6.4 Basis and dimension

7Sa

- $B = \{\underline{b}_1, \dots, \underline{b}_n\}$ is a basis of V if all elements $\underline{v} \in V$ can be written as a unique linear combination of generating set $\underline{B}$:

$$\underline{v} = \alpha_1 \underline{b}_1 + \dots + \alpha_n \underline{b}_n$$

coordinates

- Theorem: $B = \{\underline{b}_1, \dots, \underline{b}_n\}$ basis $\Leftrightarrow$ span $B = V$ and B linearly independent.

- Dimension of V: $\dim V = |B|$ ($|\cdot|$: # elements of B)

basis of V

- Theorem: Every vector space has a basis
  - ↳ every generating set contains a basis (reduction)
  - ↳ extend linearly independent subsets to basis (extension)

6.5 Matrices as representation of linear maps

Theorem: $B = \{\underline{b}_1, \dots, \underline{b}_n\}$ basis of V, $L: V \to W$ linear map.

Then: L completely/fully determined by images of B.

commuting diagram:

basis of V basis of W

$$\underline{v} = \sum_{i=1}^{n} v_i \underline{b}_i \qquad \underline{w} = \sum_{j=1}^{m} w_j \underline{c}_j$$

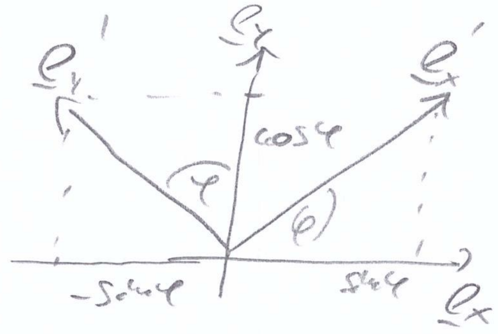
coordinate maps

$$\begin{tikzcd}
\underline{v} \in V \arrow[r, "L"] \arrow[d, "K_B"'] & \underline{w} \in W \arrow[d, "K_C"] \\
\underline{v}_B \in \mathbb{K}^n \arrow[r, "{}_C L_B"'] & \underline{w}_C \in \mathbb{K}^m_C
\end{tikzcd}$$

transformation matrix $\;{}_C L_B = K_C \circ L \circ K_B^{-1}$

Rotations in $\mathbb{R}^2$ by angle φ:

$B = \{\underline{e}_x, \underline{e}_y\}$ with $\underline{e}_x = \begin{pmatrix}1\\0\end{pmatrix}$, $\underline{e}_y = \begin{pmatrix}0\\1\end{pmatrix}$



$$\underline{e}_x' = \cos\varphi\,\underline{e}_x + \sin\varphi\,\underline{e}_y = \begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}_B$$

$$\underline{e}_y' = -\sin\varphi\,\underline{e}_x + \cos\varphi\,\underline{e}_y = \begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix}$$

$$\begin{pmatrix}\underline{e}_x'\\ \underline{e}_y'\end{pmatrix}_B = \underbrace{\begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}}_{\underline{\underline{R}}_{B\,B}} \begin{pmatrix}\underline{e}_x\\ \underline{e}_y\end{pmatrix}_B$$

6.6 Matrix calculations

(i) sum: $(\underline{\underline{A}}+\underline{\underline{B}})_{ij} = a_{ij}+b_{ij}$, $(\underline{\underline{A}}+\underline{\underline{B}})+\underline{\underline{C}} = \underline{\underline{A}}+(\underline{\underline{B}}+\underline{\underline{C}})$, $\underline{\underline{A}}+\underline{\underline{B}} = \underline{\underline{B}}+\underline{\underline{A}}$

(ii) $(\alpha \underline{\underline{A}})_{ij} = \alpha a_{ij}$

(iii) $\underline{\underline{A}} \in \mathbb{K}^{m\times n}$, $\underline{\underline{B}} = \mathbb{K}^{n\times p}$, $\underline{\underline{C}} = \mathbb{K}^{p\times q}$

$$\underline{\underline{A}} = \begin{pmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & & \vdots\\ a_{m1} & \cdots & a_{mn}\end{pmatrix} \quad \underline{\underline{B}} = \begin{pmatrix}b_{11} & \cdots & b_{1p}\\ \vdots & & \vdots\\ b_{n1} & \cdots & b_{np}\end{pmatrix} \quad \underline{\underline{C}} = \begin{pmatrix}c_{11} & \cdots & c_{1q}\\ \vdots & & \vdots\\ c_{p1} & \cdots & c_{pq}\end{pmatrix}$$

$\Rightarrow \underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{C}}$ matching dimensions!

