4.5.3 motion of planets & Kepler's laws

solving the integral $\varphi - \varphi_0 = \int_{r_0}^{r} \frac{dr'}{r'^2 \sqrt{\frac{2m}{\ell^2}(E - \tilde{V}(r'))}}$

$$= \int_{r_0}^{r} \frac{dr'}{r'^2 \sqrt{\frac{2m}{\ell^2}\left(E + \frac{k}{r'} - \frac{\ell^2}{2mr'^2}\right)}}$$

by (i) quadratic extension and (ii) substitution

(i) $\frac{2m}{\ell^2}(E - \tilde{V}(r)) = \ldots = D\left[1 - \frac{1}{D}\left(\frac{1}{r'} - \frac{mk}{\ell^2}\right)^2\right]$ with $D = \frac{2m}{\ell^2}\left(\frac{mk^2}{2\ell^2} + E\right)$

(ii) $\cos\vartheta'(r) = \frac{1}{\sqrt{D}}\left(\frac{1}{r'} - \frac{mk}{\ell^2}\right)$

$\Rightarrow \frac{d}{dr'}\cos\vartheta'(r) = \frac{d}{dr}\frac{1}{\sqrt{D}}\left(\frac{1}{r'} - \frac{mk}{\ell^2}\right)$

$\Rightarrow -\sin\vartheta' \frac{d\vartheta'}{dr'} = -\frac{1}{\sqrt{D}\,r'^2}$ $\Rightarrow \frac{dr'}{\sqrt{D}\,r'^2} = +\sin\vartheta' d\vartheta'$

$\Rightarrow \varphi - \varphi_0 = \int_{\vartheta_0}^{\vartheta} d\vartheta' \sin\vartheta' \frac{1}{\sqrt{1 - \cos^2\vartheta'}} = \int_{\vartheta_0}^{\vartheta} d\vartheta' = \vartheta - \vartheta_0$

$\Rightarrow \varphi(r) = \arccos\left[\frac{1}{\sqrt{D}}\left(\frac{1}{r'} - \frac{mk}{\ell^2}\right)\right]$

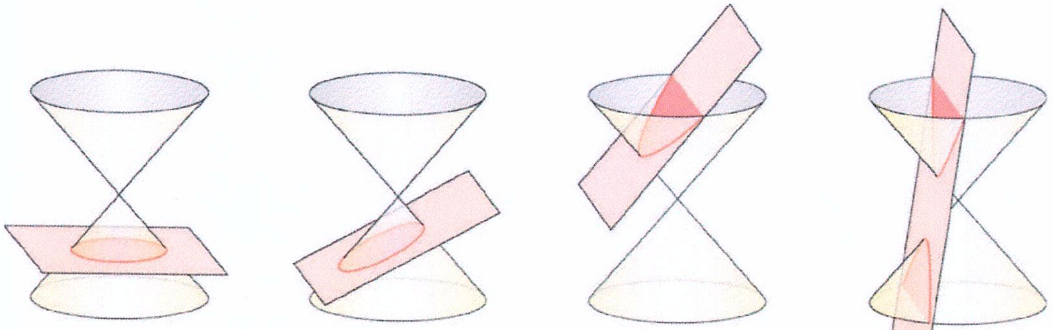
$\Rightarrow r(\varphi) = \frac{\ell^2/mk}{1 + \epsilon\cos\varphi}$

$\epsilon = 0$: circle $\qquad 0 < \epsilon < 1$: ellipse $\left(-\frac{mk^2}{2\ell^2} < E < 0\right)$

