4.11 Integration by parts

recap: $\int_a^b f'g\,dx = fg\Big|_a^b - \int_a^b f g'\,dx$, $f,g:\mathbb{R}\to\mathbb{R}$

For scalar fields $f:\mathbb{R}^n\to\mathbb{R}$ and vector fields $\underline{v}:\mathbb{R}^n\to\mathbb{R}^n$:

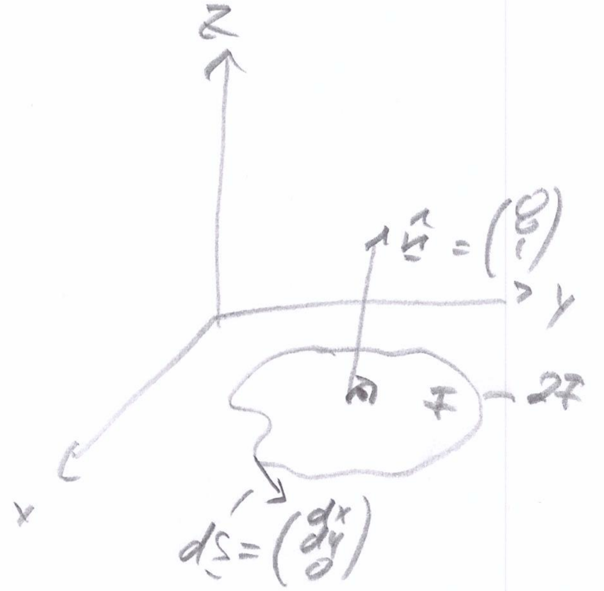
$$\int_V (\underline{\nabla} f)\cdot\underline{v}\,dV = \int_{\partial V} f\underline{v}\cdot d\underline{A} - \int_V f(\underline{\nabla}\cdot\underline{v})\,dV$$

4.12 Green's theorem

F: plane region (surface in $\mathbb{R}^2$) $\Rightarrow$ normal vector $\underline{n} = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$\underline{v}$: C^1 vector field with $\underline{v} = \begin{pmatrix}f(x,y)\\g(x,y)\\v_z\end{pmatrix}$

$$\Rightarrow \int_{\partial F} f(x,y)\,dx + g(x,y)\,dy = \int_B \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx\,dy$$



5 Complex numbers

hierarchy of sets of numbers:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

- $\mathbb{N}\to\mathbb{Z}$: add inverse of "+"
- $\mathbb{Z}\to\mathbb{Q}$: add inverse of "·"
- $\mathbb{Q}\to\mathbb{R}$: convergence of all series
- $\mathbb{R}\to\mathbb{C}$: factorization of all polynomials

Rechenregeln:

(i) Addition: $\underbrace{(a+ib)}_{z_1} + \underbrace{(c+id)}_{z_2} = \underbrace{(a+b)}_{Re(z_1+z_2)} + i\underbrace{(c+d)}_{Im(z_1+z_2)}$ $\quad i^2=-1$

(ii) Multiplikation: $(a+ib)(c+id) = (ac-bd) + i(ad+bc)$

(iii) Komplex konjugierte Zahl $z^* \equiv \bar{z} = a - ib$

$\hookrightarrow Re(z) = \frac{1}{2}(z+z^*)$, $Im(z) = \frac{1}{2i}(z-z^*)$

(iv) Division: $\frac{z_1}{z_2} = \frac{z_1}{z_2}\frac{z_2^*}{z_2^*} = \frac{(a+ib)(c-id)}{(c+id)(c-id)} = \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$

(v) Betrag: $|z| = \sqrt{a^2+b^2} = \sqrt{(a+ib)(a-ib)} = \sqrt{zz^*}$

Bsp: Löse $x^2+2x+2=0$

quadratisch ergänzen:

$x^2+2x+1-1+2=0$

$\Leftrightarrow (x+1)^2 = -1$

$\Leftrightarrow x+1 = \pm i$

$\Rightarrow x_{1,2} = -1 \pm i$

pq-Formel:

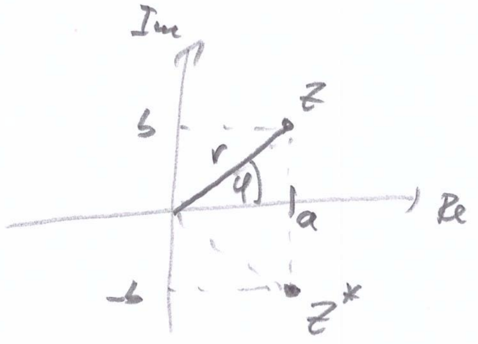
$x_{1,2} = -\frac{2}{2} \pm \sqrt{\left(\frac{2}{2}\right)^2 - 2}$

