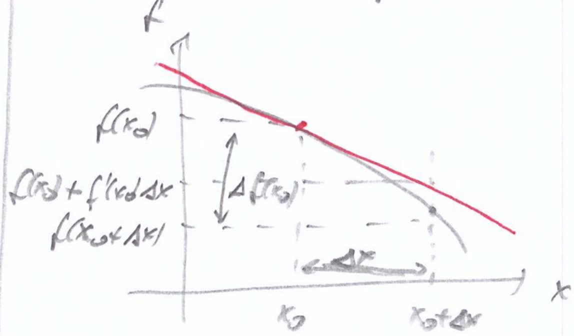
## 1.1 Introduction

## 1.2 Linear approximation of a function




idea: approximation of $f(x_0+\Delta x)$ for a given $f(x_0)$ such that the error is of order $O(\Delta x^2)$

$$\Rightarrow f(x_0+\Delta x) = f(x_0) + f'(x_0)\,\Delta x + O(\Delta x^2)$$

Def.: A function $f: D \to \mathbb{R}$, defined on an open set $D$, is said to be differentiable at $x_0 \in D$ if the derivative

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

exists.

## 1.2 Lineare Näherung (Fortsetzung)

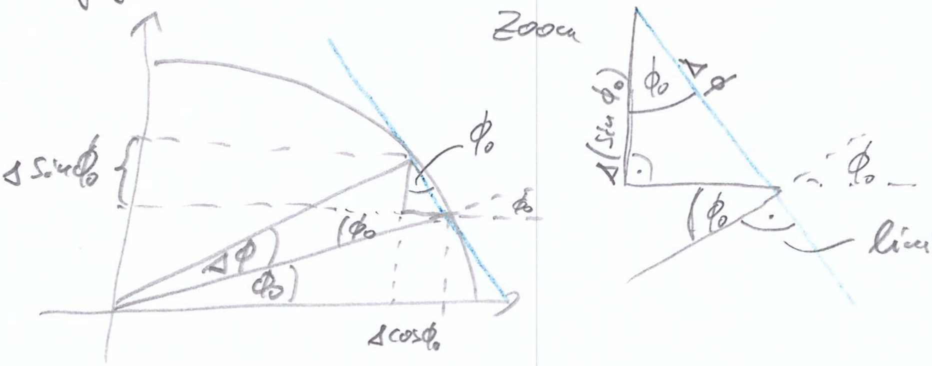
Notation in der Physik: (i) $f'(x_0) = \frac{d}{dx} f(x)\Big|_{x=x_0} = \frac{df}{dx}(x_0)$

bzw. $f'(x) = \frac{df}{dx}$

(ii) Ableitung nach der Zeit $t$: $f(t) \Rightarrow \dot{f}(t) = \frac{df}{dt}$

Bsp.: $\dot{z}(t) = \frac{dz}{dt}$

## 1.3 Example: derivative of the sine function by geometric means




$$\frac{d}{d\phi}\sin\phi_0 = \lim_{\Delta\phi\to 0} \frac{\Delta(\sin\phi)\big|_{\phi=\phi_0}}{\Delta\phi} = \cos\phi_0$$

## 1.4. Rules of differentiation

(i) linearity $\quad (\lambda f(x) + \mu g(x))' = \lambda f'(x) + \mu g'(x)$

(ii) product rule $\quad (f(x)\, g(x))' = f'(x)\, g(x) + f(x)\, g'(x)$

(iii) chain rule $\quad h(f(g(x))) = \left.\frac{df}{dy}\right|_{y=g(x)} \frac{dg}{dx} = \frac{df}{dg}\frac{dg}{dx}$

(iv) quotient rule $\quad \left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\, g(x) - f(x)\, g'(x)}{g(x)^2}$

(v) derivative of an inverse function $\quad (f^{-1}(y))' = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(y)}}$

## 1.5 Important derivatives

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cos x - \sin x\, (\cos x)'}{(\cos x)^2} = \frac{\cos x \cos x - \sin x\,(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$= \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = 1 + \tan^2 x$$

## 1.6 Taylor expansion

idea: approximate $f(x)$ by a polynomial involving its derivatives

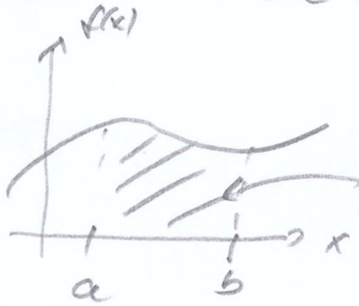
Taylor's theorem: Any $f \in C^{n+1}(D)$ defined on an open set $D \subset \mathbb{R}$ can be expressed by

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)\,(x-x_0)^k}_{\text{Taylor polynomial}} + R_n(x)$$

with $x, x_0 \in D$ and the remainder term $R_n(x) = \dfrac{f^{(n+1)}(\overbrace{\xi}^{\text{between } x \text{ and } x_0})}{(n+1)!}(x-x_0)^{n+1}$

## 1.7 1D integrals

(i) area under the curve/function



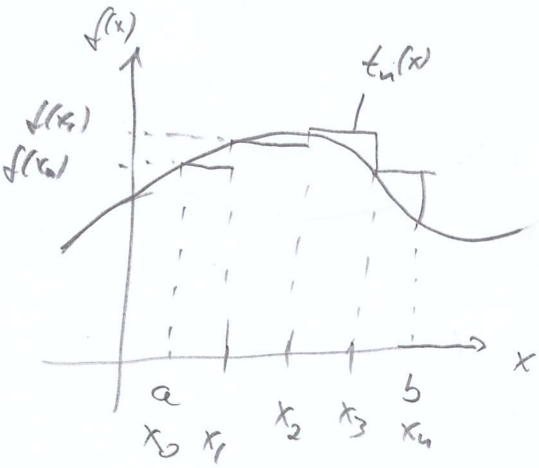

$$F = \int_a^b f(x)\,dx$$

(ii) application: antiderivative

$$x(t_0) + \int_{t_0}^{t} v(s)\,ds = x(t) \underset{\int dt}{\overset{\frac{d}{dt}}{\rightleftarrows}} \dot{x} = v(t) = \frac{d}{dt}x(t)$$

(iii) Cauchy integral

idea: approximation of $F$ by step functions




spacing of meshpoints: $\Delta x = \frac{b-a}{n}$

$$F_n = \int_a^b t_n(x)\,dx = \sum_{i=0}^{n-1} f(x_i)\,\Delta x$$

Integral:

$$F = \lim_{n\to\infty} F_n = \lim_{n\to\infty} \int_a^b t_n(x)\,dx$$

## 1.8 Fundamental theorem of calculus

(i) Let $f$ be a continuous real-valued function on a closed interval $I$. Then, for all $x_0 \in I$, the function $F: I \to \mathbb{R}$ defined as $F(x) = \int_{x_0}^{x} f(\tilde{x}) d\tilde{x}$ is differentiable with $F'(x) = f(x)$.

(ii) Let $f: [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$ and $F$ an antiderivative in $(a,b)$. Then, $\int_a^b f(x)\,dx = F(b) - F(a)$

- antiderivative not unique (differ by a constant)
- $F(x) = \int f(x) dx$ indefinite integral

## 1.9 Rules of integration

(i) same boundaries / limits of integration: $\int_a^a f(x) dx = 0$

(ii) swapping boundaries: $\int_a^b f(x)\,dx = -\int_b^a f(x) dx$

(iii) additivity of boundaries: $\int_a^b f(x) dx + \int_b^c f(x)\,dx = \int_a^c f(x) dx$

(iv) linearity: $\int_a^b (\lambda f(x) + \mu g(x)) dx = \lambda \int_a^b f(x) dx + \mu \int_a^b g(x) dx$

(v) partial integration: $\int_a^b f(x) g(x) dx = f(x) g(x)\Big|_a^b - \int_a^b f(x) g'(x) dx$

(vi) substitution:

$$\int_a^b f(x)\,dx = \int_{u^{-1}(a)}^{u^{-1}(b)} f(u(t)) \frac{du}{dt} dt\,, \quad \int_a^b f(u(x)) \frac{du}{dx} dx = \int_{u(a)}^{u(b)} f(t) dt$$

1.10 Examples & tricks

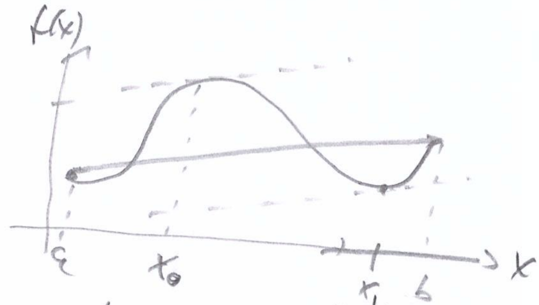
## 1.11 Improper integrals



- integration of functions with singularities
  or unbounded limits
- sometimes, it works, but sometimes, it does not

## 1.12 Mean value theorems

- Differentiation: Let $f:[a,b]\to\mathbb{R}$ be a continuous function on the closed interval $[a,b]$ and differentiable on $(a,b)$.
  Then, there exists some $x_0 \in (a,b)$ such that
$$f'(x_0) = \frac{f(b)-f(a)}{b-a}.$$




$$f'(x_1) = f'(x_0) = \frac{f(b)-f(a)}{b-a}$$

- integration: Let $f:[a,b]\to\mathbb{R}$ be a continuous function.
  Then, there exists a $x_0 \in [a,b]$ such that
$$\int_a^b f(x)\,dx = f(x_0)(b-a).$$

↳ equal area rule (Maxwell construction)

# 2. Multidimensional analysis

## 2.1. Partial derivatives

Let $f: \mathbb{R}^n \to \mathbb{R}$, $\underline{x} \mapsto f(x_1, \dots, x_n)$ be a function on $\mathbb{R}^n$ (or an open subset $\Omega \subset$ ...