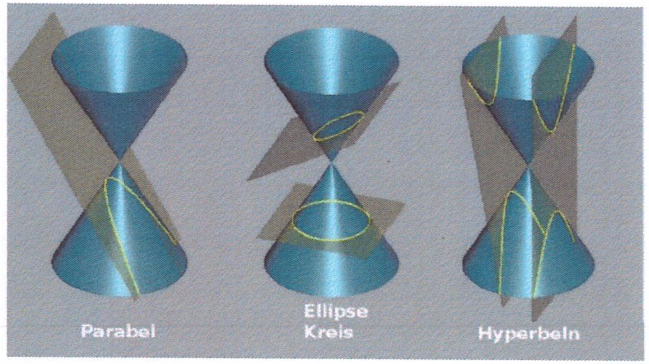
$\epsilon = 1$: parabola $\qquad \epsilon > 1$: hyperbola

ϵ = numeric excentricity

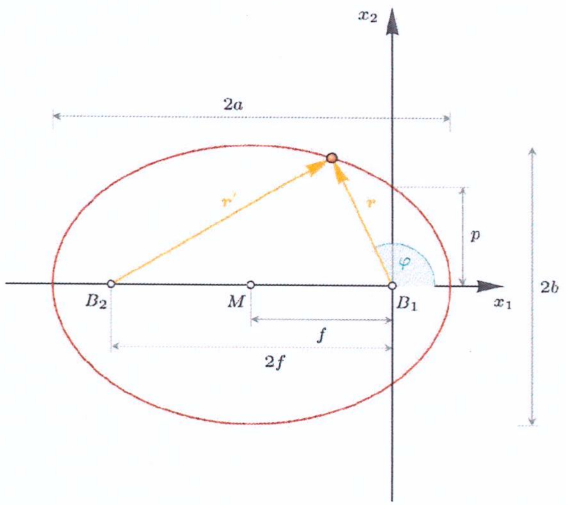
Kepler's 3rd law: $\underset{\uparrow\,\text{period}}{T^2} \sim \underset{\uparrow\,\text{semi axis of ellipse}}{a^3}$



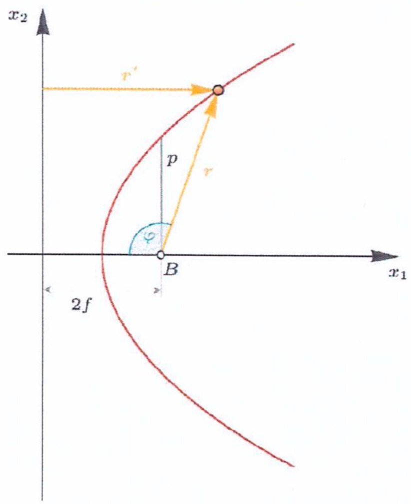
$\varepsilon = 0$ $\quad 0 < \varepsilon < 1$ $\quad \varepsilon = 1$ $\quad \varepsilon > 1$



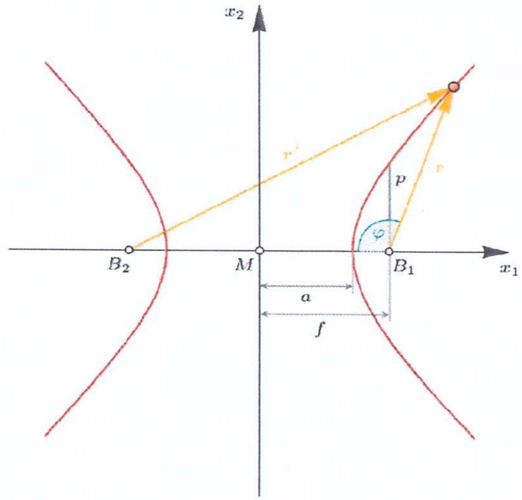
Ellipse



Parabel



Hyperbel



5. Der Hamiltonsche kanonische Formalismus

Idee: Lagrange-Theorie: generalisierte Koordinaten $q_1, \dots, q_f$

und Geschwindigkeiten $\dot q_1, \dots, \dot q_f$

$$\Rightarrow L(q_1,\dots,q_f, \dot q_1,\dots,\dot q_f, t) \Rightarrow \underbrace{\frac{d}{dt}\frac{\partial L}{\partial \dot q_k} - \frac{\partial L}{\partial q_k} = 0, \quad k=1,\dots,f}_{f \text{ Dgls } 2.\text{ Ordnung}}$$