$= -1 \pm \sqrt{-1}$

$= -1 \pm i$

5.2 Graphische Darstellung in komplexer Ebene

Idee: $x = Re(z)$, $y = Im(z)$ $\Rightarrow$ Polardarstellung



$$\left.\begin{aligned} r &= |z| = \sqrt{a^2+b^2} \\ Re\, z &= r\cos\varphi \\ Im\, z &= r\sin\varphi \end{aligned}\right\} \Rightarrow z = r(\cos\varphi + i\sin\varphi)$$

Betrag — Phase $\varphi = \arg(z)$

Achtung: $\varphi \to \varphi + n2\pi$, $n \in \mathbb{Z}$ liefert die gleiche komplexe Zahl

$\Rightarrow$ nicht eindeutig

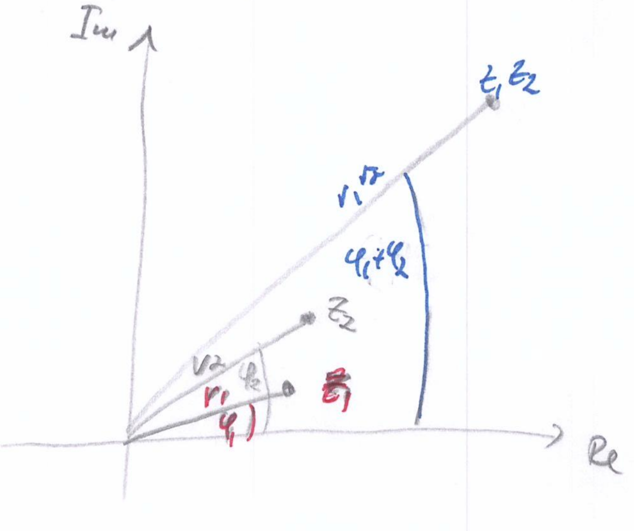
iii) $r \neq 0$: z reell $\Leftrightarrow \varphi = n\pi$, $n \in \mathbb{Z}$

Multiplikation: $z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$, $z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$

$$\Rightarrow z_1 z_2 = r_1 r_2 \Big[\underbrace{(\cos\varphi_1\cos\varphi_2 \overset{i^2}{-} \sin\varphi_1\sin\varphi_2)}_{\cos(\varphi_1+\varphi_2)} + i\underbrace{(\cos\varphi_1\sin\varphi_2 + \sin\varphi_1\cos\varphi_2)}_{\sin(\varphi_1+\varphi_2)}\Big]$$

$$= r_1 r_2\,[\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)]$$

Beträge multiplizieren sich — Phasen addieren sich



Damit lässt sich der vermeintliche Widerspruch für $i = \sqrt{-1}$ auflösen:

$$-1 = i^2 = \sqrt{-1}\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$$

$$i = 0 + i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

$$\Rightarrow i^2 = i\,i = 1\cdot 1\,(\underbrace{\cos\pi}_{=-1} + i\underbrace{\sin\pi}_{=0})$$

$$= -1$$

5.3 Komplexe Exponentialfunktion

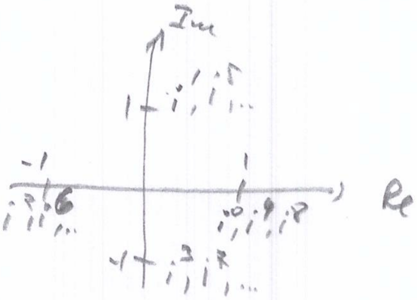
Idee: Definiere die komplexe Exponentialfunktion über ihre Potenzreihe:

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

↳ $$e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y)$$

Beweis: $e^{iy} = \sum_{n=0}^{\infty} \frac{1}{n!} i^n y^n$

n	0	1	2	3	4	5	6	7	8	…
i^n	1	i	-1	$-i$	1	i	-1	$-i$	1	…



$$= \underbrace{1 - \frac{1}{2!} y^2 + \frac{1}{4!} y^4 - \frac{1}{6!} y^6 + \frac{1}{8!} y^8 + \ldots}_{\cos y}$$
$$+ i \underbrace{\left(y - \frac{1}{3!} y^3 + \frac{1}{5!} y^5 - \frac{1}{7!} y^7 + \ldots \right)}_{\sin y}$$

(vgl. Tut 6 Auf 3)