$(\underline{\underline{A}}+\underline{\underline{B}})\underline{\underline{C}} = \underline{\underline{A}}\,\underline{\underline{C}} + \underline{\underline{B}}\,\underline{\underline{C}}$, but $\underline{\underline{A}}\,\underline{\underline{B}} \neq \underline{\underline{B}}\,\underline{\underline{A}}$ (most often)

$(\underline{\underline{A}}\,\underline{\underline{B}})\underline{\underline{C}} = \underline{\underline{A}}(\underline{\underline{B}}\,\underline{\underline{C}})$

$\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{0}} \not\Rightarrow \underline{\underline{A}} = 0$ or $\underline{\underline{B}} = 0$, e.g. $\underline{\underline{A}} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$; $\underline{\underline{B}} = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$

dot product: $\underset{\underset{n\times 1}{\uparrow}}{\underline{a}} \cdot \underline{b} = \underset{\underset{1\times n}{\uparrow}}{\underline{a}^T}\,\underline{b} = (a_1, \ldots, a_n)\begin{pmatrix}b_1\\ \vdots\\ b_n\end{pmatrix} = \sum_{i=1}^{n} a_i b_i$

$$\begin{pmatrix}w_1\\ \vdots\\ w_m\end{pmatrix} = \begin{pmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & & \vdots\\ a_{m1} & \cdots & a_{mn}\end{pmatrix}\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix} \Leftrightarrow \underline{w} = \underline{\underline{A}}\,\underline{v}$$

$$\left(\underline{\underline{A}}(\underline{\underline{B}}\,\underline{\underline{C}})\right)_{ij} = \sum_k a_{ik} (\underline{\underline{B}}\,\underline{\underline{C}})_{kj}$$

$$= \sum_k a_{ik} \sum_l b_{kl} c_{lj}$$

$$= \sum_l \underbrace{\left(\sum_k a_{ik} b_{kl}\right)}_{(\underline{\underline{A}}\,\underline{\underline{B}})_{il}} c_{lj}$$

$$= \left((\underline{\underline{A}}\,\underline{\underline{B}})\underline{\underline{C}}\right)_{ij}$$

- $\underline{\underline{A}}\,\underline{\underline{B}} \neq \underline{\underline{B}}\,\underline{\underline{A}}$ nicht kommutativ

  Bsp.: $\underline{\underline{A}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\underline{\underline{B}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

  $\hookrightarrow \underline{\underline{A}}\,\underline{\underline{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\underline{\underline{B}}\,\underline{\underline{A}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

  Kommutator $[\underline{\underline{A}}, \underline{\underline{B}}] := \underline{\underline{A}}\,\underline{\underline{B}} - \underline{\underline{B}}\,\underline{\underline{A}}$
  wichtig in Quantenmechanik

  $[\hat{x}, \hat{p}] = i\hbar \neq 0$

  Orts- Impulsoperator
  nicht gleichzeitig genau bestimmbar
  (Heisenbergsche Unschärferel.)

- nicht nullteilerfrei: $\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{0}}$ folgt nicht $\underline{\underline{A}} = \underline{\underline{0}}$ oder $\underline{\underline{B}} = \underline{\underline{0}}$

  Bsp: $\underline{\underline{A}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow \underline{\underline{A}}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ obwohl $\underline{\underline{A}} \neq \underline{\underline{0}}$

- Skalarprodukt im $\mathbb{R}^n$: $\underline{a} \cdot \underline{b} = (a_1 \dots a_n) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \underline{a}^T \underline{b} \in \mathbb{R}$

  transponiert

- Matrix mal Vektor = Vektor

$$\begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

$$\underline{v} = \underline{\underline{A}}\ \underline{u}$$

$$v_i = \sum_{k=1}^n a_{ik} u_k$$

- Matrixmultiplikation: Verkettung von 2 linearen Abbildungen

	$U \xrightarrow{L}$	$V \xrightarrow{M}$	W
Dim	n	m	l
Basis	A	B	C
Matrix	${}_BL_A \in K^{m\times n}$	${}_CM_B \in K^{l\times m}$	