Hamilton-Formalismus: $(q_k, \dot q_k, t) \longrightarrow (q_k, p_k, t)$

$$p_k = \frac{\partial L}{\partial \dot q_k}$$

$$H(q_1,\dots,q_f, p_1,\dots,p_f, t) \Rightarrow \dots \Rightarrow \underbrace{\dot q_k = \frac{\partial H}{\partial p_k}, \quad \dot p_k = -\frac{\partial H}{\partial q_k}}_{2f \text{ Dgls } 1.\text{ Ordnung}}$$

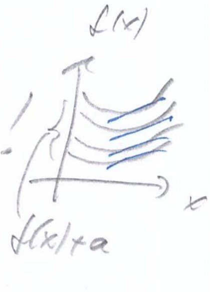
5.1 Legendre Transformation

Idee: Für $y = f(x)$ soll statt x die Variable $\frac{df}{dx} =: u$ verwendet werden

$f'(u)$

Annahme: $\frac{df}{dx} = u$ umkehrbar (bei $x = \varphi(u)$) $\Rightarrow \frac{d^2 f}{dx^2} \neq 0$

Achtung: $y = f(x) + a$ liefert auch dieselbe Steigung $\frac{df}{dx}$!

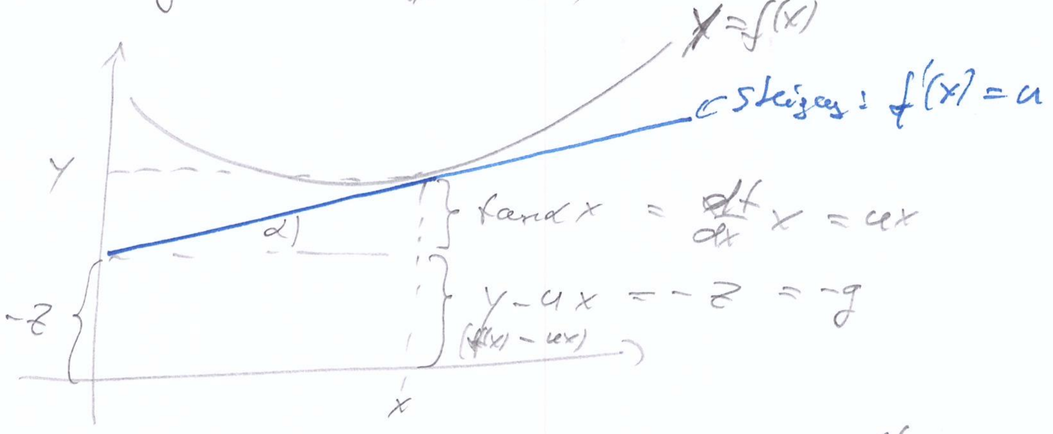


$\Rightarrow$ Legendre-Transformation

$$x \rightarrow u = \frac{df}{dx}$$

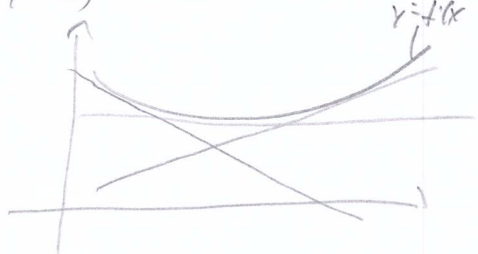
$$y \rightarrow z = xu - \underbrace{f(x)}_{y} = \varphi(u)\,u - f(\varphi(u)) \equiv g(u)$$

$$\Rightarrow g(u) = x\frac{df}{dx} - f(x)$$



Also: statt $y = f(x)$: Trafo $x \to u = \frac{df}{dx}$ und (negativen) Achsenabschnitt der Tangente $(-z)$