Eulersche Formel: $e^{i\varphi} = \cos\varphi + i \sin\varphi \quad , \varphi \in \mathbb{R}$

- $e^{i\pi} = -1$, $|e^{i\varphi}| = 1$ $\left(|e^z| = e^x \right)$, $\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}$, $\sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}$

  vgl. $\cosh x = \frac{e^x + e^{-x}}{2}$, $\sinh x = \frac{e^x - e^{-x}}{2}$

- Polardarstellung: $z = r e^{i\varphi}$

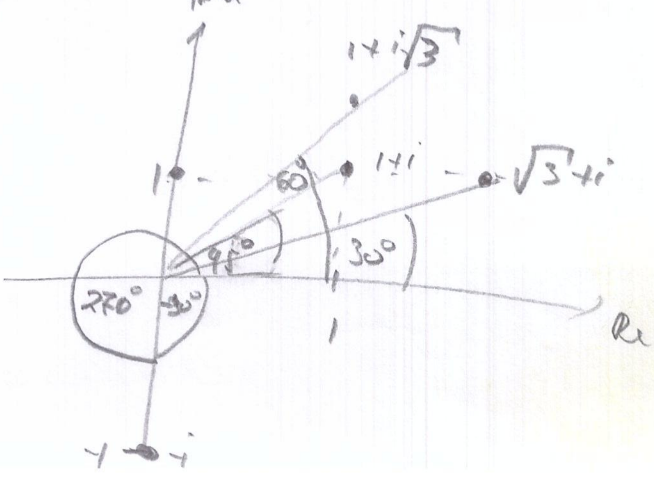
  ↳ $z_1 z_2 = r_1 e^{i\varphi_1} r_2 e^{i\varphi_2} = r_1 r_2 e^{i(\varphi_1 + \varphi_2)}$

Bsp: $1 + i = \sqrt{2}\, e^{i\frac{\pi}{4}}$

$-i = e^{-\frac{\pi}{2} i} = e^{\frac{3\pi}{2} i}$

$\sqrt{3} + i = 2 e^{i\frac{\pi}{6}}$

$1 + i\sqrt{3} = 2 e^{i\frac{\pi}{3}}$



- Komplexer Logarithmus:

$$z = re^{i\varphi} = re^{i(\varphi+n2\pi)} \Rightarrow \ln z = \ln r e^{i(\varphi+2\pi n)}$$

$$= \ln r + \ln e^{i(\varphi+2\pi n)}$$

$$= \ln r \underbrace{+ i(\varphi+2\pi n)}_{\text{mehrwertig!}}$$

- Matrizenschreibweise:

$$Z = a\underbrace{\begin{pmatrix}1&0\\0&1\end{pmatrix}}_{\underline{\underline{E}}\ \hat{=}\ \mathrm{Re}} + b\underbrace{\begin{pmatrix}0&-1\\1&0\end{pmatrix}}_{\underline{\underline{I}}\ \hat{=}\ \mathrm{Im}} = \begin{pmatrix}a&-b\\b&a\end{pmatrix} \hat{=}\ a+ib$$

Test: (i) Einheitsmatrix: $\underline{\underline{E}}\,\underline{\underline{E}} = \underline{\underline{E}}$

(ii) $\underline{\underline{I}}\,\underline{\underline{I}} = \begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix} = \begin{pmatrix}-1&0\\0&-1\end{pmatrix} = -\underline{\underline{E}} \ \hat{=}\ i^2 = -1$

(iii) Multiplikation: (oder auch mit $\left[\begin{pmatrix}a&0\\0&a\end{pmatrix}+\begin{pmatrix}0&-b\\b&0\end{pmatrix}\right]\left[\begin{pmatrix}c&0\\0&c\end{pmatrix}+\begin{pmatrix}0&-d\\d&0\end{pmatrix}\right]$

$$Z_1 Z_2 = \left[\begin{pmatrix}a_1&0\\0&a_1\end{pmatrix}+\begin{pmatrix}0&-b_1\\b_1&0\end{pmatrix}\right]\left[\begin{pmatrix}a_2&0\\0&a_2\end{pmatrix}+\begin{pmatrix}0&-b_2\\b_2&0\end{pmatrix}\right]$$

$$= \begin{pmatrix}a_1a_2&0\\0&a_1a_2\end{pmatrix} + \begin{pmatrix}0&-a_1b_2\\a_1b_2&0\end{pmatrix} + \begin{pmatrix}0&-b_1a_2\\b_2a_1&0\end{pmatrix} + \begin{pmatrix}-b_1b_2&0\\0&-b_1b_2\end{pmatrix}$$

$$= \begin{pmatrix}a_1a_2-b_1b_2&0\\0&a_1a_2-b_1b_2\end{pmatrix} + \begin{pmatrix}0&-a_1b_2-b_1a_2\\+a_1b_2+b_1a_2&0\end{pmatrix}$$

$$= (a_1a_2-b_1b_2)\,\underline{\underline{E}} + (a_1b_2+a_2b_1)\,\underline{\underline{I}} \ \hat{=}\ (ac-bd)+i(bc+ad)$$

