

Parametrization using, e.g., spherical coordinates:

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$$\underline{\Phi}: (\vartheta, \varphi) \mapsto \begin{pmatrix} R \sin \vartheta \cos \varphi \\ R \sin \vartheta \sin \varphi \\ R \cos \vartheta \end{pmatrix}, \quad \vartheta \in [0, \pi], \varphi \in [0, 2\pi)$$

$\hookrightarrow B = [0, \pi] \times [0, 2\pi)$

$\hookrightarrow$  unit vectors:  $\{\underline{e}_{\vartheta}, \underline{e}_{\varphi}, \underline{e}_r\}$

$$\underline{e}_{\vartheta} = \begin{pmatrix} \cos \vartheta \cos \varphi \\ \cos \vartheta \sin \varphi \\ -\sin \vartheta \end{pmatrix}$$

$$\underline{e}_{\varphi} = \begin{pmatrix} -\sin \varphi \\ \cos \varphi \\ 0 \end{pmatrix}$$

$$\underline{e}_r = \begin{pmatrix} \sin \vartheta \cos \varphi \\ \sin \vartheta \sin \varphi \\ \cos \vartheta \end{pmatrix}$$

derived via

$$\frac{\partial \underline{x}}{\partial \vartheta}, \frac{\partial \underline{x}}{\partial \varphi}, \frac{\partial \underline{x}}{\partial \vartheta} \times \frac{\partial \underline{x}}{\partial \varphi}$$

and normalization

surface/area element:  $\left| \frac{\partial \underline{x}}{\partial \vartheta} \times \frac{\partial \underline{x}}{\partial \varphi} \right| = R^2 \sin \vartheta$

### 4.8 Integration over areas

Surface integral of vector fields:  $\underline{\Phi}: B \rightarrow \mathbb{R}^3, (u, v) \mapsto \underline{x} = \underline{\Phi}(u, v)$

$$\int_F \underline{v}(\underline{x}) \cdot d\underline{A} := \int_B \underline{v}(\underline{x}(u, v)) \cdot \underbrace{\left( \frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} \right)}_{d\underline{A}} du dv$$

4.9 Gauss's theorem (Divergence theorem) 4.10 Stoke's theorem

$$\int_{\partial V} \underline{v} \cdot d\underline{A} = \int_V \underbrace{\operatorname{div} \underline{v}}_{\nabla \cdot \underline{v}} dV$$

$$\int_{\partial F} \underline{v} \cdot d\underline{s} = \int_F \underbrace{\operatorname{rot} \underline{v}}_{\nabla \times \underline{v}} \cdot d\underline{A}$$

flux through closed surface  $\hookrightarrow$  divergences (sources) of field enclosed

line integral around boundary  $\hookrightarrow$  curl over surface

## 9.11 Partielle Integrationen

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• Erinnerung: Skalare Funktionen:  $f, g: \mathbb{R} \rightarrow \mathbb{R}$

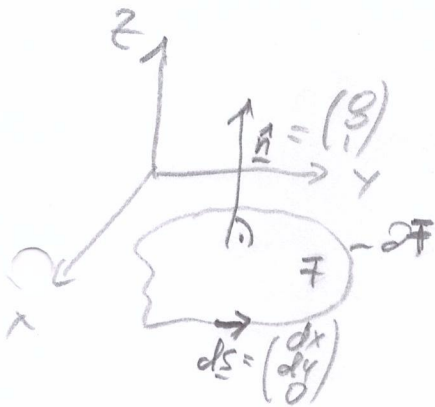
$$\int_a^b f'g \, dx = fg \Big|_a^b - \int_a^b fg' \, dx$$

• Für Felder (Skalar:  $f: \mathbb{R}^n \rightarrow \mathbb{R}$ , vektoriell:  $\underline{v}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ ) gilt:

$$\int_V \underbrace{\nabla \cdot (f\underline{v})}_{\text{div}(f\underline{v})} \, dV \stackrel{(\text{Kap. 9.2})}{=} \int_V \underbrace{(\nabla f)}_{\text{grad} f} \cdot \underline{v} \, dV + \int_V f \underbrace{(\nabla \cdot \underline{v})}_{\text{div} \underline{v}} \, dV$$

$$\stackrel{\text{Satz v. Green}}{\Rightarrow} \int_V (\nabla f) \cdot \underline{v} \, dV = \underbrace{\int_{\partial V} f \underline{v} \cdot d\underline{A}}_{\text{Randterm}} - \int_V f (\nabla \cdot \underline{v}) \, dV$$

## 9.12 Satz von Green



$F$ : Fläche in  $(x, y)$ -Ebene

$$\underline{v} = \begin{pmatrix} f(x, y) \\ g(x, y) \\ v_z \end{pmatrix} \quad C^1\text{-Vektorfeld}$$

$$\Rightarrow \int_{\partial F} f(x, y) dx + g(x, y) dy = \int_B \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy$$

↳ Übrig bleibt  $g$

# 5. Komplexe Zahlen

5.1 Definitionen & formales Rechnen

5.2 Graphische Darstellung in komplexer Ebene

5.3 Komplexe Exponentialfunktionen

## 5.1 Definitionen & formales Rechnen

Recap: (i) **natürliche Zahlen**  $\mathbb{N} = \{1, 2, 3, \dots\}$

$+, \cdot$   $\forall a, b \in \mathbb{N} \Rightarrow a+b, a \cdot b \in \mathbb{N}$ , aber kein Inverses

(ii) **ganze Zahlen**  $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

$\hookrightarrow$  Inverses Element bzgl. Addition, aber nicht Multiplikation

(iii) **rationale Zahlen**  $\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}$

$\hookrightarrow$  Inverses Element bzgl. Multiplikation

$\hookrightarrow$  Grenzwert konvergenter Reihen ggf. nicht in  $\mathbb{Q}$

(iv) **reelle Zahlen**  $\mathbb{R}$

$\hookrightarrow$  Anders als  $\mathbb{Q}$  durch irrationale Zahlen wie  $\sqrt{2}, \pi, e, \dots$

$\hookrightarrow$  nicht alle Polynome in Linearform zerlegbar:

$$x^2 + 1 = 0 \text{ nicht in } \mathbb{R} \text{ lösbar}$$

Idee: **imaginäre Einheit**  $i^2 := -1$  (häufig eingeführt als  $i = \sqrt{-1}$ )  
aber Vorsicht!

$\Rightarrow$  Menge der **komplexen Zahlen**  $\mathbb{C} := \{z = a + ib \mid a, b \in \mathbb{R}\}$

**Realteil**  $a = \operatorname{Re}(z)$ , **Imaginärteil**  $b = \operatorname{Im}(z)$

$\Rightarrow \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$