$(u, -z)$ bestimmen die Schar der Einhüllenden



Anwendung auf (i) $L(q, \dot q, t)$:

$$y = L,\quad x = \dot q,\quad u = \frac{\partial L}{\partial \dot q} = p \Rightarrow z = \underset{\uparrow \atop \dot q(q,p)}{\dot q}\, p - L = H(q,p,t)$$

Hamilton-Funktion

(ii) $L(q_1, \dots, q_f, \dot q_1, \dots, \dot q_f, t)$:

$$p_k = \frac{\partial L}{\partial \dot q_k} \Rightarrow H(q_1, \dots, q_f, p_1, \dots, p_f, t) = \sum_{j=1}^{f} \dot q_j p_j - L(q_1, \dots, q_f, \dot q_1, \dots, \dot q_f, t)$$

$\left(\det \frac{\partial^2 L}{\partial \dot q_k \partial \dot q_j} \neq 0\right.$, sonst $p_k = \frac{\partial L}{\partial \dot q_k}$ nicht nach $\dot q_k$ auflösbar$\left.\right)$

5.2 Die Hamiltonschen Gleichungen

Ziel: Bewegungsgleichungen für $q_1,\dots,q_f,\ p_1,\dots,p_f$:

$$H(q_1,\dots,q_f,p_1,\dots,p_f,t) = \sum_{j=1}^{f} \dot q_j\, p_j - L(q_1,\dots,q_f,\dot q_1,\dots,\dot q_f,t)$$

totales Differential

$$\Rightarrow dH = \sum_{j=1}^{f} \left\{ \frac{\partial H}{\partial q_j}\, dq_j + \frac{\partial H}{\partial p_j}\, dp_j \right\} + \frac{\partial H}{\partial t}\, dt \qquad \text{(linke Seite)}$$

$$= \sum_{j=1}^{f} \Big\{ \dot q_j\, dp_j + \cancel{p_j\, d\dot q_j} \underbrace{- \frac{\partial L}{\partial q_j}\, dq_j - \cancel{\overbrace{\frac{\partial L}{\partial \dot q_j}}^{p_j} d\dot q_j} \Big\} - \frac{\partial L}{\partial t}\, dt}_{-dL} \qquad \text{(rechte Seite)}$$

Vergleich LS mit RS liefert:

$$dq_j: \quad \dot q_j = \frac{\partial H}{\partial p_j}$$

$$dp_j: \quad \frac{\partial H}{\partial q_j} = -\frac{\partial L}{\partial q_j} = -\frac{d}{dt}\frac{\partial L}{\partial \dot q_j} = -\frac{d}{dt} p_j = -\dot p_j$$

$$dt: \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

$$\dot q_j = \frac{\partial H}{\partial p_j}, \qquad \dot p_j = -\frac{\partial H}{\partial q_j}$$

Hamiltonsche Gleichungen

Bei zeitlicher Translationsinvarianz (skleronome Zwangsbedingungen: $\frac{\partial}{\partial t}\mathbf{r}_i(q_1,\dots,q_f) = 0$, $\frac{\partial L}{\partial t} = 0$)

$$\sum_{j=1}^{f} \frac{\partial T}{\partial \dot q_j}\dot q_j = 2T \Rightarrow H = \sum_j \dot q_j \underbrace{\frac{\partial L}{\partial \dot q_j}}_{\frac{\partial T}{\partial \dot q_j}} - L = 2T - (T-V) = T+V$$

$\Rightarrow \quad H = T+V$: Gesamtenergie

$\Rightarrow$ Energieerhaltung:
$$\frac{dH}{dt} = \underbrace{\sum_{j=1}^{f} \Big\{ \frac{\partial H}{\partial q_j}\underbrace{\dot q_j}_{\frac{\partial H}{\partial p_j}} + \frac{\partial H}{\partial p_j}\underbrace{\dot p_j}_{-\frac{\partial H}{\partial q_j}} \Big\}}_{=0} + \underbrace{\frac{\partial H}{\partial t}}_{-\frac{\partial L}{\partial t} = 0} = 0$$

- rheonome Zwangsbedingungen: $H \neq T+V$

Anwendungsschema:

Kochrezept der Hamiltonschen Gleichungen:

1. Auswählen generalisierter Koordinaten: $q_1, \ldots, q_f$
2. Transformation $\mathbf{r}_i = \mathbf{r}_i(q_1, \ldots, q_f, t)$ und $\dot{\mathbf{r}}_i = \dot{\mathbf{r}}_i(q_1, \ldots, q_f, \dot{q}_1, \ldots, \dot{q}_f, t)$
3. Berechnen der Lagrange-Funktion $L(q_1, \ldots, q_f, \dot{q}_1, \ldots, \dot{q}_f, t)$
4. Berechnen der generalisierten Impulse $p_j = \frac{\partial L}{\partial \dot{q}_j}$
   und deren Umkehrung $\dot{q}_j = \dot{q}_j(q_1, \ldots, q_f, p_1, \ldots, p_f, t)$
5. Legendre-Transformation $H(q_1, \ldots, q_f, p_1, \ldots, p_f, t) = \sum_{j=1}^{f} \dot{q}_j p_j - L(q_1, \ldots, q_f, \dot{q}_1, \ldots, \dot{q}_f, t)$
6. Aufstellung der Bewegungsgleichungen ($j = 1, \ldots, f$):
$$\dot{q}_j = \frac{\partial H}{\partial p_j} \qquad \text{und} \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$
7. Lösen der Bewegungsgleichungen

Bsp: (a) 1 Teilchen in Zylinderkoordinaten ohne Zwangsbedingungen:

(i) $q_1 = \rho$, $q_2 = \varphi$, $q_3 = z$

(ii)
$$x = \rho \cos\varphi \qquad \dot{x} = \dot{r}\cos\varphi - r\dot{\varphi}\sin\varphi$$
$$y = \rho \sin\varphi \qquad \dot{y} = \dot{r}\sin\varphi + r\dot{\varphi}\cos\varphi$$
$$z = z \qquad \dot{z} = \dot{z}$$

(iii) $T = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2)$, $V = V(\rho, \varphi, z)$

$\Rightarrow L(\rho, \varphi, z, \dot{\rho}, \dot{\varphi}, \dot{z}) = T - V$

(iv) $p_\rho = \frac{\partial L}{\partial \dot{\rho}} = m\dot{\rho} \Rightarrow \dot{\rho} = \frac{p_\rho}{m}$ Radialimpuls

$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m\rho^2\dot{\varphi} \Rightarrow \dot{\varphi} = \frac{p_\varphi}{m\rho^2}$ z-Komponente des Drehimpulses (l_z)

$p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \Rightarrow \dot{z} = \frac{p_z}{m}$ z-Komponente des Impulses

(v) $H = p_\rho\dot{\rho} + p_\varphi\dot{\varphi} + p_z\dot{z} - L = \frac{1}{m}\left(p_\rho^2 + \frac{p_\varphi^2}{\rho^2} + p_z^2\right) - \frac{1}{2m}\left(p_\rho^2 + \frac{p_\varphi^2}{\rho^2} + p_z^2\right) + V(\rho, \varphi, z)$

$= \frac{1}{2m}\left(p_\rho^2 + \frac{p_\varphi^2}{\rho^2} + p_z^2\right) + V(\rho, \varphi, z)$

(vi)
$$\dot{\rho} = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m}, \qquad \dot{p}_\rho = -\frac{\partial H}{\partial \rho} = (+)\frac{p_\varphi^2}{m\rho^3} - \frac{\partial V}{\partial \rho}$$
$$\dot{\varphi} = \frac{\partial H}{\partial p_\varphi} = \frac{p_\varphi}{mr^2}, \qquad \dot{p}_\varphi = -\frac{\partial H}{\partial \varphi} = -\frac{\partial V}{\partial \varphi}$$
$$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}, \qquad \dot{p}_z = -\frac{\partial H}{\partial z} = -\frac{\partial V}{\partial z}$$