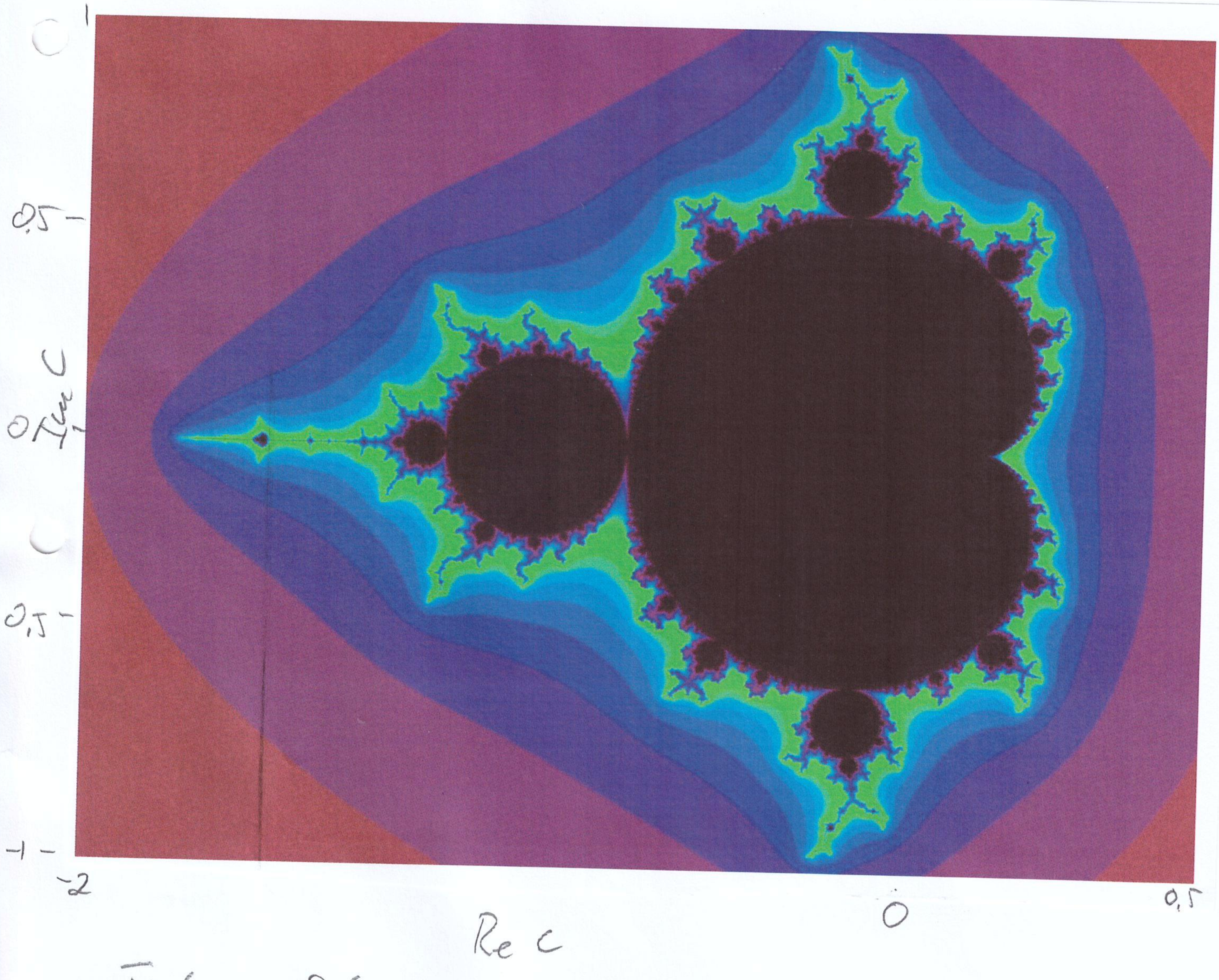
$$z=\begin{pmatrix}a&0\\0&a\end{pmatrix}+\begin{pmatrix}0&-b\\b&0\end{pmatrix} \Rightarrow z^*=\begin{pmatrix}a&0\\0&a\end{pmatrix}+\begin{pmatrix}0&b\\-b&0\end{pmatrix}$$

$$z=\begin{pmatrix}a&-b\\b&a\end{pmatrix} \quad \text{vgl.:} \begin{pmatrix}r\cos\varphi & -r\sin\varphi\\ r\sin\varphi & r\cos\varphi\end{pmatrix}$$ Drehstreckungen!

