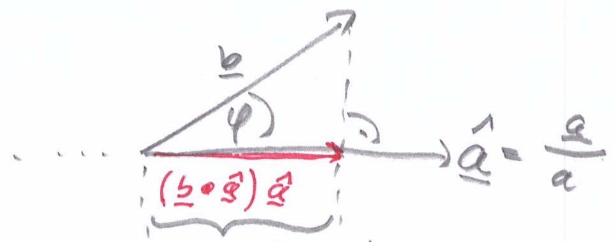
# 3.3 Dot product (continued)

$(\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ (most often $n=3$)

$(\underline{a},\underline{b}) \longmapsto \underline{a}\cdot\underline{b} \equiv \langle \underline{a},\underline{b} \rangle = a_1 b_1 + \dots + a_n b_n$

projection of $\underline{b}$ onto $\underline{a}$:




length: $|\underline{b}\cdot\hat{\underline{a}}| = b\cos\varphi$

orthonormal basis: $\{\underline{e}_1, \dots, \underline{e}_n\}$ with $\underline{e}_i \cdot \underline{e}_j = \delta_{ij}$ (orthogonal)

and $|\underline{e}_i| = 1$ (normalized)

# 3.4 Cross product

The crossproduct of 2 vectors $\underline{a}, \underline{b} \in \mathbb{R}^3$ is defined as

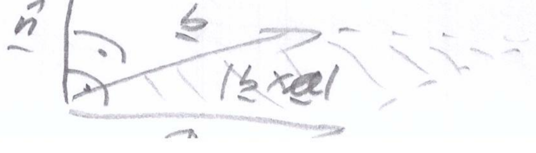
$(\cdot,\cdot) : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$

$$(\underline{a},\underline{b}) \longmapsto \underline{a}\times\underline{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

Properties:

- antisymmetric: $\underline{a}\times\underline{b} = -\underline{b}\times\underline{a}$
- bilinear : $\underline{a}\times(\alpha\underline{b} + \beta\underline{c}) = \alpha(\underline{a}\times\underline{b}) + \beta(\underline{a}\times\underline{c})$, $\alpha,\beta \in \mathbb{R}$, $\underline{a},\underline{b},\underline{c} \in \mathbb{R}^3$
- not associative: $\underline{a}\times(\underline{b}\times\underline{c}) \neq (\underline{a}\times\underline{b})\times\underline{c}$
- $(\underline{a}\times\underline{b})_i = \sum_{j,k=1}^{3} \varepsilon_{ijk}\, a_j b_k$ with Levi-Civita-Symbol $\varepsilon_{ijk}$
- geometric interpretation (i) $\underline{a}\times\underline{b} \perp \underline{a} \wedge \underline{a}\times\underline{b} \perp \underline{b}$

  (ii) $|\underline{a}\times\underline{b}|$ = area of parallelogram spanned by $\underline{a}, \underline{b}$
- $\underline{a}\times\underline{b} = \underline{0} \Leftrightarrow \underline{a} \parallel \underline{b}$
- $\underline{a}\times\underline{b} = |a|\,|b| \sin\varphi\, \hat{\underline{n}}$



# 3.5 Levi-Civita-Symbol

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{for } ijk \text{ even permutation of } (1,2,3) \\ -1 & \text{—"— odd —"—} \\ 0 & \text{otherwise} \end{cases}$$

| | i=1 | | | i=2 | | | i=3 | | |
|---|---|---|---|---|---|---|---|---|---|
| j \ k | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 1 | 0 |
| 2 | 0 | 0 | 1 | 0 | 0 | 0 | -1 | 0 | 0 |
| 3 | 0 | -1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |

## 3.5 Levi-Civita-Symbol

Def: Das **Levi-Civita-Symbol** ist definiert als

$$\varepsilon_{ijk} := \begin{cases} 1 & \text{falls } ijk \text{ eine gerade Permutation von } (1,2,3) \\ -1 & \text{---"--- ungerade ---"--- ---} \\ 0 & \text{sonst} \end{cases}$$

gerade Permutationen: (1 2 3), (3 1 2), (2 3 1)

ungerade Permutationen: (1 3 2), (3 2 1), (2 1 3)

$$\Rightarrow \quad (\underline{a}\times\underline{b})_i = \sum_{jk=1}^{3} \varepsilon_{ijk}\, a_j b_k \equiv \varepsilon_{ijk}\, a_j b_k$$

↑ Einsteinsche Summenkonvention: über doppelte Indizes wird summiert!

⇒ **Graßmann-Identität:**

$$\Rightarrow \quad \sum_{k=1}^{3} \varepsilon_{ijk}\, \varepsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$

links-rechts  außen-innen

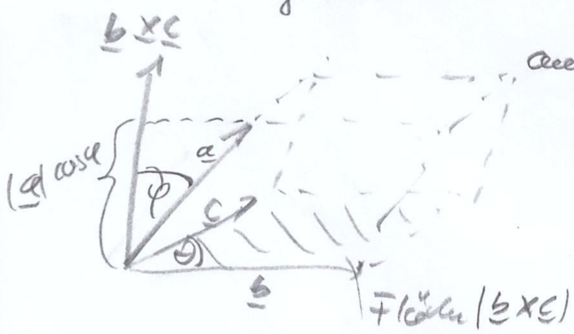
## 3.6 Mehrfachprodukte

(i) doppeltes Kreuzprodukt: **bac-cab Regel**

$$\underline{a}\times(\underline{b}\times\underline{c}) = \underline{b}\,(\underline{a}\cdot\underline{c}) - \underline{c}\,(\underline{a}\cdot\underline{b})$$

(ii) **Spatprodukt**: $\underline{a}\cdot(\underline{b}\times\underline{c})$

geometrische Interpretation: Volumen des von $\underline{a},\underline{b},\underline{c}$ aufgespannten Parallelepipeds.




$$V = \underbrace{|\underline{b}\times\underline{c}|}_{\text{Fläche}}\ \underbrace{|\underline{a}|\cos\varphi}_{\text{Höhe}}$$

$$= abc \sin\theta \cos\varphi$$

↳ gleiches Volumen bei zyklischer Vertauschung:

$$\underline{a}\cdot(\underline{b}\times\underline{c}) = \underline{b}\cdot(\underline{c}\times\underline{a}) = \underline{c}\cdot(\underline{a}\times\underline{b})$$

↳ Berechnung über Determinante:

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1 (b_2 c_3 - b_3 c_2) + a_2 (b_3 c_1 - b_1 c_3) + a_3 (b_1 c_2 - b_2 c_1$$

$$= \sum_{i=1}^{3} a_i \, (\underline{b} \times \underline{c})_i$$

$$= \sum_{i,j,k=1}^{3} \varepsilon_{ijk} \, a_i b_j c_k$$

(iii) $(\underline{a} \times \underline{b}) \cdot (\underline{c} \times \underline{d}) = (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d}) - (\underline{a} \cdot \underline{d})(\underline{b} \cdot \underline{c})$

links rechts    außen innen

↳ $(\underline{a} \times \underline{b})^2 = (\underline{a} \cdot \underline{a})(\underline{b} \cdot \underline{b}) - (\underline{a} \cdot \underline{b})(\underline{b} \cdot \underline{a})$

$$= a^2 \; b^2 - (\underline{a} \cdot \underline{b})^2$$

$$= a^2 b^2 - (ab \cos\varphi)^2$$

$$= a^2 b^2 (1 - \cos^2\varphi)$$

$$= a^2 b^2 \sin^2\varphi$$

↳ $\underline{a} \times \underline{b} = a b \sin\varphi \; \hat{\underline{n}}$