5.3 Symplectic structure of the phase space

idea: find canonical transformations that leave the Hamiltonian equations

$$\dot q_j = \frac{\partial H(q_1,\dots,q_f,p_1,\dots,p_f,t)}{\partial p_j}, \quad \dot p_j = -\frac{\partial H}{\partial q_j}$$

form invariant: $(q,p) \longrightarrow (P,Q)$

$$H(q,p,t) \longrightarrow \bar H(P,Q,t)$$

- use Hamilton principle: $\delta \int_{t_1}^{t_2} dt\, L = \delta \int_{t_1}^{t_2} dt \left\{ \sum_{j=1}^{f} p_j \dot q_j - H(q,p,t) \right\} = 0$

4 equivalent forms of generating functions of canonical transformations:

(i) $M_1(q,Q,t)$: $\quad p_j = \frac{\partial M_1}{\partial q_j}, \quad P_j = -\frac{\partial M_1}{\partial Q_j}$

$\bar H = H + \frac{\partial M_1}{\partial t}$

$$\Rightarrow \frac{\partial p_j}{\partial Q_k} = \frac{\partial^2 M_1}{\partial Q_k \partial q_j} = -\frac{\partial P_k}{\partial q_j}$$

(ii) $M_2(q,P,t)$: $\quad p_j = \frac{\partial M_2}{\partial q_j}, \quad Q_j = \frac{\partial M_2}{\partial P_j}$

$= M_1(q,Q,t) - \sum_{j=1}^{f} \underbrace{\frac{\partial M_1}{\partial Q_j}}_{-P_j} Q_j$

Legendre bzgl. Q_j

$$\Rightarrow \frac{\partial p_j}{\partial P_k} = \frac{\partial^2 M_2}{\partial P_k \partial q_j} = \frac{\partial Q_k}{\partial q_j}$$

(iii) $M_3(p,Q,t)$: $\quad q_j = -\frac{\partial M_3}{\partial p_j}, \quad P_j = -\frac{\partial M_3}{\partial Q_j}$

$= M_1 - \sum_{j=1}^{f} \underbrace{\frac{\partial M_1}{\partial q_j}}_{p_j} q_j$

Legendre bzgl. q_j

$$\Rightarrow \frac{\partial q_j}{\partial Q_k} = -\frac{\partial^2 M_3}{\partial Q_k \partial p_j} = \frac{\partial P_k}{\partial p_j}$$

(iv) $M_4(p,P,t)$: $\quad q_j = -\frac{\partial M_4}{\partial p_j}, \quad Q_j = \frac{\partial M_4}{\partial P_j}$

Legendre bzgl. Q_j und q_j

$$\Rightarrow \frac{\partial q_j}{\partial P_k} = -\frac{\partial^2 M_4}{\partial P_k \partial p_j} = -\frac{\partial Q_k}{\partial p_j}$$

$$\underline{x} = \begin{pmatrix} q \\ p \end{pmatrix}, \quad \underline{H}_x = \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix}, \quad \underline{\underline{J}} = \begin{pmatrix} \underline{\underline{0}} & \underline{\underline{1}} \\ -\underline{\underline{1}} & \underline{\underline{0}} \end{pmatrix}$$

$$\Rightarrow \dot{\underline{x}} = \underline{\underline{J}}\,\underline{H}_x \iff \dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

- $\underline{\underline{J}}^2 = -\underline{\underline{1}}, \quad \underline{\underline{J}}^{-1} = \underline{\underline{J}}^T = -\underline{\underline{J}}$

$$\Rightarrow \underbrace{-\underline{\underline{J}}}_{=\underline{\underline{J}}^T}\dot{\underline{x}} = \underline{H}_x$$

mit $\underline{x} = \begin{pmatrix} q_1 \\ \vdots \\ q_f \\ p_1 \\ \vdots \\ p_f \end{pmatrix}$, $\underline{y} = \begin{pmatrix} Q_1 \\ \vdots \\ Q_f \\ P_1 \\ \vdots \\ P_f \end{pmatrix}$ und $M_{\alpha\beta} := \frac{\partial x_\alpha}{\partial y_\beta}$, $(M^{-1})_{\alpha\beta} = \frac{\partial y_\alpha}{\partial x_\beta}$ folgt:

$\alpha, \beta = 1, \dots, 2f$

$$M_{\alpha\beta} = \sum_{\mu,\nu=1}^{2f} J_{\alpha\mu} J_{\beta\nu} (M^{-1})_{\nu\mu}$$

$$\Leftrightarrow \underline{\underline{M}} = \underline{\underline{J}}\left(\underline{\underline{J}}\,\underline{\underline{M}}^{-1}\right)^T \Leftrightarrow \underline{\underline{M}}^T \underline{\underline{J}}\,\underline{\underline{M}} = \underline{\underline{J}}$$

metrischer Tensor

Matrix der 2. Ableitungen der Erzeugenden der kanonischen Trafo

$\underline{\underline{J}}$ definiert „Metrik" über das verallgemeinerte schiefsymmetrische Skalarprodukt $(\underline{x},\underline{y}) = \underline{x}^T \underline{\underline{J}}\,\underline{y} = \sum_{i,k=1}^{2f} x_i J_{ik} y_k$

(i) schiefsymmetrisch: $(\underline{y},\underline{x}) = -(\underline{x},\underline{y})$

(ii) bilinear: $(\underline{x}, \lambda_1 \underline{y}_1 + \lambda_2 \underline{y}_2) = \lambda_1 (\underline{x},\underline{y}_1) + \lambda_2 (\underline{x},\underline{y}_2)$

(iii) nichtentartet: $(\underline{x},\underline{y}) = 0 \ \forall \underline{y} \Rightarrow \underline{x} = 0$

(iv) $(\underline{x},\underline{x}) = 0$ selbstorthogonal

Fazit: Die (Form-)Invarianz der kanonischen Gleichungen $\dot{\underline{x}} = \underline{\underline{J}}\,\underline{H}_x$ unter kanonischen Transformationen kann durch die symplektische Struktur des Phasenraums beschrieben werden.

$$\dot y_j = \sum_k \frac{\partial y_j}{\partial x_k} \dot x_k \Leftrightarrow \dot{\underline{y}} = \underline{\underline{M}}^{-1} \dot{\underline{x}} = \left(\underline{\underline{J}}^{-1} \underline{\underline{M}}^T \underbrace{\underline{\underline{J}}\right) \underline{\underline{J}}}_{-\mathbb{1}} \overbrace{(\underline{\underline{M}}^T)^{-1} \underline{H}_y}^{\underline{\underline{J}}\,\underline{H}_x} = \underline{\underline{J}}\,\underline{H}_y$$

$-\underline{\underline{J}}^{-1} = \underline{\underline{J}}$

$$\frac{\partial H}{\partial y_j} = \sum_k \frac{\partial H}{\partial x_k} \frac{\partial x_k}{\partial y_j} \Leftrightarrow \underline{H}_y = \underline{\underline{M}}^T \underline{H}_x$$

$\underline{H}_x = (\underline{\underline{M}}^T)^{-1} \underline{H}_y$

$$\underline{\underline{M}}^{-1} \underline{\underline{M}} = \left(\underline{\underline{J}}^{-1} \underline{\underline{M}}^T \underline{\underline{J}}\right) \underline{\underline{M}} = \underline{\underline{J}}^{-1} \left(\underline{\underline{M}}^T \underline{\underline{J}}\,\underline{\underline{M}}\right) = \underline{\underline{J}}^{-1} \underline{\underline{J}} = \mathbb{1}$$

$$\underline{\underline{M}} = \underline{\underline{J}}\left(\underline{\underline{J}}\,\underline{\underline{M}}^{-1}\right)^T$$

$$\underline{\underline{M}} = \left(\begin{array}{cccc|ccc}
\frac{\partial q_1}{\partial Q_1} & \frac{\partial q_1}{\partial Q_2} & \cdots & \frac{\partial q_1}{\partial Q_f} & \frac{\partial q_1}{\partial P_1} & \cdots & \frac{\partial q_1}{\partial P_f} \\
\frac{\partial q_2}{\partial Q_1} & \vdots & & \vdots & \vdots & & \vdots \\
\vdots & & & & & & \\
\frac{\partial q_f}{\partial Q_1} & \frac{\partial q_f}{\partial Q_2} & & \frac{\partial q_f}{\partial Q_f} & \frac{\partial q_f}{\partial P_f} & & \frac{\partial q_f}{\partial P_f} \\
\hline
\frac{\partial p_1}{\partial Q_1} & \frac{\partial p_1}{\partial Q_2} & & \frac{\partial p_1}{\partial Q_f} & \frac{\partial p_1}{\partial P_1} & \cdots & \frac{\partial p_1}{\partial P_f} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
\frac{\partial p_f}{\partial Q_1} & \frac{\partial p_f}{\partial Q_2} & \cdots & \frac{\partial p_f}{\partial Q_f} & \frac{\partial p_f}{\partial P_1} & \cdots & \frac{\partial p_f}{\partial P_f}
\end{array}\right)$$