Then, the partial derivative of $f$ with respect to the $i$-th variable is defined as

$$\frac{\partial f}{\partial x_i} := \lim_{\Delta x_i \to 0} \frac{f(x_1, \dots, x_i + \Delta x_i, \dots, x_n) - f(x_1, \dots, x_n)}{\Delta x_i}$$

notation $\frac{\partial f}{\partial x_i} \equiv \partial_{x_i} f \equiv f_{x_i} \equiv D_{x_i} f$

Schwarz's theorem: (Clairaut's) For $f: \Omega \to \mathbb{R}$ defined on $\Omega \subset \mathbb{R}^n$, if $\underline{p} \in \Omega$ and $f$ has continuous second partial derivatives on a neighbourhood of $\underline{p}$, then

$$\frac{\partial^2}{\partial x_i \partial x_j} f(\underline{p}) = \frac{\partial^2}{\partial x_j \partial x_i} f(\underline{p}) \quad \forall\, i, j \in \{1, \dots, n\}$$

## 2.2 Total differential

The sum of all partial differentials w.r.t. all independent variables $x_1, \dots, x_n$ is called total differential:

$$df = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

## 2.3 Total differentiation and rules of differentiation

- chain rule: $f: \mathbb{R}^n \to \mathbb{R}$, $f = f(x_1, \dots, x_n)$ with $x_i = x_i(t)$.

  Then: $\frac{df}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial t}$

## 2.4 Gradient

The gradient of a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ at $\underline{a} \in \Omega$ is defined as

$$\operatorname{grad} f(\underline{a}) = \begin{pmatrix} \partial_{x_1} f(\underline{a}) \\ \vdots \\ \partial_{x_n} f(\underline{a}) \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} f(\underline{a})$$

- Using vector notation, the total differential can be rewritten as

$$df(\underline{a}) = \operatorname{grad} f(\underline{a}) \underset{\substack{\uparrow \\ \text{scalar} \\ \text{product}}}{\bullet} d\underline{x} \quad \text{with } d\underline{x} = \begin{pmatrix} dx_1 \\ \vdots \\ dx_n \end{pmatrix}$$

$$= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(\underline{a}) \bullet dx_i$$

- notation: $\operatorname{grad} f(\underline{a}) \equiv \underline{\nabla} f(\underline{a})$ with the Nabla operator $\underline{\nabla} = \begin{pmatrix} \partial_{x_1} \\ \vdots \\ \partial_{x_n} \end{pmatrix}$

  $\hookrightarrow df(\underline{a}) = \underline{\nabla} f(\underline{a}) \bullet d\underline{x}$

- geometric intuition: (i) $\underline{\nabla} f(\underline{a})$ points in the direction of steepest slope of $f$ at $\underline{a}$.

  (ii) $$\underset{\substack{\uparrow \\ \text{change of } f}}{\Delta f} = \underline{\nabla} f \cdot \Delta \underline{x} = |\underline{\nabla} f| \cdot |\Delta \underline{x}| \cos \underset{\substack{\uparrow \\ \text{angle between } \underline{\nabla} f \text{ and } \Delta \underline{x}}}{\Theta}$$

## 2.5 High(er)dimensional Taylor expansion

reminder: $f: \mathbb{R} \to \mathbb{R}$: $f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0) \underbrace{(x-x_0)^k}_{\Delta x^k}$

generalization to $f: \mathbb{R}^n \to \mathbb{R}$ using vector notation:

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \Delta x = \begin{pmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{pmatrix}, \quad x_0 \in \mathbb{R} \longrightarrow \underline{x}_0 \in \mathbb{R}^n, \quad \frac{d}{dx} \to \underline{\nabla} = \begin{pmatrix} \partial_{x_1} \\ \vdots \\ \partial_{x_n} \end{pmatrix}$$

$$\Rightarrow f(\underline{x}) = \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{\left( \sum_{i=1}^{n} \Delta x_i \frac{\partial}{\partial x_i} \right)^k}_{(\Delta \underline{x} \bullet \underline{\nabla})^k} f(\underline{x}_0)$$

$\uparrow$ scalar product

up to order 2: $k = 0, 1, 2$

transposed: $\Delta \underline{x}^T = (x_1, \dots, x_n)$

$$f(\underline{x}) = f(\underline{x}_0) + \underline{\nabla} f(\underline{x}_0) \cdot \Delta \underline{x} + \frac{1}{2} \Delta \underline{x}^T \underline{\underline{H}}(\underline{x}_0) \Delta \underline{x} + O(|\Delta \underline{x}|^3)$$

with the Hessian / Hesse matrix (of $2^{nd}$ derivatives)

$$\underline{\underline{H}}(\underline{x}_0) = \begin{pmatrix} \partial_{x_1} \partial_{x_1} f(\underline{x}_0) & \cdots & \partial_{x_1} \partial_{x_n} f(\underline{x}_0) \\ \vdots & & \vdots \\ \partial_{x_n} \partial_1 f(\underline{x}_0) & \cdots & \partial_{x_n} \partial_{x_n} f(\underline{x}_0) \end{pmatrix}$$

## 2.6 Extreme values

reminder: $f: \mathbb{R} \to \mathbb{R}$ with extreme value/extremum at $a$: $f'(a)=0$

↳ $f''(a)>0$ ☺ : Minimum (positive curvature)

↳ $f''(a)<0$ ☹ : Maximum (negative curvature)

↳ $f''(a)=0$ : anything is possible (including saddles)

generalization to $f: \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto f(x,y)$

$$df(\underline{a}) = \frac{\partial}{\partial x} f(\underline{a})\, dx + \frac{\partial}{\partial y} f(\underline{a})\, dy = 0 \Rightarrow \frac{\partial}{\partial x} f(\underline{a}) = 0 \wedge \frac{\partial f}{\partial y}(\underline{a}) = 0$$

↳ Minimum: $\partial_x^2 f(\underline{a}) > 0$, $\partial_y^2 f(\underline{a}) > 0$, $D := (\partial_x^2 f)(\partial_y^2 f) - (\partial_x \partial_y f)^2 > 0$

$\partial_x \partial_x f = \frac{\partial^2}{\partial x^2} f$

↳ Maximum: $\partial_x^2 f(\underline{a}) < 0$, $\partial_y^2 f(\underline{a}) < 0$, $D > 0$

↳ Saddle: $D < 0$

↳ unclear (higher orders needed): $D = 0$

$\boxed{\begin{array}{l} n>2: \\ df(\underline{a}) = 0 \\ \text{Eigenvalues of } \underline{H}(\underline{a}): \\ \hookrightarrow \text{all negative: Maximum} \\ \hookrightarrow \text{all positive: Minimum} \end{array}}$

## 2.7 Extremum with constraints

idea: extremum of $f(x,y)$ for a constraint $g(x,y)=0$

↳ $\underline{\nabla} f(x,y) = -\lambda \underline{\nabla} g(x,y)$ (parallel gradients)

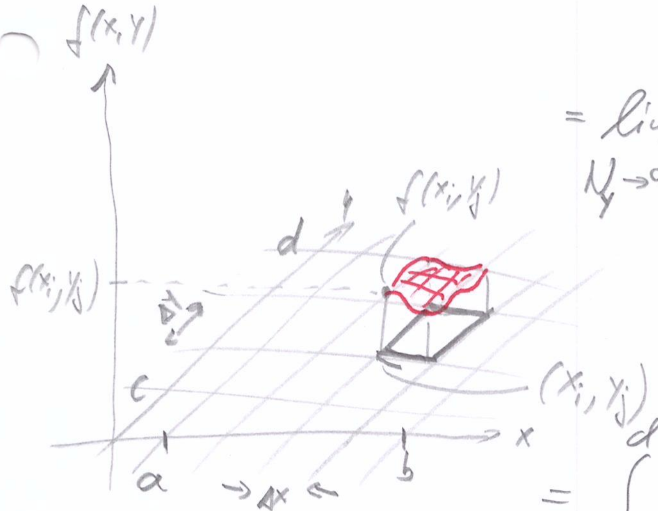
Lagrange multiplier

## 2.8 Integration (of scalar functions) in $\mathbb{R}^n$

**Def.:** The 2-dimensional integral of $f: \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ (continuous) over $\Omega = [a,b] \times [c,d]$ is defined as:

$$\int_\Omega f(x,y)\,dx\,dy := \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \sum_{j=0}^{\frac{d-c}{\Delta y}-1} \sum_{i=0}^{\frac{b-a}{\Delta x}-1} f(x_i, y_j)\,\Delta x\,\Delta y .$$

$$\left( \begin{array}{llll} \text{with} & x_i = a + i\,\Delta x & \text{and} & N_x = \frac{b-a}{\Delta x} \\ & y_j = c + j\,\Delta y & & N_y = \frac{d-c}{\Delta y} \end{array} \right)$$




$$= \lim_{N_y \to \infty} \lim_{N_x \to \infty} \sum_{j=0}^{N_y - 1} \sum_{i=0}^{N_x - 1} f(x_i, y_j)\,\Delta x\,\Delta y$$

volume of cuboid: $f(x_i, y_j)\,\Delta x\,\Delta y$

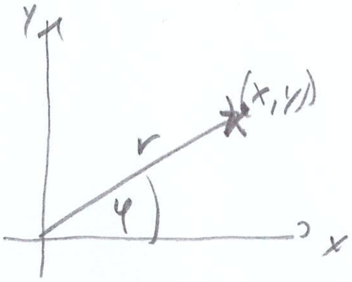
$$= \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Fubini's theorem:

$$\int_\Omega f(x,y)\,dx\,dy = \int_c^d \left[ \int_a^b f(x,y)\,dx \right] dy = \int_a^b \left[ \int_c^d f(x,y)\,dy \right] dx$$

## 2.9 Coordinate transforms

### 2.9.1 Polar coordinates in $\mathbb{R}^2$

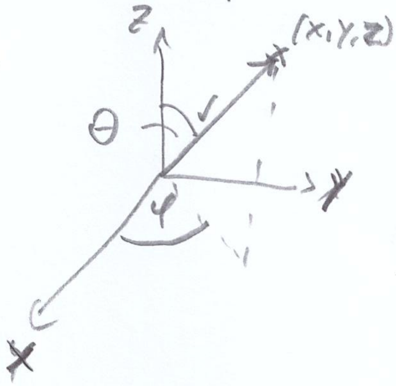



$$x = x(r,\varphi) = r\cos\varphi, \quad y = y(r,\varphi) = r\sin\varphi, \qquad r \in [0,\infty),\ \varphi \in [0,2\pi)$$

$$\Rightarrow \underline{\underline{J}} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$$

$\Rightarrow$ area element: $dA = \det \underline{\underline{J}}\, dr\, d\varphi = r\, dr\, d\varphi$

### 2.9.2 Spherical coordinates

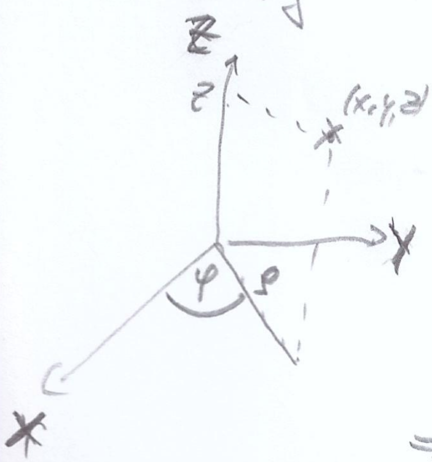



$$x = x(r,\Theta,\varphi) = r\sin\Theta\cos\varphi$$
$$y = y(r,\Theta,\varphi) = r\sin\Theta\sin\varphi, \qquad r\in[0,\infty),\ \Theta\in[0,\pi],\ \varphi\in[0,2\pi)$$
$$z = z(r,\Theta,\varphi) = r\cos\Theta$$

$$\Rightarrow \underline{\underline{J}} = \begin{pmatrix} \sin\Theta\cos\varphi & r\cos\Theta\cos\varphi & -r\sin\Theta\sin\varphi \\ \sin\Theta\sin\varphi & r\cos\Theta\sin\varphi & r\sin\Theta\cos\varphi \\ \cos\Theta & -r\sin\Theta & 0 \end{pmatrix}$$

$\Rightarrow$ volume element: $dV = r^2 \sin\Theta\, dr\, d\Theta\, d\varphi$

### 2.9.3 Cylindrical coordinates




$$x = x(\rho,\varphi) = \rho\cos\varphi$$
$$y = y(\rho,\varphi) = \rho\sin\varphi$$
$$z = z = z$$

$$\Rightarrow \underline{\underline{J}} = \begin{pmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow \quad dV = \rho\, d\rho\, dz\, d\varphi$

## 3.1 Scalars

scalars = numbers (plus unit)

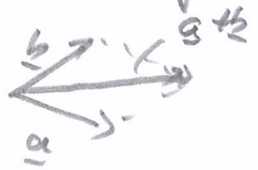
## 3.2 Vectors in Cartesian spaces

- vector: $\underline{a}$ shift in $\mathbb{R}^n$ (most often forces: $n=3$)
  ↳ length/magnitude $|\underline{a}| = \|\underline{a}\| \equiv a$ and direction
- Cartesian bases $\{\underline{e}_1, \underline{e}_2, \underline{e}_3\}$:

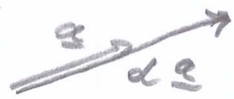
Cartesian coordinates

$$\underline{a} = a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3 = \sum_{i=1}^{3} a_i \underline{e}_i = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

- adding 2 vectors




- multiplying a vector with a scalar




## 3.3 Dot product (scalar product)

- The dot product of 2 vectors $\underline{a}, \underline{b} \in \mathbb{R}^3$ with Cartesian coordinates $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, respectively, is defined as a algebraic operation:

$$(\circ, \circ): \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$$

$$(\underline{a}, \underline{b}) \longmapsto \underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

- The length/magnitude/norm of a vector is defined as

$$a \equiv \|\underline{a}\| \equiv |\underline{a}| := \sqrt{\underline{a} \cdot \underline{a}} = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\sum_{i=1}^{3} a_i^2}$$

- normalized vector: $\hat{\underline{a}} = \dfrac{\underline{a}}{a}$

- properties of the dot product:
  (i) bilinear: $\alpha \in \mathbb{R}$, $(\alpha \underline{a}, \underline{b}) = \alpha (\underline{a}, \underline{b})$
  $(\underline{a}, \underline{b} + \underline{c}) = (\underline{a}, \underline{b}) + (\underline{a}, \underline{c})$
  (ii) symmetric: $(\underline{a}, \underline{b}) = (\underline{b}, \underline{a})$
  (iii) positive definit: $(\underline{a}, \underline{a}) \geq 0 \wedge (\underline{a}, \underline{a}) = a^2 = 0 \Leftrightarrow \underline{a} = \underline{0}$

- geometric interpretation:

$\underline{c} = \underline{b} - \underline{a}$

$$\underline{c}\cdot\underline{c} = |\underline{c}|^2 = c^2 = (\underline{b}-\underline{a})\cdot(\underline{b}-\underline{a})$$

$$= b^2 + a^2 - 2\,\underline{a}\cdot\underline{b}$$

Law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\varphi$

$\Big\}\ \underline{a}\cdot\underline{b} = ab\cos\varphi$

$\Rightarrow$ for $\underline{a},\underline{b} \neq \underline{0}$: $\underline{a}\cdot\underline{b} = 0 \Leftrightarrow \underline{a}\perp\underline{b}$ $\quad (\varphi = \pm\frac{\pi}{2})$

$\Rightarrow$ orthogonality

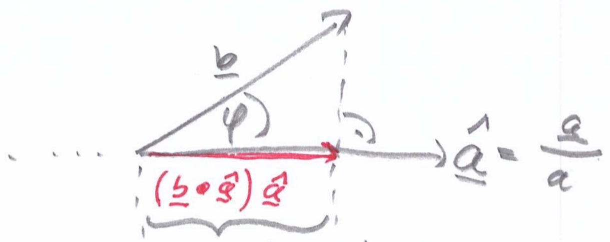
$$\Rightarrow \varphi = \arccos\frac{\underline{a}\cdot\underline{b}}{ab} = \arccos\, \underline{\hat{a}}\cdot\underline{\hat{b}}$$

# 3.3 Dot product (continued)

$(\cdot,\cdot): \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ (most often $n=3$)

$(\underline{a},\underline{b}) \longmapsto \underline{a}\cdot\underline{b} \equiv \langle \underline{a},\underline{b}\rangle = a_1 b_1 + \dots + a_n b_n$

projection of $\underline{b}$ onto $\underline{a}$:




length: $|\underline{b}\cdot\hat{\underline{a}}| = b\cos\varphi$

orthonormal basis: $\{\underline{e}_1,\dots,\underline{e}_n\}$ with $\underline{e}_i\cdot\underline{e}_j=\delta_{ij}$ (orthogonal)

and $|\underline{e}_i|=1$ (normalized)

## 3.4 Cross product

The crossproduct of 2 vectors $\underline{a},\underline{b}\in\mathbb{R}^3$ is defined as