Skalierung r, Drehwinkel φ

$\hat{=}\ re^{i\varphi}$ in komplexer Ebene

Mandelbrot-Menge: Iteration von $f_c(z)=z^2+c$: $z_0=0,\ z_1=c,\ z_2=c^2+c$, $z_3=f_c(z_2)\dots$

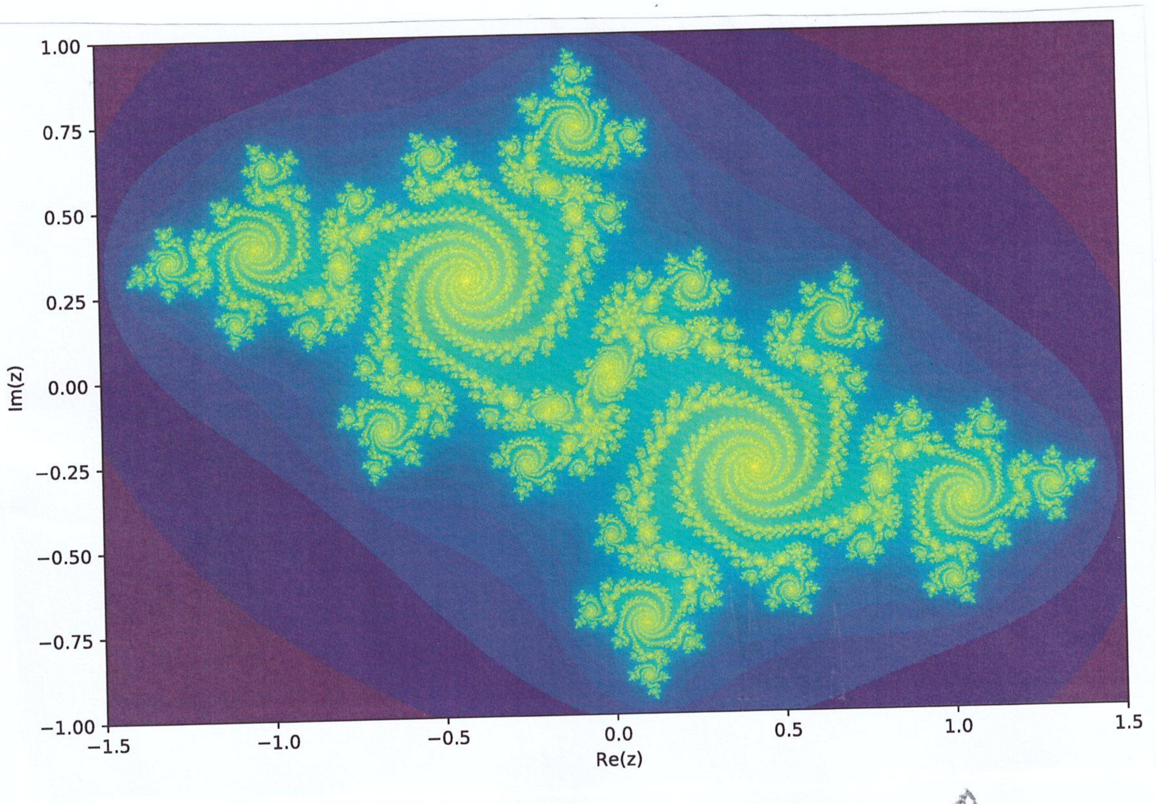


Farbcode: Schwarz für $|z_n|<1000$

bunt für $|z_n|\geq 1000$, in welchem Schritt n

(hier: rot $\hat{=}\ n=5$)

Julia-Menge:



hier: c fest (etwa: $c = -0.5125149 + i\,0.5212955$)

und z variable: $f_c(z) = z^2 + c$

Farbcode: gelb ≙ beschränkt $z_{n+1} = z_n^2 + c$ bleibt $|z| \leq 2$

bunt: Iterationsschritt, bei dem $|z| > 2$

Erweiterung auf andere Polynome möglich

↳ Fraktale Struktur (Selbstähnlichkeit bei Zoom)

↳ Anknüpfung an dynamische Systeme und Chaos