5 Hamiltonformalism

idea: instead of $(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f)$ choose $(q_1, \dots, q_f, p_1, \dots, p_f)$

with $\frac{\partial L}{\partial \dot{q}_j} = p_j$

5.1 Legendre transformation

idea: $x \to u = \frac{df}{dx} \Rightarrow$ inverse: $\varphi(u) = x$

$y = f(x)$

$$y \Rightarrow z = xu - f(x) = x\frac{df}{dx} - f(x) = \varphi(u)u - f(\varphi(u)) = g(u)$$

applied to Lagrangian:

$$y = L \quad , x = \dot{q} \Rightarrow u = \frac{\partial L}{\partial \dot{q}} = p \Rightarrow z = \dot{q}p - L(q,\dot{q},t) = H(q,p,t)$$

similarly: $H(q_1, \dots, q_f, p_1, \dots, p_f, t) = \sum_{j=1}^{f} \dot{q}_j p_j - L(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f, t)$

5.2 Hamilton equations

calculate dH and equate term dq_j, dp_j, dt

$$\Rightarrow \quad \dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad \frac{\partial H}{\partial t} = \frac{\partial L}{\partial t}$$

application protocol

(i) $q_1, \dots, q_f$

(ii) $\underline{r}_i = \underline{r}_i(q_1, \dots, q_f, t)$, $\dot{\underline{r}}_i = \dot{\underline{r}}_i(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f, t)$

(iii) $L(q_1, \dots, q_f, \dot{q}_1, \dots, \dot{q}_f, t) \Rightarrow$ (iv) $p_j = \frac{\partial L}{\partial \dot{q}_j}$

(v) $H = \sum_{j=1}^{f} \dot{q}_j p_j - L$

(vi) $\dot{q}_j = \frac{\partial H}{\partial p_j}$, $\dot{p}_j = -\frac{\partial H}{\partial q_j}$

$\dot p_\varphi = 0 \Rightarrow p_\varphi = m\rho^2\dot\varphi = \text{const}$ Drehimpulserhaltung

$\dot p_z = 0 \Rightarrow p_z = \text{const} = 0$ (ebene Bewegung)

(b) 1D harmonischer Oszillator

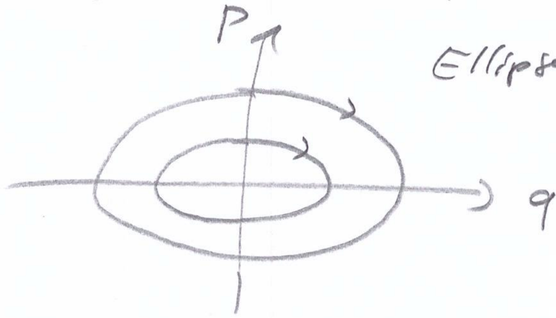
(i) – (iii) $L = \frac{m}{2}\left(\dot q^2 - \omega^2 q^2\right)$

(iv) $p = \frac{\partial L}{\partial \dot q} = m\dot q$

(v) $H = p\dot q - L = \frac{p^2}{m} - \frac{m}{2}\left(\frac{p^2}{m^2} - \omega^2 q^2\right) = \frac{p^2}{2m} + \frac{m}{2}\omega^2 q^2$

$\frac{\partial L}{\partial t} = 0$: skleronomes System $\Rightarrow E = H = T + V = \text{const}$

$\Rightarrow \frac{p^2}{2mE} + \frac{q^2}{\frac{2E}{m\omega^2}} = 1$



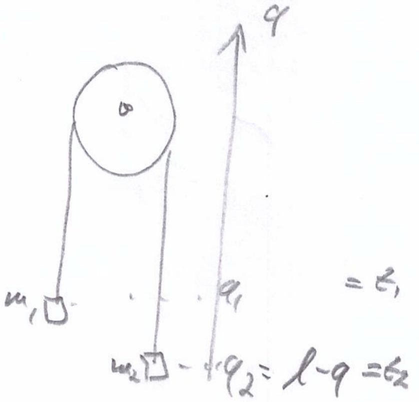
Ellipsen!