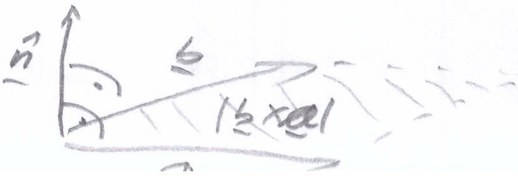
$(\cdot,\cdot): \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$

$$(\underline{a},\underline{b}) \longmapsto \underline{a}\times\underline{b} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

Properties:

- antisymmetric: $\underline{a}\times\underline{b} = -\underline{b}\times\underline{a}$
- bilinear : $\underline{a}\times(\alpha\underline{b}+\beta\underline{c}) = \alpha(\underline{a}\times\underline{b})+\beta(\underline{a}\times\underline{c})$, $\alpha,\beta\in\mathbb{R}$, $\underline{a},\underline{b},\underline{c}\in\mathbb{R}^3$
- not associative: $\underline{a}\times(\underline{b}\times\underline{c}) \neq (\underline{a}\times\underline{b})\times\underline{c}$
- $(\underline{a}\times\underline{b})_i = \sum_{j,k=1}^{3} \varepsilon_{ijk}\, a_j b_k$ with Levi-Civita-Symbol $\varepsilon_{ijk}$
- geometric interpretation (i) $\underline{a}\times\underline{b} \perp \underline{a} \wedge \underline{a}\times\underline{b} \perp \underline{b}$

  (ii) $|\underline{a}\times\underline{b}|$ = area of parallelogram spanned by $\underline{a},\underline{b}$
- $\underline{a}\times\underline{b} = \underline{0} \iff \underline{a} \parallel \underline{b}$
- $\underline{a}\times\underline{b} = |a|\,|b| \sin\varphi\, \hat{\underline{n}}$



# 3.5 Levi-Civita-Symbol

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{for } ijk \text{ even permutation of } (1,2,3) \\ -1 & \text{—"— odd —"—} \\ 0 & \text{otherwise} \end{cases}$$

| | i=1 | | | i=2 | | | i=3 | | |
|---|---|---|---|---|---|---|---|---|---|
| j \ k | 1 | 2 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 1 | 0 | 0 | 0 | 0 | 0 | -1 | 0 | 1 | 0 |
| 2 | 0 | 0 | 1 | 0 | 0 | 0 | -1 | 0 | 0 |
| 3 | 0 | -1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |

## 3.6 Products involving 3 vectors / triple product

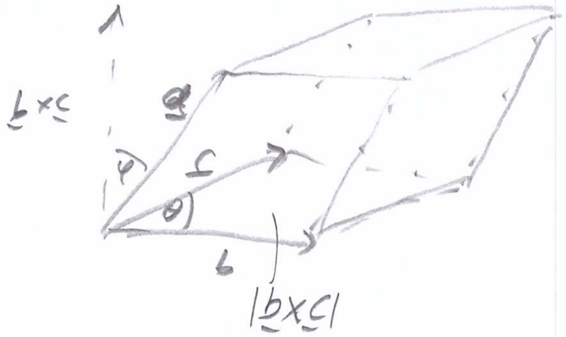
(i) bac-cab rule : $\underline{a} \times (\underline{b} \times \underline{c}) = \underline{b}\,(\underline{a}\cdot\underline{c}) - \underline{c}\,(\underline{a}\cdot\underline{b})$

"bac minus cab" rule

(ii) (Scalar) triple product : $\underline{a} \cdot (\underline{b} \times \underline{c})$ = volume of parallelepiped

box product

spanned by $\underline{a}, \underline{b}, \underline{c}$




$= \underline{b} \cdot (\underline{c} \times \underline{a}) = \underline{c} \cdot (\underline{a} \times \underline{b})$

$= \det \begin{pmatrix} | & | & | \\ \underline{a} & \underline{b} & \underline{c} \\ | & | & | \end{pmatrix}$

$= \sum_{ijk=1}^{3} \varepsilon_{ijk}\, a_i\, b_j\, c_k$

(iii) Lagrange's identity :

$(\underline{a} \times \underline{b}) \cdot (\underline{c} \times \underline{d}) = (\underline{a}\cdot\underline{c})\,(\underline{b}\cdot\underline{d}) - (\underline{a}\cdot\underline{d})\,(\underline{b}\cdot\underline{c})$

left right outer inner

# 4. Vector calculus / analysis

## 4.1 (Conservative) vector fields

- Vectorfield: $\underline{v} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, $\underline{x} \mapsto \underline{v}(\underline{x}) = \begin{pmatrix} v_1(\underline{x}) \\ \vdots \\ v_n(\underline{x}) \end{pmatrix}$

  $\uparrow$ potentially $\dim \Omega < n$

- $C^s$-vectorfields: $v_1, \dots, v_n$ $s$-times continuously differentiable.

- Conservative vectorfield: $\underline{v}$ is gradient of scalar potential $\phi$

$$\underline{v}(\underline{x}) = \underline{\nabla} \phi(\underline{x})$$

## 4.2 Derivations of vectorfields

- (Rotation) curl of $\underline{v}(\underline{x}) = \begin{pmatrix} v_1(\underline{x}) \\ v_2(\underline{x}) \\ v_3(\underline{x}) \end{pmatrix}$:

$$\operatorname{rot} \underline{v} \equiv \underline{\nabla} \times \underline{v} := \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{pmatrix}$$

$$(\operatorname{rot} \underline{v})_i = \sum_{jk=1}^{3} \varepsilon_{ijk} \partial_j v_k$$

- Divergence of $\underline{v}(\underline{x})$: $\operatorname{div} \underline{v}(\underline{x}) = \underline{\nabla} \cdot \underline{v} := \frac{\partial}{\partial x_1} v_1 + \frac{\partial}{\partial x_2} v_2 + \frac{\partial}{\partial x_3} v_3$

## 4.2 Derivations of vector fields

(iii) Laplace-Operator: (a) $\Delta\phi := \operatorname{div}\operatorname{grad}\phi = \underline{\nabla}\cdot(\underline{\nabla}\phi)$ scalar field potential

$$= \frac{\partial^2}{\partial x^2}\phi + \frac{\partial^2}{\partial y^2}\phi + \frac{\partial^2}{\partial z^2}\phi$$

(b) $\Delta\underline{A} = \frac{\partial^2}{\partial x^2}\underline{A} + \frac{\partial^2}{\partial y^2}\underline{A} + \frac{\partial^2}{\partial z^2}\underline{A} \overset{?}{=} \operatorname{grad}\operatorname{div}\underline{A} - \operatorname{rot}\operatorname{rot}\underline{A}$

$$= \underline{\nabla}(\underline{\nabla}\cdot\underline{A}) - \underline{\nabla}\times(\underline{\nabla}\times\underline{A})$$

- rot, grad, div : linear operators
- product rules: (a) $\operatorname{div}(\phi\underline{v}) = \phi\operatorname{div}\underline{v} + (\operatorname{grad}\phi)\cdot\underline{v}$

$$\underline{\nabla}\cdot(\phi\underline{v}) = \phi(\underline{\nabla}\cdot\underline{v}) + (\underline{\nabla}\phi)\cdot\underline{v}$$

(b) $\operatorname{rot}(\phi\underline{A}) = \phi\operatorname{rot}\underline{A} + \operatorname{grad}\phi\times\underline{A}$

$$= \phi(\underline{\nabla}\times\underline{A}) + (\underline{\nabla}\phi)\times\underline{A}$$

## 4.3 Gradient field and curl

Theorem: (i) Gradient fields have no curl

$\underline{v} = \underline{\nabla}\phi \Rightarrow \operatorname{rot}\underline{v} = \underline{0}$

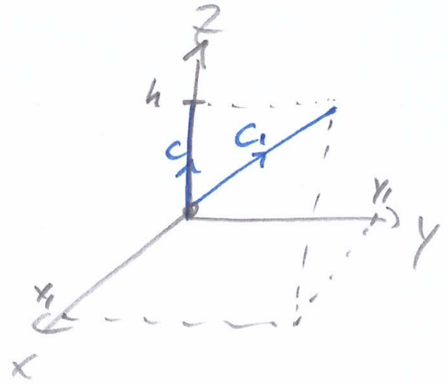
(ii) $\operatorname{rot}\underline{v} = \underline{0} \Rightarrow \underline{v} = \underline{\nabla}\phi$

$\Big\}$ $\underline{v}$ conservative $\Leftrightarrow \operatorname{rot}\underline{v} = 0$

Theorem: (i) $\underline{v} = \operatorname{rot}\underline{A} \Rightarrow$ no sink / no source $(\operatorname{div}\underline{v} = 0)$

(ii) $\operatorname{div}\underline{v} = 0 \Rightarrow \underline{v} = \operatorname{rot}\underline{A}$

⊛ mechanische Arbeit entlang eines Weges C:




Parametrisierung von C: $\underline{\alpha}(t) = ht\,\hat{\underline{e}}_z$, $t \in [0,1]$

—— " —— $C_1$: $\underline{\beta}(t) = ht\,\hat{\underline{e}}_z + x_1 t\,\hat{\underline{e}}_x + y_1 t\,\hat{\underline{e}}_y$

$$W = \int_C \underline{F} \cdot d\underline{s} = \ldots = -mgh \quad (\text{s. S. 53})$$

$$W_1 = \int_{C_1} \underline{F} \cdot d\underline{s} = \int_0^1 (-mg)\,\hat{\underline{e}}_z \cdot \dot{\underline{\beta}}(t)\,dt$$

$$= \int_0^1 dt\,(-mg)\left[\hat{\underline{e}}_z \cdot h\,\hat{\underline{e}}_z + \underbrace{\hat{\underline{e}}_z \cdot x_1\,\hat{\underline{e}}_x}_{=0} + \underbrace{\hat{\underline{e}}_z \cdot y_1\,\hat{\underline{e}}_y}_{=0}\right]$$

$\hat{\underline{e}}_z \perp \hat{\underline{e}}_x$, $\hat{\underline{e}}_z \perp \hat{\underline{e}}_y$

oder auch $\hat{\underline{e}}_i \cdot \hat{\underline{e}}_j = \delta_{ij}$

$$= \int_0^1 dt\,(-mg)\,h = -mgh = W$$

gilt auch für noch kompliziertere Wege von $z=0$ auf Höhe $h$.

