left(\frac{1}{2}x + \frac{1}{3}\right)$$

$$= \frac{1}{4}x^2 + \frac{1}{3}x\Big|_{x=0}^{2}$$

$$= \frac{1}{4}4 + \frac{1}{3}2 - 0$$

$$= \frac{5}{3}$$

$$= \int_0^1 dy \int_0^2 dx\,(xy+y^2)$$

$$= \int_0^1 dy \left(\frac{1}{2}x^2y + xy^2\right)\Bigg|_{x=0}^{2}$$

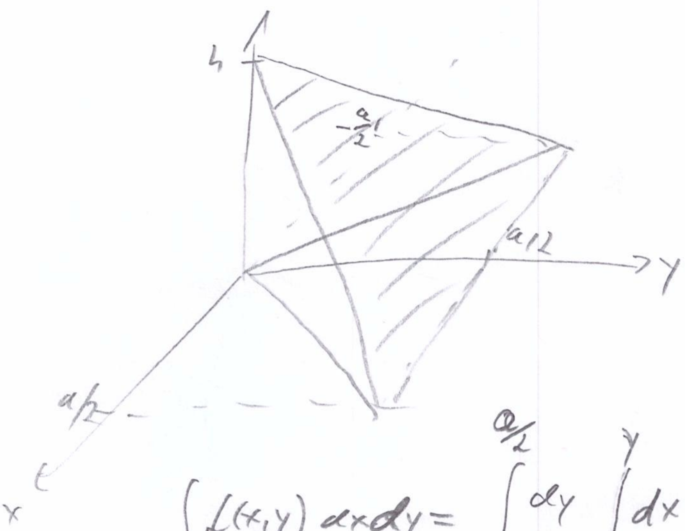
$$= \int_0^1 dy\,(2y+2y^2)$$

$$= y^2 + \frac{2}{3}y^3\Big|_{y=0}^{1}$$

$$= 1 + \frac{2}{3} = \frac{5}{3}$$

(erst x- dann y-Integration liefert dasselbe Ergebnis)

(ii) Dreieck:



$$f(x,y) = h - \frac{h}{a/2}\,y$$

$$\Omega = \left\{(x,y)\,\middle|\, 0 \le y \le \frac{a}{2} \wedge -y \le x \le y\right\}$$

x-Richtung hängt von y ab!

$$\int_\Omega f(x,y)\,dx\,dy = \int_0^{a/2} dy \int_{-y}^{y} dx \left(-\frac{2h}{a}y + h\right) = h\int_0^{\frac{a}{2}} dy \left(-\frac{2}{a}y+1\right)x\Big|_{x=-y}^{x=y}$$

$$= h\int_0^{\frac{a}{2}} dy\left(-\frac{4}{a}y^2 + 2y\right) = h\left(-\frac{4}{3}y^3 + y^2\right)\Bigg|_{y=0}^{y=\frac{a}{2}}$$

$$= \frac{1}{12}a^2h$$ (Vgl. Viertel Pyramide $\frac{1}{3}a^2h$)