Halbachsen:

$a = \sqrt{\frac{2E}{m\omega^2}}\,, \quad b = \sqrt{2mE}$

(vi) $\left.\begin{aligned} \dot p &= -\frac{\partial H}{\partial q} = -m\omega^2 q \\ \dot q &= \frac{\partial H}{\partial p} = \frac{p}{m} \end{aligned}\right\}$ definiert Richtung in (q,p)-Ebene

(c) Atwoodsche Fallmaschine: (vgl. 3.4, ~~SMC~~d, U4 S 126)



(i)-(iii) $L = T - V = \frac{1}{2}(m_1+m_2)\dot q^2 - m_1 g q - m_2 g (l-q)$

(iv) $p = \frac{\partial L}{\partial \dot q} = (m_1+m_2)\dot q$

(v) $H = \dot q p - L = \frac{p^2}{m_1+m_2} - \frac{m_1+m_2}{2}\frac{p^2}{(m_1+m_2)^2} + m_1 g q + m_2 g (l-q)$

$= \frac{p^2}{2(m_1+m_2)} + m_1 g q + m_2 g (l-q)$

(vi) $\left.\begin{aligned} \dot p &= -\frac{\partial H}{\partial q} = -(m_1 g - m_2 g) = -(m_1 - m_2) g \\ \dot q &= \frac{\partial H}{\partial p} = \frac{p}{m_1+m_2} \end{aligned}\right\}$ check: $\ddot q = \frac{\dot p}{m_1+m_2} = -\frac{m_1-m_2}{m_1+m_2} g$

5.3 Symplektische Struktur des Phasenraums

Beobachtung: verallgemeinerte Koordinaten (q, p) nicht eindeutig

vgl.: Forminvarianz der Lagrange-Gleichungen $\tilde{L}(Q, \dot{Q}, t) = L(q(Q,t), \dot{q}(Q, \dot{Q}, t), t) + \frac{d}{dt} M(q,t)$
(3.3 Eichinvarianz)

Frage: Unter welchen Transformationen sind die Hamiltonschen Gleichungen forminvariant? $(q,p) \to (Q,P)$

$$\hookrightarrow \dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j} \longrightarrow \dot{Q}_j = \frac{\partial \tilde{H}}{\partial P_j}, \quad \dot{P}_j = -\frac{\partial \tilde{H}}{\partial Q_j}$$

Idee: Für zyklische Variablen: Finde geeignete Trafos der (q,p), s.d. alle Q zyklisch sind:

Bsp: reduziertes 2-Körperproblem: $L(r, \dot{r}, \varphi, \dot{\varphi}) = \frac{m}{2}(\dot{r}^2 + r^2 \dot{\varphi}^2) - V(r)$

φ zyklisch $\Leftrightarrow \frac{\partial L}{\partial \varphi} = 0 \Rightarrow \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \dot{\varphi} = p_\varphi = \ell = \text{const}$

Hamiltonsche Gleichungen: $p_r = \frac{\partial L}{\partial \dot{r}} = m \dot{r} \Rightarrow \dot{r} = \frac{p_r}{m}$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \dot{\varphi} \Rightarrow \dot{\varphi} = \frac{p_\varphi}{m r^2}$$

$$\Rightarrow H = p_r \dot{r} + p_\varphi \dot{\varphi} - L = \frac{m}{2}(\dot{r}^2 + r^2 \dot{\varphi}^2) + V(r) = \frac{p_r^2}{2m} + \frac{\ell^2}{2 m r^2} + V(r)$$