## 4.4 (Parameterization of) curves

- curve $C$ (set of points in $\mathbb{R}^n$): parameterized by a function

$$\underline{\alpha} : \underbrace{[a,b]}_{I} \to \mathbb{R}^n , \quad t \mapsto \underline{\alpha}(t) = \begin{pmatrix} \alpha_1(t) \\ \vdots \\ \alpha_n(t) \end{pmatrix}$$

- Derivative: $\dot{\underline{\alpha}}(t) = \begin{pmatrix} \dot{\alpha}_1(t) \\ \vdots \\ \dot{\alpha}_n(t) \end{pmatrix}$ vector tangential on curve $C$ at $\underline{\alpha}(t)$

- regular curve: $\dot{\underline{\alpha}}(t) \neq \underline{0} \quad \forall t \in I$

## 4.5 Length of a curve

- length of $C$: $L(C) = \int_a^b |\dot{\underline{\alpha}}(t)| \, dt$

## 4.5 Integration along curves

- scalar $\left( \begin{array}{l} f : \mathbb{R}^n \to \mathbb{R} \\ \underline{x} \mapsto f(\underline{x}) \end{array} \right)$: $\int_C f(\underline{x}) \, ds = \int_a^b f(\underline{\alpha}(t)) |\dot{\underline{\alpha}}(t)| \, dt$

- vectorial $\left( \begin{array}{l} \underline{v} : \mathbb{R}^n \to \mathbb{R}^n \\ \underline{x} \mapsto \underline{v}(\underline{x}) \end{array} \right)$: $\int_C \underline{v}(\underline{x}) \cdot d\underline{s} = \int_a^b \underline{v}(\underline{\alpha}(t)) \cdot \dot{\underline{\alpha}}(t) \, dt$

## 4.7 Parametrization of areas

$[a,b]\times[c,d]$

- area $F$ : parametrized by $\underline{\Phi}: B \to \mathbb{R}^3$

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \underline{x} = \underline{\Phi}(u,v)$$

- tangential vectors: $\dfrac{\partial \underline{\Phi}}{\partial u}, \dfrac{\partial \underline{\Phi}}{\partial v}$

$$\left| \frac{\partial \underline{\Phi}}{\partial u} \times \frac{\partial \underline{\Phi}}{\partial v} \right|$$

## 4.8 Integration over areas

- scalar $\left( \begin{array}{l} f: \mathbb{R}^4 \to \mathbb{R} \\ \underline{x} \mapsto f(\underline{x}) \end{array} \right)$ : $\displaystyle\int_F f(\underline{x})\, dA = \int_B f(\underbrace{\underline{x}(u,v)}_{\underline{\Phi}(u,v)}) \underbrace{\left| \frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} \right|}_{\left| \frac{\partial \underline{\Phi}}{\partial u} \times \frac{\partial \underline{\Phi}}{\partial v} \right|} du\,dv$

- recap: spherical coordinates: $r = R$

$$\underline{\Phi}: (\theta, \varphi) \longmapsto \begin{pmatrix} R \sin\theta \cos\varphi \\ R \sin\theta \sin\varphi \\ R \cos\theta \end{pmatrix}, \quad \theta \in [0,\pi],\ \varphi \in [0, 2\pi)$$

$$\Rightarrow B = [0,\pi] \times [0, 2\pi)$$

$$\Rightarrow \frac{\partial \underline{x}}{\partial \theta} = \begin{pmatrix} R\cos\theta \cos\varphi \\ R\cos\theta \sin\varphi \\ -R \sin\varphi \end{pmatrix} = R\, \vec{\underline{e}}_\theta$$

$$\frac{\partial \underline{x}}{\partial \varphi} = \begin{pmatrix} -R\sin\theta \sin\varphi \\ R \sin\theta \cos\varphi \\ 0 \end{pmatrix} = R \sin\theta\, \vec{\underline{e}}_\varphi$$

$$\Rightarrow \frac{\partial \underline{x}}{\partial \theta} \times \frac{\partial \underline{x}}{\partial \varphi} = \ldots = R^2 \sin\theta \begin{pmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{pmatrix} = R^2 \sin\theta\, \vec{\underline{e}}_r$$

$$\Rightarrow \left| \frac{\partial \underline{x}}{\partial \theta} \times \frac{\partial \underline{x}}{\partial \varphi} \right| = \ldots = R^2 \sin\theta$$

- $\{\vec{\underline{e}}_\theta, \vec{\underline{e}}_\varphi, \vec{\underline{e}}_r\}$ : orthonormal basis of $\mathbb{R}^3$ (equivalent to $\vec{\underline{e}}_x, \vec{\underline{e}}_y, \vec{\underline{e}}_z$

Parametrization using, e.g., spherical coordinates:

$$\underline{\Phi}: (\theta, \varphi) \longmapsto \begin{pmatrix} R \sin\theta \cos\varphi \\ R \sin\theta \sin\varphi \\ R\cos\theta \end{pmatrix}, \quad \theta \in [0,\pi],\ \varphi \in [0, 2\pi)$$

$\hookrightarrow B = [0,\pi] \times [0, 2\pi)$

$\hookrightarrow$ unit vectors : $\{\underline{\hat{e}}_\theta, \underline{\hat{e}}_\varphi, \underline{\hat{e}}_r\}$

$$\left.\begin{aligned} \underline{\hat{e}}_\theta &= \begin{pmatrix} \cos\theta \cos\varphi \\ \cos\theta \sin\varphi \\ -\sin\varphi \end{pmatrix} \\ \underline{\hat{e}}_\varphi &= \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \\ \underline{\hat{e}}_r &= \begin{pmatrix} \sin\theta \cos\varphi \\ \sin\theta \sin\varphi \\ \cos\theta \end{pmatrix} \end{aligned}\right\}$$

derived via $\frac{\partial \underline{x}}{\partial \theta}, \frac{\partial \underline{x}}{\partial \varphi}, \frac{\partial \underline{x}}{\partial \theta} \times \frac{\partial \underline{x}}{\partial \varphi}$ and normalization

surface/area element: $\left| \frac{\partial \underline{x}}{\partial \theta} \times \frac{\partial \underline{x}}{\partial \varphi} \right| = R^2 \sin\theta$

## 4.8 Integration over areas

Surface integral of vector fields: $\underline{\Phi}: B \to \mathbb{R}^3, (u,v) \mapsto \underline{x} = \underline{\Phi}(u,v)$

$$\int_F \underline{v}(\underline{x}) \cdot d\underline{A} := \int_B \underline{v}(\underline{x}(u,v)) \cdot \underbrace{\left( \frac{\partial \underline{x}}{\partial u} \times \frac{\partial \underline{x}}{\partial v} \right) du\, dv}_{d\underline{A}}$$

## 4.9 Gauss's theorem (Divergence theorem)

$$\int_{\partial V} \underline{v} \cdot d\underline{A} = \int_V \underbrace{\operatorname{div} \underline{v}}_{\underline{\nabla} \cdot \underline{v}}\, dV$$

flux through closed surface $\leftrightarrow$ divergences (sources) of field enclosed

## 4.10 Stoke's theorem

$$\int_{\partial F} \underline{v} \cdot d\underline{s} = \int_F \underbrace{\operatorname{rot} \underline{v}}_{\underline{\nabla} \times \underline{v}} \cdot d\underline{A}$$

line integral around boundary $\leftrightarrow$ curl over surface

## 4.11 Integration by parts

recap: $\int_a^b f'g\,dx = fg\Big|_a^b - \int_a^b f g'\,dx \quad , f,g: \mathbb{R} \to \mathbb{R}$

For scalar fields $f: \mathbb{R}^n \to \mathbb{R}$ and vector fields $\underline{v}: \mathbb{R}^n \to \mathbb{R}^n$:

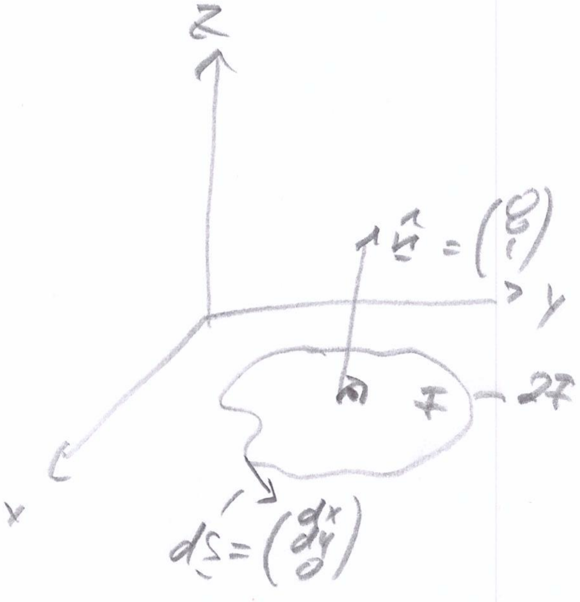
$$\int_V (\underline{\nabla} f) \cdot \underline{v}\, dV = \int_{\partial V} f\underline{v} \cdot d\underline{A} - \int_V f(\underline{\nabla} \cdot \underline{v})\, dV$$

## 4.12 Green's theorem

$F$: plane region (surface in $\mathbb{R}^2$) $\Rightarrow$ normal vector $\underline{\hat{n}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\underline{v}$: $C^1$ vector field with $\underline{v} = \begin{pmatrix} f(x,y) \\ g(x,y) \\ v_z \end{pmatrix}$

$$\Rightarrow \int_{\partial F} f(x,y)\,dx + g(x,y)\,dy = \int_B \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx\,dy$$




## 5 Complex numbers

hierarchy of sets of numbers:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

add inverse of "+" | add inverse of "·" | convergence of all series | factorization of all polynomials

## 5.1 Complex numbers (definitions and basic calculations)

$z = a+ib = re^{i\varphi} \in \mathbb{C}$ with $a,b,r,\varphi \in \mathbb{R}$, $r \geq 0$, $i^2=-1$

$a = \mathrm{Re}(z)$ : real part, $b = \mathrm{Im}(z)$ : imaginary part

$r = |z|$ : absolute value, $\varphi = \arg(z) = \arctan\left(\frac{b}{a}\right)$ : argument

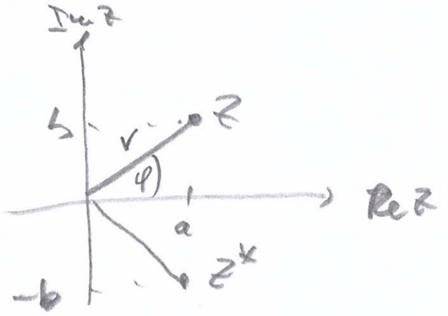
$= \sqrt{a^2+b^2}$

watch out of signs of a, b

$= \sqrt{zz^*}$ , $z^* = a-ib$ complex conjugate ↳ arctan2-function! : arctan2(x, y)

## 5.2 complex plane

multi-valued phase: $z = re^{i\varphi} = re^{i(\varphi + n2\pi)}$ , $n \in \mathbb{Z}$




multiplication:

$$z_1 z_2 = (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2)$$
$$= r_1 r_2 e^{i(\varphi_1+\varphi_2)}$$

## 5.3 complex exponential

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = e^x e^{iy} = e^x(\cos y + i \sin y)$$

Euler's formula : $e^{i\varphi} = \cos\varphi + i\sin\varphi$

multi-valued!

complex logarithm: $\ln z = \ln r + i(\varphi + n2\pi)$ , $n \in \mathbb{Z}$

Mandelbrot set : $f_c(z) = z^2 + c$ with varying c and $z_0 = 0$

↳ $z_{n+1} = f_c(z_n)$

Julia set : $f_c(z) = z^2 + c$ with fixed c and varying $z_0$

complex numbers as matrices:

$$z = a \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\underline{\underline{E}}} + b \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\underline{\underline{I}} \;\Rightarrow\; \underline{\underline{I}}^2 = -\underline{\underline{E}}} \mathrel{\hat{=}} a+ib$$

# 6 Elements of linear algebra

## 6.1 Linear equations

linear equations $L(u)=v$ with linear operator $L$:

$$\text{(i)}\quad L(u+v)=L(u)+L(v) \qquad \text{(ii)}\quad L(\alpha u)=\alpha L(u)$$

## 6.2 Vektorspaces (linear spaces)

a non-empty set $V$ with a binary operation $+$ and a binary function $\cdot$ is called a vector space over a field $\mathbb{K}$, if the following holds for all $\underline{v}, \underline{w}, \in V$ and $\lambda, \mu \in \mathbb{K}$:

(V1) $\underline{v}+\underline{w} \in V$, $\lambda \cdot \underline{v} \in V$

(V2) $(\underline{u}+\underline{v})+\underline{w} = \underline{u}+(\underline{v}+\underline{w})$ (associativity)

(V3) $\exists\, \underline{0} \in V : \underline{u}+\underline{0}=\underline{u}$ (neutral element)

(V4) $\exists\, \underline{u}' \in V : \underline{u}+\underline{u}'=\underline{0}$ (inverse element)

(V5) $\underline{u}+\underline{v} = \underline{v}+\underline{u}$ (commutativity)

(V6) $\lambda \cdot (\underline{u}+\underline{v}) = \lambda \cdot \underline{u} + \lambda \cdot \underline{v}$ (distributivity)

(V7) $(\lambda+\mu)\,\underline{u} = \lambda \underline{u} + \mu \underline{u}$

(V8) $(\lambda\mu)\cdot \underline{u} = \lambda \cdot (\mu \cdot \underline{u})$

(V9) $1 \cdot \underline{u} = \underline{u}$

vector: element of a vector space

linear subspace: $U \subset V$ with $V$ $\mathbb{K}$-vectorspace if

for $\underline{u}, \underline{v} \in U$, $\lambda \in \mathbb{K}$: (U1) $\underline{u}+\underline{v} \in U$

(U2) $\lambda\, \underline{u} \in U$

## 6.3 linear combination, span, generating set

linear combination $\alpha_1 \underline{v}_1 + \ldots + \alpha_n \underline{v}_n = \sum_{i=1}^{n} \alpha_i \underline{v}_i$

span : set of all linear combinations of $\underline{v}_1, \ldots, \underline{v}_n$ : $\text{span}\{\underline{v}_1, \ldots, \underline{v}_n\}$

generating set: $X \subseteq V$, $\text{span}(X)=U$ : $\forall \underline{u} \in U$ : $\underline{u}=\alpha_1 \underline{x}_1 + \ldots + \alpha_n \underline{x}_n$, $\underline{x}_i \in X$, $\alpha_i \in \mathbb{K}$

linear independence: $\sum_{i=1}^{n} \alpha_i \underline{v}_i = \underline{0} \Rightarrow \alpha_1 = \ldots = \alpha_n = 0$

alternatively: no vector of a set can be written as a linear combination of the other elements

## 6.4 Basis and dimension

- $B = \{\underline{b}_1, \dots, \underline{b}_n\}$ is a basis of $V$ if all elements $\underline{v} \in V$ can be written as a unique linear combination of generating set $\underline{B}$:

$$\underline{v} = \underbrace{\alpha_1}_{\text{coordinates}} \underline{b}_1 + \dots + \alpha_n \underline{b}_n$$

- Theorem: $B = \{\underline{b}_1, \dots, \underline{b}_n\}$ basis $\Leftrightarrow$ span $B = V$ and $B$ linearly independent.
- Dimension of $V$: $\dim V = |\underbrace{B}_{\text{basis of } V}|$ ($|\cdot|$: # elements of $B$)
- Theorem: Every vector space has a basis
  - ↳ every generating set contains a basis (reduction)
  - ↳ extend linearly independent subsets to basis (extension)

## 6.5 Matrices as representation of linear maps

Theorem: $B = \{\underline{b}_1, \dots, \underline{b}_n\}$ basis of $V$, $L: V \to W$ linear map.