Radialpotential: $\dot{p}_\rho = \frac{p_\varphi^2}{m\rho^3} - \frac{\partial V}{\partial \rho}$

$\dot{p}_\varphi = 0 = \dot{p}_z$

$\dot p_\varphi = 0 \Rightarrow p_\varphi = m\rho^2\dot\varphi = \text{const}$ Drehimpulserhaltung

$\dot p_z = 0 \Rightarrow p_z = \text{const} = 0$ (ebene Bewegung)

(b) 1D harmonischer Oszillator

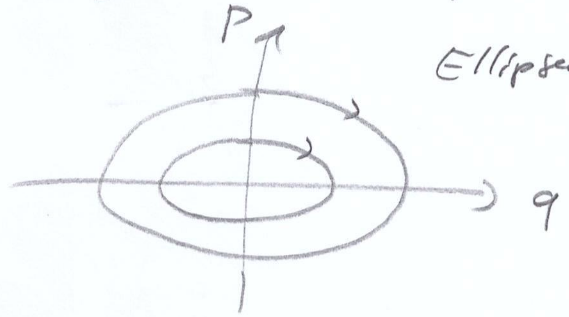
(i) – (iii) $L = \frac{m}{2}\left(\dot q^2 - \omega^2 q^2\right)$

(iv) $p = \frac{\partial L}{\partial \dot q} = m\dot q$

(v) $H = p\dot q - L = \frac{p^2}{m} - \frac{m}{2}\left(\frac{p^2}{m^2} - \omega^2 q^2\right) = \frac{p^2}{2m} + \frac{m}{2}\omega^2 q^2$

$\frac{\partial L}{\partial t} = 0$: skleronomes System $\Rightarrow E = H = T + V = \text{const}$

$\Rightarrow \frac{p^2}{2mE} + \frac{q^2}{\frac{2E}{m\omega^2}} = 1$



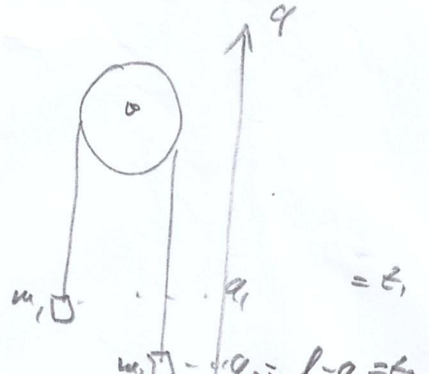
Ellipsen!

Halbachsen:

$a = \sqrt{\frac{2E}{m\omega^2}}$, $b = \sqrt{2mE}$

(vi) $\left.\begin{aligned} \dot p &= -\frac{\partial H}{\partial q} = -m\omega^2 q \\ \dot q &= \frac{\partial H}{\partial p} = \frac{p}{m} \end{aligned}\right\}$ definiert Richtung in (q,p)-Ebene

(c) Atwoodsche Fallmaschine: (vgl. z.B., Skd, M4 S 126)



(i)–(iii) $L = T - V = \frac{1}{2}(m_1+m_2)\dot q^2 + m_1 g q + m_2 g (l-q)$

(iv) $p = \frac{\partial L}{\partial \dot q} = (m_1+m_2)\dot q$

(v) $H = \dot q p - L = \frac{p^2}{m_1+m_2} - \frac{m_1+m_2}{2}\frac{p^2}{(m_1+m_2)^2} - m_1 g q - m_2 g(l-q)$

$= \frac{p^2}{2(m_1+m_2)} - m_1 g q - m_2 g (l-q)$

(vi) $\left.\begin{aligned} \dot p &= -\frac{\partial H}{\partial q} = m_1 g - m_2 g = (m_1-m_2) g \\ \dot q &= \frac{\partial H}{\partial p} = \frac{p}{m_1+m_2} \end{aligned}\right\}$ check: $\ddot q = \frac{\dot p}{m_1+m_2} = \frac{m_1-m_2}{m_1+m_2} g$