Def: **Kanonische Transformationen** sind diffeomorphe Transformationen (umkehrbar eindeutig, differenzierbar)

$$(q, p) \longrightarrow (Q, P)$$
$$H(q, p, t) \longrightarrow \tilde{H}(Q, P, t)$$

welche die Hamiltonschen Gleichungen forminvariant lassen

Bsp: Hamiltonsches Prinzip

$$\{q,p,H\}: \quad \delta\int_{t_1}^{t_2} dt\, L = \delta\int_{t_1}^{t_2} dt \left\{\sum_{j=1}^{f} \dot q_j p_j - H(q,p,t)\right\} = 0$$

$$\{\underline{Q},\underline{P},t\}: \quad \delta\int_{t_1}^{t_2} dt \left\{\sum_{j=1}^{f} \dot Q_j P_j - \bar H(\underline{Q},\underline{P},t)\right\} = 0$$

$$\Rightarrow \sum_{j=1}^{f} \dot q_j p_j - H(q,p,t) = \sum_{j=1}^{f} \dot Q_j P_j - \bar H(\underline{Q},\underline{P},t) + \frac{d}{dt}\underbrace{M_1(q,Q,t)}_{\text{Erzeugende der kanon. Trafo}}$$

Bew: $$\frac{d}{dt} M_1(q,Q,t) = \sum_{j=1}^{f}\left\{\frac{\partial M_1}{\partial q_j}\dot q_j + \frac{\partial M_1}{\partial Q_j}\dot Q_j\right\} + \frac{\partial M_1}{\partial t}$$

$$\Rightarrow \sum_{j=1}^{f}\left(p_j - \frac{\partial M_1}{\partial q_j}\right)\dot q_j = \sum_{j=1}^{f}\left(P_j + \frac{\partial M_1}{\partial Q_j}\right)\dot Q_j + H - \bar H + \frac{\partial M_1}{\partial t}$$

$$\Rightarrow \underbrace{p_j = \frac{\partial M_1}{\partial q_j}}, \quad \underbrace{P_j = -\frac{\partial M_1}{\partial Q_j}}, \quad \bar H = H + \frac{\partial M_1}{\partial t}$$

$$\begin{tikzcd}
\xrightarrow{\text{Umkehrung}} Q_j(q,p,t) \arrow[d] \arrow[rd, "M_1(q(Q,P),Q,t)"] & q_j(\underline{Q},\underline{P},t) \arrow[d] \arrow[ld, "M(q,Q(q,p,t))"'] \\
P_j(q,p,t) & p_j(\underline{Q},\underline{P},t)
\end{tikzcd}$$

weitere erzeugende Funktionenklassen:

- $$M_2(q,\underline{P},t) = M_1(q,\underline{Q},t) - \sum_{j=1}^{f} \overbrace{\frac{\partial M_1}{\partial Q_j}}^{-P_j} Q_j \qquad (\text{Legendre-Trafo})$$

$$\Rightarrow p_j = \frac{\partial M_2}{\partial q_j}, \quad Q_j = \frac{\partial M_2}{\partial P_j}, \quad \bar H = H + \frac{\partial M_2}{\partial t}$$

- $M_3(p,Q,t) = M_1(q,Q,t) - \sum_{j=1}^{f} \underbrace{\frac{\partial M_1}{\partial q_j}}_{p_j} q_j$

$$\Rightarrow \quad q_j = -\frac{\partial M_3}{\partial p_j} \qquad P_j = -\frac{\partial M_3}{\partial Q_j} \qquad , \; \bar{H} = H + \frac{\partial M_3}{\partial t}$$

- $M_4(p,P,t) = M_1(q,Q,t) - \sum_{j=1}^{f} \left(\underbrace{\frac{\partial M_1}{\partial q_j}}_{p_j} q_j + \underbrace{\frac{\partial M_1}{\partial Q_j}}_{-P_j} Q_j \right)$

$$\Rightarrow \quad q_i = -\frac{\partial M_4}{\partial p_j} \qquad Q_j = \frac{\partial M_4}{\partial P_j} \qquad \bar{H} = H + \frac{\partial M_4}{\partial t}$$