Then: $L$ completely/fully determined by images of $B$.

commuting diagram:

basis of $V$ — basis of $W$ — coordinate maps

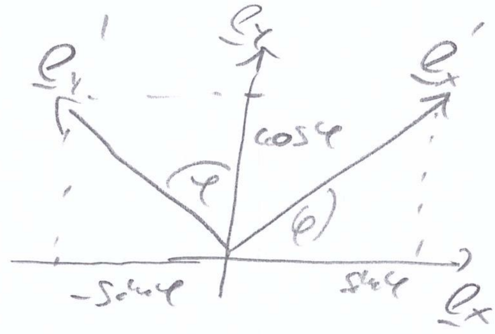
$$\underline{v} = \sum_{i=1}^{n} v_i \underline{b}_i \qquad \underline{w} = \sum_{j=1}^{m} w_j \underline{c}_j$$

$$\begin{tikzcd}
\underline{v} \in V \arrow[r, "L"] \arrow[d, "\kappa_B"'] & \underline{w} \in W \arrow[d, "\kappa_C"] \\
\underline{v}_B \in \mathbb{K}^n \arrow[r, "{}_C L_B"'] & \underline{w}_C \in \mathbb{K}^m_C
\end{tikzcd}$$

transformation matrix $\quad {}_C L_B = \kappa_C \circ L \circ \kappa_B^{-1}$

Rotations in $\mathbb{R}^2$ by angle $\varphi$:

$B = \{\underline{e}_x, \underline{e}_y\}$ with $\underline{e}_x = \begin{pmatrix}1\\0\end{pmatrix}$, $\underline{e}_y = \begin{pmatrix}0\\1\end{pmatrix}$




$$\underline{e}_x' = \cos\underline{e}_x + \sin\varphi\,\underline{e}_y = \begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix}_B$$

$$\underline{e}_y' = -\sin\varphi\,\underline{e}_x + \cos\varphi\,\underline{e}_y = \begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix}$$

$$\begin{pmatrix}\underline{e}_x'\\ \underline{e}_y'\end{pmatrix}_B = \underbrace{\begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}}_{\underset{B=B}{R}} \begin{pmatrix}\underline{e}_x\\ \underline{e}_y\end{pmatrix}_B$$

## 6.6 Matrix calculations

(i) sum: $(\underline{\underline{A}}+\underline{\underline{B}})_{ij} = a_{ij}+b_{ij}$, $(\underline{\underline{A}}+\underline{\underline{B}})+\underline{\underline{C}} = \underline{\underline{A}}+(\underline{\underline{B}}+\underline{\underline{C}})$, $\underline{\underline{A}}+\underline{\underline{B}} = \underline{\underline{B}}+\underline{\underline{A}}$

(ii) $(\alpha A)_{ij} = \alpha a_{ij}$

(iii) $\underline{\underline{A}} \in \mathbb{K}^{m\times n}$, $\underline{\underline{B}} = \mathbb{K}^{n\times p}$, $\underline{\underline{C}} = \mathbb{K}^{p\times q}$

$$\underline{\underline{A}} = \begin{pmatrix}a_{11} & \dots & a_{1n}\\ \vdots & & \vdots\\ a_{m1} & \dots & a_{mn}\end{pmatrix} \quad \underline{\underline{B}} = \begin{pmatrix}b_{11} & \dots & b_{1p}\\ \vdots & & \vdots\\ b_{n1} & \dots & b_{np}\end{pmatrix} \quad \underline{\underline{C}} = \begin{pmatrix}C_{11} & \dots & C_{1q}\\ \vdots & & \vdots\\ C_{p1} & \dots & C_{pq}\end{pmatrix}$$

$\hookrightarrow \underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{C}}$ matching dimensions!

$(\underline{\underline{A}}+\underline{\underline{B}})\underline{\underline{C}} = \underline{\underline{A}}\,\underline{\underline{C}} + \underline{\underline{B}}\,\underline{\underline{C}}$, but $\underline{\underline{A}}\,\underline{\underline{B}} \neq \underline{\underline{B}}\,\underline{\underline{A}}$ (most often)

$(\underline{\underline{A}}\,\underline{\underline{B}})\underline{\underline{C}} = \underline{\underline{A}}(\underline{\underline{B}}\,\underline{\underline{C}})$

$\underline{\underline{A}}\,\underline{\underline{B}} = \underline{\underline{0}} \not\Rightarrow \underline{\underline{A}} = 0$ or $\underline{\underline{B}} = 0$, e.g. $\underline{\underline{A}} = \begin{pmatrix}0&1\\0&0\end{pmatrix}$; $\underline{\underline{B}} = \begin{pmatrix}0&0\\1&0\end{pmatrix}$

dot product: $\underset{n\times 1}{\underline{a}}\cdot\underline{b} = \underset{1\times n}{\underline{a}^T}\underline{b} = (a_1,\dots,a_n)\begin{pmatrix}b_1\\ \vdots\\ b_n\end{pmatrix} = \sum_{i=1}^{n} a_i b_i$

$$\begin{pmatrix}w_1\\ \vdots\\ w_m\end{pmatrix} = \begin{pmatrix}a_{11} & \dots & a_{1n}\\ \vdots & & \vdots\\ a_{m1} & \dots & a_{mn}\end{pmatrix}\begin{pmatrix}v_1\\ \vdots\\ v_n\end{pmatrix} \Leftrightarrow \underline{w} = \underline{\underline{A}}\,\underline{v}$$

# 6.7 Rank of a matrix

linear map $L: V \to W$ with representing matrix $\underline{\underline{M}}_L$

- column rank: number of linearly independent columns
- row rank: —"— —————— rows

$\hookrightarrow$ row rank = column rank = rang $(\underline{\underline{M}}_L) \equiv$ rank $(\underline{\underline{M}}_L)$

- $L: \mathbb{K}^n \to \mathbb{K}^m$ injective $\Leftrightarrow$ rank $\underline{\underline{M}}_L = n$ (full column rank)
  (one-to-one)

$$\underline{\underline{M}}_L = \begin{pmatrix} m_{11} & \dots & m_{1n} \\ \vdots & & \\ m_{m1} & \dots & m_{mn} \end{pmatrix} \in \mathbb{K}^{m \times n}$$

- $L: \mathbb{K}^n \to \mathbb{K}^m$ surjective $\Leftrightarrow$ rank $\underline{\underline{M}}_L = m$ (full row rank)
  (onto/full image)
- $L: \mathbb{K}^n \to \mathbb{K}^m$ bijective $\Leftrightarrow$ rank $\underline{\underline{M}}_L = m = n$

- transposed matrix: $\underline{\underline{A}} \in \mathbb{R}^{m \times n} \Rightarrow \underline{\underline{A}}^T \in \mathbb{R}^{n \times m}$ with $(\underline{\underline{A}}^T)_{ij} = a_{ji}$

  rank $\underline{\underline{A}}^T$ = rank $\underline{\underline{A}}$

- adjoint matrix: $\underline{\underline{C}} \in \mathbb{C}^{m \times n} \Rightarrow \underline{\underline{C}}^+ \in \mathbb{C}^{n \times m}$ with $(\underline{\underline{C}}^+)_{mn} = a_{nm} - i b_{nm}$

$$\underline{\underline{C}}^+ = (\underline{\underline{C}}^*)^T = (\underline{\underline{C}}^T)^* = \underline{\underline{C}}^H \leftarrow \text{Hermitian}$$

- orthogonal matrix: $\underline{\underline{A}} \in \mathbb{R}^{n \times n}$ with $\underline{\underline{A}}^T \underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^T = \underline{\underline{1}}$

- unitary matrix: $\underline{\underline{C}} \in \mathbb{C}^{n \times n}$ with $\underline{\underline{C}}^+ \underline{\underline{C}} = \underline{\underline{C}}\,\underline{\underline{C}}^+ = \underline{\underline{1}}$

## 6.8 Systems of linear equations (Gaussian elimination)

Solve $\underline{\underline{A}}\,\underline{x} = \underline{b}$ by transforming $\underline{\underline{A}}$ to row echelon form

$$\left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right) \longrightarrow \dots \longrightarrow \left(\begin{array}{ccc|c} 1 & & * & \tilde{b}_1 \\ 0 & \ddots & & \vdots \\ 0 & \dots\, 0 & 1 & \tilde{b}_m \end{array}\right)$$

Does a solution $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ exist for $m=n$?

(i) $\operatorname{rank} \underline{\underline{A}} = n$ : $\operatorname{image}(\underline{\underline{A}}) = \mathbb{K}^n$ $\Rightarrow$ unique solution exists.

(ii) $\operatorname{rank} \underline{\underline{A}} < n$ : solution exists (but is not unique) if $\underline{b} \in \operatorname{image}(\underline{\underline{A}})$

—"— does not exist if $\underline{b} \notin \operatorname{image}(\underline{\underline{A}})$

## 6.9 Inverse matrix

Solution of $\underline{\underline{A}}\,\underline{x} = \underline{b}$ given by $\underline{x} = \underline{\underline{A}}^{-1}\underline{b}$ with $\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \mathbb{1}$

Def.: A square matrix $\underline{\underline{A}} \in \mathbb{K}^{n\times n}$ is called invertible if a matrix $\underline{\underline{A}}^{-1}$ exists with $\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \mathbb{1}$.

condition: $\operatorname{rank} \underline{\underline{A}} = n \Leftrightarrow \det \underline{\underline{A}} \neq 0$

- $\underline{\underline{A}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \underline{\underline{A}}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ Cramer's rule (if $ad-bc \neq 0$)
- Find inverse $\underline{\underline{A}}^{-1}$ by transforming

$$(\underline{\underline{A}}\,|\,\mathbb{1}) \longrightarrow (\mathbb{1}\,|\,\underline{\underline{A}}^{-1})$$