$M \circ L : U \to W$

$\underline{u} \in U \mapsto \underline{w} = M(L(\underline{u})) = (M \circ L)(\underline{u}) \in W$

$$\underline{w}_C = {}_CM_B \left({}_BL_A\, \underline{u}_A\right)$$

$$= \underbrace{\left({}_CM_B\, {}_BL_A\right)}_{\in K^{l\times n}} \underline{u}_A$$

$$= (M \circ L)\ \underline{u}$$

6.7 Rang einer Matrix

lin. Abbildung $L: V \to W$ mit Matrix $\underline{\underline{M}}_L$:

- Zeilenrang := # linear unabhängiger Zeilenvektoren
- Spaltenrang := # —— " —— Spaltenvektoren

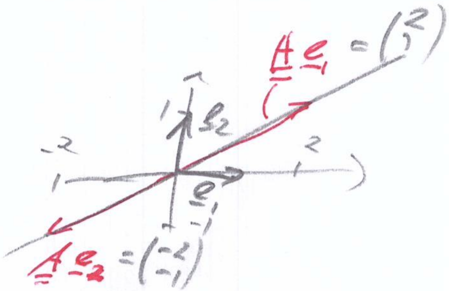
Es gilt: Zeilenrang = Spaltenrang =: Rang der Matrix M_L: $\text{rang}(M_L)$

$\text{Bild}(L) = \{L(\underline{u}) \mid \underline{u} \in V\} \subset W$

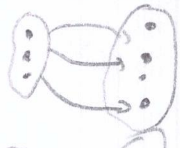
$\Rightarrow \text{rang}\, M_L = \dim(\text{Bild}(L))$

Bsp: (i) $\text{rang}\begin{pmatrix} 1 & 2 & 2 & 4 \\ 1 & 1 & -1 & -1 \\ 2 & 2 & -2 & -2 \end{pmatrix} = 2$ ↑ lin. abhängig

(ii) $\underline{\underline{A}} = \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} \in \mathbb{R}^{2\times 2} \Rightarrow \text{rang}\, \underline{\underline{A}} = 1$

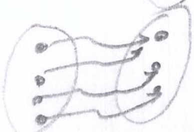


- $L: K^n \to K^m$ injektiv $\Leftrightarrow \text{rang}\, \underline{\underline{M}}_L = n$



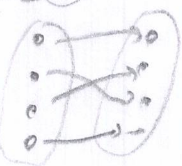
linkseindeutig

- $L: K^n \to K^m$ surjektiv $\Leftrightarrow \text{rang}\, \underline{\underline{M}}_L = m$



jedes Element hat ein Urbild

- $L: K^n \to K^m$ bijektiv $\Leftrightarrow n = m = \text{rang}(\underline{\underline{M}}_L)$



- $\underline{\underline{A}}^T$ (Transponierte) von $\underline{\underline{A}} \in \mathbb{R}^{m\times n}$: $(\underline{\underline{A}}^T)_{ij} = a_{ji} \Rightarrow \underline{\underline{A}}^T \in \mathbb{R}^{n\times m} \Rightarrow \text{rang}\, \underline{\underline{A}} = \text{rang}\, \underline{\underline{A}}^T$

- $\underline{\underline{C}}^+$ (Adjungierte von $\underline{\underline{C}} \in \mathbb{C}^{m\times n}$): $(\underline{\underline{C}}^+)_{ij} = c^*_{ji} \Rightarrow \text{rang}\, \underline{\underline{C}} = \text{rang}\, \underline{\underline{C}}^+$

- Wenn $\underline{\underline{A}}^T = \underline{\underline{A}}^{-1}$: orthogonal $\underline{\underline{A}}$

- wenn $\underline{\underline{C}}^+ = \underline{\underline{C}}^{-1}$: unitär $\underline{\underline{C}}$