Bsp: Harmonischer Oszillator: $H = \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2$

$$M_1(q,Q) = \frac{m\omega}{2} q^2 \cot Q$$

$$\Rightarrow \quad \left.\begin{aligned} p &= \frac{\partial M_1}{\partial q} = m\omega q \cot Q \\ P &= -\frac{\partial M_1}{\partial Q} = \frac{m\omega}{2} \frac{q^2}{\sin^2 Q} \end{aligned}\right\} \quad \begin{aligned} q &= \left(\frac{2P}{m\omega}\right)^{1/2} \sin Q \\ p &= m\omega \left(\frac{2P}{m\omega}\right)^{1/2} \sin Q \; \frac{\cos Q}{\sin Q} \\ &= (2m\omega P)^{1/2} \cos Q \end{aligned}$$

$$\Rightarrow \quad \bar{H} = H + \underbrace{\frac{\partial M_1}{\partial t}}_{=0} = \frac{1}{2m} 2m\omega P \cos^2 Q + \frac{m\omega^2}{2} \frac{2P}{m\omega} \sin^2 Q$$

$$= \omega P (\cos^2 Q + \sin^2 Q) = \omega P$$

$$\Rightarrow \quad \bar{H}(Q,P) = \omega P \; : \; Q \text{ zyklisch!} \Rightarrow \dot{P} = -\frac{\partial \bar{H}}{\partial Q} = 0 \Rightarrow P = \text{const} = \alpha$$

$$\dot{Q} = \frac{\partial \bar{H}}{\partial P} = \omega \Rightarrow Q = \omega t + \beta$$

$$\Rightarrow \quad q = \sqrt{\frac{2\alpha}{m\omega}} \sin(\omega t + \beta)$$

iii) $M_1(q,Q,t) = -\sum_j q_j Q_j \Rightarrow p_j = \frac{\partial M_1}{\partial q_j} = -Q_j \quad , \; P_j = -\frac{\partial M_1}{\partial Q_j} = q_j$

(iii) $M_2(q,P,t) = \sum_j q_j P_j \Rightarrow p_j = \frac{\partial M_2}{\partial q_j} = P_j \quad , \; Q_j = \frac{\partial M_2}{\partial P_j} = q_j$

Fazit: q und p können ineinander transformiert werden

$\Rightarrow$ neue Notation:

$\underline{x} = \begin{pmatrix} q \\ p \end{pmatrix}$: Vektor im Phasenraum

$\underline{H}_x = \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}$: Ableitungsvektor

mit $\underline{\underline{J}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ metrik im Phasenraum / symplektischer Tensor folgt:

kanonische Gleichungen $\quad \dot{\underline{x}} = \underline{\underline{J}}\,\underline{H}_x \Leftrightarrow \begin{array}{l} \dot{q} = \frac{\partial H}{\partial p} \\ \dot{p} = -\frac{\partial H}{\partial q} \end{array}$

$$\Leftrightarrow -\underline{\underline{J}}\,\dot{\underline{x}} = \underline{H}_x$$

Eigenschaften: $\underline{\underline{J}}^2 = -\mathbb{1}$, $\quad \underline{\underline{J}}^{-1} = \underline{\underline{J}}^T = -\underline{\underline{J}}$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{\underline{\underline{J}}}\underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\underline{\underline{J}}^T} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbb{1}}$$

allgemein:

$$\underline{x} = \begin{pmatrix} q_1 \\ \vdots \\ q_f \\ p_1 \\ \vdots \\ p_f \end{pmatrix}, \quad \underline{H}_x = \begin{pmatrix} \frac{\partial H}{\partial q_1} \\ \vdots \\ \frac{\partial H}{\partial q_f} \\ \frac{\partial H}{\partial p_1} \\ \vdots \\ \frac{\partial H}{\partial p_f} \end{pmatrix} \qquad \underline{\underline{J}} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \begin{matrix} \}f \\ \}f \end{matrix}$$

$\Rightarrow \dot{\underline{x}} = \underline{\underline{J}}\,\underline{H}_x \quad \Leftrightarrow \quad -\underline{\underline{J}}\,\dot{\underline{x}} = \underline{H}_x$