$$= \left(\begin{array}{c|c} \frac{\partial q}{\partial Q} & \frac{\partial q}{\partial P} \\ \hline \frac{\partial p}{\partial Q} & \frac{\partial p}{\partial P} \end{array}\right) \qquad \underline{A} = \begin{pmatrix} a_{11} & \cdots & a_{n1} \\ a_{21} & & \vdots \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

$$\underline{\underline{J}}\left(\underline{\underline{J}}\,\underline{\underline{M}}^{-1}\right)^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial q}{\partial Q} & \frac{\partial q}{\partial P} \\ \frac{\partial p}{\partial Q} & \frac{\partial p}{\partial P} \end{pmatrix}^{-1} \right]^T$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} \right]^T$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \\ -\frac{\partial Q}{\partial q} & -\frac{\partial Q}{\partial p} \end{pmatrix}^T$$

$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \left(\frac{\partial P}{\partial q}\right)^T & \left(-\frac{\partial Q}{\partial q}\right)^T \\ \left(\frac{\partial P}{\partial p}\right)^T & \left(-\frac{\partial Q}{\partial p}\right)^T \end{pmatrix} = \begin{pmatrix} \left(\frac{\partial P}{\partial p}\right)^T & -\left(\frac{\partial Q}{\partial p}\right)^T \\ -\left(\frac{\partial P}{\partial q}\right)^T & \left(\frac{\partial Q}{\partial q}\right)^T \end{pmatrix}$$

Test: oben links Quadrant

$$\frac{\partial q_j}{\partial Q_k} = -\left(\frac{\partial P_j}{\partial p_k}\right)^T = \frac{\partial P_k}{\partial p_j} \qquad (M_3)$$

oben rechts:

$$\frac{\partial q_j}{\partial P_k} = -\left(\frac{\partial Q_j}{\partial p_k}\right)^T = -\frac{\partial Q_k}{\partial p_j} \qquad (M_4)$$

unten links:

$$\frac{\partial p_j}{\partial Q_k} = -\left(\frac{\partial P_j}{\partial q_k}\right)^T = -\frac{\partial P_k}{\partial q_j} \qquad (M_1)$$

unten rechts:

$$\frac{\partial p_j}{\partial P_k} = \left(\frac{\partial Q_j}{\partial q_k}\right)^T = \frac{\partial Q_k}{\partial q_j} \qquad (M_2)$$

$$\underline{\underline{M}} = \underline{\underline{J}}\left(\underline{\underline{J}}\,\underline{\underline{M}}^{-1}\right)^T \qquad \Big|\cdot \underline{\underline{J}} \qquad (\underline{\underline{J}}^2 = -\underline{\underline{1}}$$

$$\underline{\underline{J}}\,\underline{\underline{M}} = -\underline{\underline{1}}\left(\underline{\underline{J}}\,\underline{\underline{M}}^{-1}\right)^T$$

$$= -\left(\underline{\underline{M}}^{-1}\right)^T \underline{\underline{J}}^T \qquad \Big|\cdot \underline{\underline{M}}^T$$

$$\underline{\underline{M}}^T \underline{\underline{J}}\,\underline{\underline{M}} = -\underbrace{\underline{\underline{M}}^T \left(\underline{\underline{M}}^{-1}\right)^T}_{\left(\underline{\underline{M}}^{-1}\underline{\underline{M}}\right)^T = \underline{\underline{1}}} \underline{\underline{J}}^T = -\underline{\underline{J}}^T = \underline{\underline{J}}$$

$$\Rightarrow \quad \underline{\underline{M}}^T \underline{\underline{J}}\,\underline{\underline{M}} = \underline{\underline{J}}$$

JJ

Def: Die Menge der Matrizen $\underline{\underline{M}}$ (kanonische Trafo) mit $\underline{\underline{M}}^T \underline{\underline{J}}\, \underline{\underline{M}} = \underline{\underline{J}}$ bildet die reelle symplektische Gruppe S über $\mathbb{R}^{2n}$ (Symmetriegruppe der symplektischen Struktur)