## 6.10 Determinant

Def: $\underline{\underline{A}} = \{a_{ij}\}_{i,j=1,\dots,n} \in \mathbb{K}^{n\times n}$

remove ith row, jth column

We define: for $n=1$ : $\det \underline{\underline{A}} = a_{11}$

$$\text{for } n \geq 2 : \det \underline{\underline{A}} = \sum_{i=1}^{n} (-1)^{i+1} a_{i1} \det(\underline{\underline{A}}_{i1})$$

$$= \sum_{s=1}^{n} (-1)^{r+s} a_{rs} \det(\underline{\underline{A}}_{rs}) \qquad \in \mathbb{K}^{(n-1)\times(n-1)}$$

$$= \sum_{r=1}^{n} (-1)^{r+s} a_{rs} \det(\underline{\underline{A}}_{rs})$$

- Sarrus rule : $\underline{\underline{A}} \in \mathbb{K}^{3\times 3}$

$$\det \underline{\underline{A}} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

$-\ +\ -\ +\ -\ +$

- $\det \underline{\underline{A}}^T = \det \underline{\underline{A}}$
- $\det \underline{\underline{A}}\,\underline{\underline{B}} = \det \underline{\underline{A}} \det \underline{\underline{B}}$
- $\det \underline{\underline{A}}^{-1} = \frac{1}{\det \underline{\underline{A}}}$
- $\det \underline{\underline{A}} = 0 \Leftrightarrow \text{rank} \underline{\underline{A}} < n$ (singular)
- $\det \underline{\underline{A}} \neq 0 \Leftrightarrow \text{rank} \underline{\underline{A}} = n$ (regular)

ex. $\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = 0$ columns linearly dependent

$$2\begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} + \begin{pmatrix} 3 \\ 6 \\ 9 \end{pmatrix}$$

- $\det \underline{\underline{A}} = \begin{pmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{pmatrix} = \underline{b} \cdot (\underline{c} \times \underline{d})$ (triple product)

# 6.11 Coordinate transformation (change of basis)

$\mathbb{K}$ vector space $V$ with 2 basis sets: $B = \{\underline{b}_1, \dots, \underline{b}_n\}$, $C = \{\underline{c}_1, \dots, \underline{c}_n\}$

$$\Rightarrow \quad \underline{v} = \sum_{i=1}^{n} \beta_i \underline{b}_i = \sum_{i=1}^{n} \gamma_i \underline{c}_i \iff \underline{v}_B = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}, \quad \underline{v}_C = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix}$$

coordinates w.r.t. $B$ and $C$, resp.

- transformation matrix: $\underline{v}_C = {}_C\underline{\underline{M}}_B\, \underline{v}_B =: \underline{\underline{S}}^{-1} \underline{v}_B$ given by

$$ {}_C\underline{\underline{M}}_B = \begin{pmatrix} | & & | \\ (\underline{b}_1)_C & \cdots & (\underline{b}_n)_C \\ | & & | \end{pmatrix} \in \mathbb{K}^{n \times n}$$

- $L : V \to V$ linear map:

$$\begin{tikzcd}
V \arrow[r, "L"] \arrow[d, "\kappa_B"] & V \arrow[r, "\mathrm{id}_V"] \arrow[d, "\kappa_B"] & V \arrow[d, "\kappa_C"] \\
\underline{u}_B \in \mathbb{K}^n \arrow[r, "{}_B\underline{\underline{L}}_B"'] & \underline{v}_B \in \mathbb{K}^n \arrow[r, "{}_C\underline{\underline{M}}_B = \underline{\underline{S}}^{-1}"'] & \underline{v}_C \in \mathbb{K}^n
\end{tikzcd}$$

$$\Rightarrow \quad \underline{v}_B = {}_B\underline{\underline{L}}_B\, \underline{u}_B$$

$$\Rightarrow \quad \underline{\underline{S}}^{-1} \underline{v}_B = \underline{\underline{S}}^{-1}\, {}_B\underline{\underline{L}}_B\, \overbrace{\underline{\underline{S}}\, \underline{\underline{S}}^{-1}}^{\mathbb{1}} \underline{u}_B$$

$$\underline{v}_C = {}_C\underline{\underline{L}}_C \qquad \underline{u}_C$$

## 6.12 Eigenvalue problem

- $\underline{\underline{A}} \in \mathbb{K}^{n\times n} \Rightarrow \lambda \in \mathbb{K}$ eigenvalue of $\underline{\underline{A}}$ if $\underline{v} \in \mathbb{K}^n \setminus \{\underline{0}\}$ exists with

$$\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}$$

($\underline{v}$: Eigenvektor)

- subspace $\mathrm{Eig}_{\underline{\underline{A}}}(\lambda) = \{\underline{v} \in \mathbb{K}^n \,|\, \underline{\underline{A}}\,\underline{v} = \lambda \underline{v}\}$ : eigenspace (Eigenvektorraum)

- Characteristic polynomial: $\chi_{\underline{\underline{A}}}(x) = \det(\underline{\underline{A}} - \lambda \underline{\underline{1}})$

$$= \det \begin{pmatrix} a_{11}-\lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-\lambda & & \vdots \\ \vdots & & \ddots & a_{n-1\,n} \\ a_{n1} & \dots & a_{n\,n-1} & a_{nn}-\lambda \end{pmatrix}$$

- $\lambda$ eigenvalue of $\underline{\underline{A}} \Leftrightarrow \chi_{\underline{\underline{A}}}(\lambda) = 0$

- $\chi_{\underline{\underline{A}}}(x) = \underbrace{(\lambda_1 - x)^{k_1} \dots (\lambda_r - x)^{k_r}}_{\text{factorization of } \chi_{\underline{\underline{A}}}}$ with $k_1 + \dots + k_r = n$

- $k_i$ : algebraic multiplicity

- $\dim \mathrm{Eig}_{\underline{\underline{A}}}(\lambda)$ : geometric multiplicity

- $\dim \mathrm{Eig}_{\underline{\underline{A}}}(\lambda) > 1$ : $\lambda$ is called degenerate (entartet).

6.13 Diagonalization

- $L: V \to V$ with representing ~~matrix~~ $\underset{C=C}{L}$ (w.r.t. basis $C$)
- $B = \{\underline{b}_1, \dots, \underline{b}_n\}$ alternative basis of eigenvectors of $\underset{C=C}{L}$ : $\underset{C=C}{L}\, \underline{b}_i = \lambda_i \underline{b}_i$
- $\underset{B=B}{L} = \underline{\underline{S}}^{-1} \underset{C=C}{L}\, \underline{\underline{S}} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ with $\underline{\underline{S}} = \begin{pmatrix} | & & | \\ \underline{b}_1 & \dots & \underline{b}_n \\ | & & | \end{pmatrix}$

Thm: $\underline{\underline{A}}$ diagonalisable $\Leftrightarrow$ There is a basis of eigenvectors

7 Differential equations

7.1 Example

$\ddot{z}(t) = -g \Rightarrow z(t) = -\frac{1}{2} g t^2 + v_0 t + z_0$ , $v_0, z_0$ initial conditions

$z_0 = z(0)$, $v_0 = \dot{z}(0)$

7.2 Nomenclature

- ordinary DEq
- partial DEq
- DEq. of n-th order
- general solution contain n constants (determined by initial or boundary values)

- linear DEq. $a_n(x) \frac{d^n y}{dx^n} + \dots + a_1(x) \frac{dy}{dx} = b(x)$
- inhomogeneous DEq : $b(x) \neq 0$
- homogeneous DEq : $b(x) = 0$ ($y(t) = 0$ is trivial solution

7.3 Solving Differential equations

(i) direct integration

(ii) separation of variables

(iii) ansatz

(iv) variation of constants (linear inhomogeneous DEq)

(v) power series

# 7.4 Linear inhomogeneous DEqs

- $L[y(x)] = b(x) \Rightarrow y(t) = y_h(x) + y_p(x)$

$L[y_h(x)] = 0$ homogeneous solution

$L[y_p(x)] = b(x)$ particular solution

- ex: $\ddot{y}(t) + \omega_0^2 y(t) = f_0 \cos(\Omega t)$

(a) homogeneous: $\ddot{y}(t) + \omega_0^2 y(t) = 0 \Rightarrow y_h(t) = C_1 \sin(\omega_0 t) + C_2 \cos(\omega_0 t)$

$= A \sin(\omega_0 t + \varphi)$

(b) particular solution: ansatz: $y_p(t) = \alpha \cos(\omega t)$

$\Rightarrow \omega = \Omega \quad , \quad \alpha = \frac{f_0}{\omega_0^2 - \Omega^2} \Rightarrow y(t) = A \sin(\omega_0 t + \varphi) + \frac{f_0}{\omega_0^2 - \Omega^2} \cos(\Omega t)$

what about $\omega_0 = \Omega$?

7.4 Linear inhomogeneous DEs

$$L[y(x)] = b(x) \Rightarrow y(x) = y_h(x) + y_p(x)$$

homogeneous solution $L[y_h(x)] = 0$ particular solution $L[y_p(x)] = b(x)$

7.3 (v) Power series

ansatz: $y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \Rightarrow y'(x) = \sum_{n=1}^{\infty} a_n n (x-x_0)^{n-1}, y''(x) = \sum_{n=2}^{\infty} a_n n(n-1) (x-x_0)^n$

↳ insert into DE

↳ sort by powers of x

↳ determine $a_n$ by comparison of coefficients

↳ recurrence equations for $a_n$